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A DIFFERENTIAL GEOMETRIC APPROACH TO ELECTROMAGNETIC LENS DESIGN

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ABSTRACT

A lens design technique developed by Carl E. Baum for transitioning TEM waves, ideally with no reflection or distortion, between cylindrical and conical transmission lines is investigated. This method uses a differential geometric approach combined with Maxwell's equations and the constitutive parameters ϵ and μ in an orthogonal curvilinear coordinate system. Isotropic but inhomogeneous media are considered. It is shown that rotational coordinate systems obtained from complex analytic transformations in the plane may be utilized in the design, and that a class of solutions to the design problem exists. This class of solutions is based on a Riccati type of differential equation.

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1. INTRODUCTION

The differential geometric approach to transient lens design is one of several promising approaches to electromagnetic problems. In general one starts with Maxwell's equations together with boundary conditions and general theorems such as conservation of energy and reciprocity and looks for various mathematical concepts for representing the solution of an EM problem. For example, operator diagonalization and the use of the complex frequency plane have proved to be extremely useful in the analysis and synthesis of EM devices (see [1]). Other promising approaches, which remain to be thoroughly investigated, but which could have important application to EMP simulators and energy transport in pulse power equipment include topological properties of scatterers and group theoretic properties as well as the differential geometric method considered in this paper.

This paper concerns inhomogeneous TEM plane waves which propagate on ideal cylindrical transmission lines with two or more independent perfectly conducting boundaries. These types of inhomogeneous media can be used to define lenses for transitioning TEM waves, without reflection or distortions, between conical and cylindrical transmission lines. While there are practical limitations (e.g., the properties of materials used to obtain the desired permittivity and permeability of the inhomogeneous medium) perfect characteristics are not really necessary. This differential geometric approach to lens design was initiated by C.E. Baum in [2] and further aspects of this method appeared in [3]. Specifically, the differential geometric scaling method creates a class of equivalent electromagnetic problems each having a complicated geometry and medium from an electromagnetic problem having a simple (Cartesian) geometry and medium. Included in these papers were examples of lenses which provided a perfect matching section between conical and cylindrical coaxial wave guides. In this paper a general design procedure for such lenses is specified and it is shown that the class of solutions of a certain Riccati equation yield a suitable lens design. This class includes in particular the examples given in [2]. Whether or not this class is in any sense unique is an open question at this time.

2. THE SCALING METHOD

We first consider a Cartesian coordinate system (x,y,z) and an orthogonal curvilinear coordinate system (u_1,u_2,u_3) with line element

$$(ds)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2 \tag{2.1}$$

with the scale factors h_i , $i = 1,2,3$, given by

$$h_i^2 = \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2 \tag{2.2}$$

If certain combinations of the h_i , assumed positive, are defined as

$$(\alpha_{ij}) = \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix}$$

$$(\beta_{ij}) = \begin{bmatrix} h_2h_3 & 0 & 0 \\ 0 & h_1h_3 & 0 \\ 0 & 0 & h_1h_2 \end{bmatrix} \tag{2.3}$$

$$(\gamma_{ij}) = \begin{bmatrix} h_2h_3/h_1 & 0 & 0 \\ 0 & h_1h_3/h_2 & 0 \\ 0 & 0 & h_1h_2/h_3 \end{bmatrix}$$

then one can write out the usual expressions for $\nabla\Phi$, $\nabla\times\vec{E}$, and $\nabla\cdot\vec{E}$ in the u_i coordinate system in terms of physical components of the vector $\vec{E}$.

Another set of vectors and operators (formal vectors and formal operators) may be written out in terms of tensor components. For example, if $\vec{E}$ has components E_1 , E_2 , E_3 referred to the u_i coordinates, then $\vec{E}'$, the formal vector, has components E'_1 , E'_2 , E'_3 and we write

$$\vec{E}' = (\alpha_{ij}) \cdot \vec{E} \tag{2.4}$$

while

$$\nabla\Phi = (\alpha_{ij})^{-1} \cdot \nabla\Phi' \text{ , where } \Phi = \Phi' \text{ ,} \tag{2.5}$$

and

$$\nabla\times\vec{E} = (\beta_{ij})^{-1} \cdot \nabla'\times E' \text{ .} \tag{2.6}$$

The result one obtains is that Maxwell's equations

$$\begin{aligned} \nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \nabla\times\vec{H} &= \vec{J} + \frac{\partial\vec{D}}{\partial t} \\ \nabla\cdot\vec{D} &= \rho \\ \nabla\cdot\vec{B} &= 0 \end{aligned} \tag{2.7}$$

together with the constitutive relations

$$\begin{aligned} \vec{D} &= (\epsilon_{ij}) \cdot \vec{E} \\ \vec{B} &= (\mu_{ij}) \cdot \vec{H} \end{aligned} \tag{2.8}$$

and the equation of continuity

$$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$$

can be rewritten in the form

$$\nabla' \times \vec{E}' = - \frac{\partial \vec{B}'}{\partial t}$$

$$\nabla' \times \vec{H}' = \vec{J}' + \frac{\partial \vec{D}'}{\partial t}$$

$$\nabla' \cdot \vec{D}' = \rho'$$

$$\nabla' \cdot \vec{B}' = 0 \qquad (2.9)$$

$$\vec{D}' = (\epsilon'_{ij}) \cdot \vec{E}'$$

$$\vec{B}' = (\mu'_{ij}) \cdot \vec{H}'$$

$$\nabla' \cdot \vec{J}' = - \frac{\partial \rho'}{\partial t}$$

where

$$(\epsilon'_{ij}) = (\gamma_{ij}) \cdot (\epsilon_{ij})$$

$$(\mu'_{ij}) = (\gamma_{ij}) \cdot (\mu_{ij}) \qquad (2.10)$$

and

$$(\alpha'_{ij}) = (\gamma_{ij}) \cdot (\alpha_{ij})$$

where it is assumed that (ϵ_{ij}) , (μ_{ij}) , (α_{ij}) are real constant diagonal matrices which are independent of frequency, though possibly functions of position. Note that the primed equations (2.9) are of the same form as equations (2.7) and (2.8), and so if we think of the u_i as a Cartesian coordinate system, a known solution of Maxwell's equations referred to Cartesian coordinates can be taken and if primed quantities are substituted for unprimed quantities, solutions to (2.9) can be found. The result turns out to be that we have a solution to Maxwell's equations for which (ϵ_{ij}) , (μ_{ij}) and (α_{ij}) may be anisotropic and/or inhomogeneous. The basic idea is then to pick (ϵ'_{ij}) , (μ'_{ij}) , and (α'_{ij}) and boundary surfaces of convenient forms in the u_i coordinate system so that a solution can be obtained in terms of the formal quantities. If some particular relationship between the u_i coordinates and the x,y,z coordinates is chosen, then the parameters (ϵ_{ij}) , (μ_{ij}) and the geometry of the boundary surfaces are determined and the solution can be applied to the case under study.

Since we are concerned with problems related to inhomogeneous isotropic media in this paper, the constitutive parameter matrices are diagonal matrices of the form

$$(\epsilon_{ij}) = \epsilon(\delta_{ij}) \ , \ (\mu_{ij}) = \mu(\delta_{ij}) \tag{2.11}$$

where ϵ and μ are scalar functions of position. The formal quantities then look like

$$(\epsilon'_{ij}) = \epsilon(\gamma_{ij}) \ , \ (\mu'_{ij}) = \mu(\gamma_{ij}) \ . \tag{2.12}$$

We impose the restriction that $\sigma = 0$, so that the conductivity matrix is the zero matrix. Note that an inhomogeneous TEM wave with subscripts 1 and 2 only has no interaction with ϵ'_{33} or μ'_{33} in which case each of the matrices (ϵ'_{ij}) and (μ'_{ij}) has constant and equal diagonal entries in the first two diagonal positions. Hence such TEM solutions may be used to define lenses to match waves onto cylindrical and/or conical transmission lines.

3. THREE-DIMENSIONAL TEM WAVES

As indicated in the previous section we consider inhomogeneous TEM plane waves which propagate on ideal cylindrical transmission lines. Such structures are assumed to have two or more separate perfect conductors in a homogeneous medium, and these conductors form a cross section in a plane perpendicular to the z axis independent of z .

If this situation is considered with reference to the formal fields discussed in Section 2 and we assume the wave to be propagating in the positive u_3 direction with formal constitutive parameters given as

$$(\epsilon'_{//}) = \begin{bmatrix} \epsilon' & 0 & 0 \\ 0 & \epsilon' & 0 \\ 0 & 0 & \epsilon'_3 \end{bmatrix} \tag{3.1}$$

$$(\mu'_{//}) = \begin{bmatrix} \mu' & 0 & 0 \\ 0 & \mu' & 0 \\ 0 & 0 & \mu'_3 \end{bmatrix}$$

where ϵ' and μ' are constants, then the dependence of ϵ'_3 and μ'_3 on coordinates doesn't matter. Hence, since only ϵ' and μ' are relevant, we may assume the medium to be formally isotropic and homogeneous. We also note that by a direct application of known results for cylindrical transmission lines to the formal setting we have, for $i = 1,2$

$$E'_i = E'_{i_0}(u_1,u_2)f(t-u_3/c')$$

$$H'_i = H'_{i_0}(u_1,u_2)f(t-u_3/c') \tag{3.2}$$

and

$$E'_3 = 0 \ , \ H'_3 = 0$$

where $c' = (\mu'\epsilon')^{-1/2}$ and $c = (\mu_0\epsilon_0)^{-1/2}$ and $f(t-u_3/c')$ can be chosen to specify the waveform. These formal fields are related by

$$E'_1 = Z'_0 H'_2 \quad \text{and} \quad E'_2 = -Z'_0 H'_1 \tag{3.3}$$

where $Z'_0 = (\mu'/\epsilon')^{1/2}$ is the formal wave impedance. These results all require that the conducting boundaries be represented in terms of only their u_1 and u_2 coordinates and lead us to the conclusion that it is only necessary to restrict the first two diagonal components of the formal matrices given in equations (3.1). The constitutive parameter matrices given in equations (2.11) still correspond to isotropic inhomogeneous media and so the formal constitutive parameter matrices have the form (2.12). If equations (2.11) and (3.1) are combined, we obtain

$$(\gamma_{ij}) = \begin{bmatrix} h_2h_3/h_1 & 0 & 0 \\ 0 & h_3h_1/h_2 & 0 \\ 0 & 0 & h_1h_2/h_3 \end{bmatrix} = \frac{1}{\epsilon}(\epsilon'_{ij}) = \frac{1}{\mu}(\mu'_{ij}) \ . \tag{3.4}$$

Since equations (3.4) imply

$$\frac{h_2h_3}{h_1} = \frac{h_3h_1}{h_2} = \frac{\epsilon'}{\epsilon} = \frac{\mu'}{\mu} \tag{3.5}$$

$$\frac{h_1h_2}{h_3} = \frac{\epsilon'_3}{\epsilon} = \frac{\mu'_3}{\mu}$$

we obtain the result that

$$h_1 = h_2 \tag{3.6}$$

and also that ϵ and μ are given by

$$\epsilon = \epsilon'/h_3 \quad \text{and} \quad \mu = \mu'/h_3 \ . \tag{3.7}$$

Clearly ϵh_3 and μh_3 are constants and the formal wave impedance is equal to the physical wave impedance since

$$Z'_0 = \left(\frac{\mu'}{\epsilon'}\right)^{1/2} = \left(\frac{\mu h_3}{\epsilon h_3}\right)^{1/2} = \left(\frac{\mu}{\epsilon}\right)^{1/2} = Z_0 \ . \tag{3.8}$$

Finally, since ϵ'_3 and μ'_3 are arbitrary, we look for orthogonal curvilinear coordinate systems for which the scale factors h_1 and h_2 are equal. For these systems the scale factor h_3 determines ϵ and μ in view of equations

(3.7). It is important to note, however, that there is another restriction on our coordinate system which results from the condition that $h_1 = h_2$. Equation (3.6), as we shall see in the next section, yields the result that the surfaces of constant u_3 must be spheres or planes.

4. COORDINATE SYSTEMS AND DIFFERENTIAL GEOMETRY

A (u_1, u_2, u_3) coordinate system with the property that $h_1 = h_2$ is constructed as follows by considering any complex analytic transformation in the plane, say

$$z = \boldsymbol{F}(w) = f(u,v) + jg(u,v) = x + jy, \quad j = \sqrt{-1}. \tag{4.1}$$

The (x,y,z) coordinates are then obtained in the form

$$\begin{aligned} x &= f(u,v)\cos(\varphi) \\ y &= f(u,v)\sin(\varphi) \\ z &= g(u,v) \end{aligned} \tag{4.2}$$

by performing a rotation about the y-axis. The line element which results is

$$(ds)^2 = h_u^2 (du)^2 + f^2 (d\varphi)^2 + h_u^2 (dv)^2, \tag{4.3}$$

where

$$h_u = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial u}\right)^2}. \tag{4.4}$$

Note that any plane orthogonal differentiable transformation, say $x = \eta(u,v)$, $y = \xi(u,v)$ would yield on rotation about the y-axis the line element

$$(ds)^2 = h_u^2 (du)^2 + \eta^2 (d\varphi)^2 + h_v^2 (dv)^2. \tag{4.5}$$

The condition that $h_u = h_v$ then implies η and ξ are the real and imaginary parts of an analytic function. This fact results from the equations

$$(\frac{\partial \eta}{\partial u})^2 + (\frac{\partial \xi}{\partial u})^2 = (\frac{\partial \eta}{\partial v})^2 + (\frac{\partial \xi}{\partial v})^2 \tag{4.6}$$

and

$$(\frac{\partial \eta}{\partial u})(\frac{\partial \eta}{\partial v}) + (\frac{\partial \xi}{\partial u})(\frac{\partial \xi}{\partial v}) = 0 \tag{4.7}$$

which yield, after a little manipulation, the Cauchy-Riemann equations

$$\frac{\partial \eta}{\partial u} = \pm \frac{\partial \xi}{\partial v} \tag{4.8}$$

$$\frac{\partial \eta}{\partial v} = \mp \frac{\partial \xi}{\partial u} . \tag{4.9}$$

Conversely, if η and ξ are the real and imaginary parts of an analytic function, then we find $h_u = h_v$.

Next, because of the requirements on the constitutive parameters we must have the scale factors h_1 and h_2 equal for the (u_1,u_2,u_3) coordinates, which are constructed from the (u,φ,v) coordinates. As a consequence additional restrictions will be imposed on our coordinate systems. However before investigating this, let us introduce some differential geometric terminology.

If one considers surfaces of constant v , then our line element (given in equation (4.3) becomes

$$(ds)^2 = E(du)^2 + G(d\varphi)^2 \tag{4.10}$$

where $E = h_u^2$ and $G = f^2$, and this equation defines the first fundamental form for these surfaces. This form may also be calculated directly from the formula

$$(ds)^2 = d\vec{R} \cdot d\vec{R} \tag{4.11}$$

where $\vec{R} = [f(u,v)\cos(\varphi)]\hat{i} + [f(u,v)\sin(\varphi)]\hat{j} + g(u,v)\hat{k}$ is the position vector of any point on the surface. The second fundamental form of a surface is denoted by II and is calculated from the formula

$$\mathit{II} = -d\vec{R} \cdot d\vec{N} \tag{4.12}$$

where $\vec{N}$ is a unit normal to our surface. For the surfaces of constant v one obtains

$$\mathit{II} = [(-\frac{\partial^2 f}{\partial u^2}\frac{\partial g}{\partial u} + \frac{\partial^2 g}{\partial u^2}\frac{\partial f}{\partial u})(du^2 + (f\frac{\partial g}{\partial u})(d\varphi)^2]/h_u \tag{4.13}$$

which we write in a simpler form as

$$\mathit{II} = L(du)^2 + N(d\varphi)^2 \tag{4.14}$$

where the coefficients L and N have the obvious meaning. When the surfaces of constant v are planes, we clearly have $L = N = 0$, and $\mathit{II} = 0$. A well known result in classical differential geometry is the fact that the surfaces of constant v are spheres or planes if and only if the coefficients of the first and second fundamental forms are in proportion. That is, if and only if

$$\frac{L}{E} = \frac{N}{G}. \tag{4.15}$$

The reader should see references [4] or [5] for a proof. One may then obtain by direct calculations the result that the surfaces of constant v are spheres or planes if and only if $h_u/f(u,v)$ is independent of v. That is, if $\frac{L}{h_u^2} = \frac{N}{f^2}$, then one obtains

$$f(\frac{\partial f}{\partial u}\frac{\partial^2 g}{\partial u^2} - \frac{\partial g}{\partial u}\frac{\partial^2 f}{\partial u^2}) = \frac{\partial g}{\partial u} h_u^2. \tag{4.16}$$

With the aid of the Cauchy-Riemann equations one may then easily obtain the result that

$$\frac{\partial}{\partial v}\,\ell n\,(h_u) = -\frac{1}{h_u^2}\left(\frac{\partial f}{\partial u}\,\frac{\partial^2 g}{\partial u^2} - \frac{\partial g}{\partial u}\,\frac{\partial^2 f}{\partial u^2}\right)$$

and hence obtain

$$\frac{\partial}{\partial v}\,\ell n\,(h_u) = \frac{\partial}{\partial v}\,\ell n\,(f)\ .$$

Thus $\frac{\partial}{\partial v}\,\ell n\,(\frac{h_u}{f}) = 0$ and we must then have $\frac{h_u}{f}$ independent of v . The preceding steps are obviously reversible and thus we have an if and only if statement.

The (u,φ,v) coordinates are now utilized to define new coordinates (u_1,u_2,u_3) as follows. We set

$$\begin{aligned}
u_1 &= \alpha_0\left(\exp \int_{b_0}^{u} \frac{h_u(u',v)}{f(u',v)}\,du'\right)(\cos(\varphi)) \\
u_2 &= \alpha_0\left(\exp \int_{b_0}^{u} \frac{h_u(u',v)}{f(u',v)}\,du'\right)(\sin(\varphi)) \qquad (4.17) \\
u_3 &= F(v) = \int_0^{v} h_u(\bar{u},v')\,dv'
\end{aligned}$$

where α_0 and b_0 are positive constants at our disposal. There are several reasons for introducing this particular choice of coordinates. First of all, we will have an orthogonal system of coordiates in which surfaces of contant u_3 are also surfaces of constant v which must also be spheres or planes. There is also some flexibility in choosing h_3 which then determines ϵ and μ . Finally, the choice of u_1 and u_2 is one which will make $h_1 = h_2$. In equation (4.17) we will take $\bar{u} = 0$ if h_u is decreasing (for fixed v) in the range of interest for the coordinates (u, φ, v) , while $\bar{u} = u_0$ if h_u is increasing (for fixed v) in the range of interest. It is also assumed that $g(u, 0) = 0$ since this will be seen to guarantee continuity of conductors across one of the boundaries of the lens, say at $v = 0$. The line element is then calculated with the result that

$$(ds)^2 = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2 \quad . \tag{4.18}$$

where

$$h_1 = h_2 = (f(u,v))/(\alpha_0 \exp \int_{b_0}^{u} \frac{h_u(u',v)}{f(u',v)} du') \tag{4.19}$$

$$h_3 = \frac{h_u(u,v)}{h_u(\bar{u},v)} \quad .$$

The scale factor h_3 will satisfy the condition that $h_3 \leq 1$ in the range of interest, and the equation (4.16) will guarantee continuity of the conductors across the other lens boundary say at $v = v_0$. Knowledge of h_3 then permits calculation of the constitutive parameters for the lens.

Let us illustrate the use of equations (4.17) in the case where the (u, φ, v) coordinates are the spherical coordinates (θ, φ, r) defined by

$$\begin{aligned} x &= r \sin(\theta) \cos(\varphi) \\ y &= r \sin(\theta) \sin(\varphi) \\ z &= z_1 + r \cos(\theta) \end{aligned} \tag{4.20}$$

where $r > 0$, $0 \leq \theta < 2\pi$, $0 \leq \varphi < \pi$ and z_1 is a constant which we can specify. Since spherical coordinates are not obtainable directly from complex analytic transformations of the form specified by equation (4.1), the condition that $h_u = h_v$ is not satisfied and we will replace h_u by h_v in the equation which defines u_3 in (4.17). We note that surfaces of constant $v = r$ are spheres and propation is in the r direction. Since the line element is

$$(ds)^2 = r^2(d\theta)^2 + r^2\sin^2(\theta)(d\varphi)^2 + (dr)^2 \tag{4.21}$$

we have

$$h_\theta = r \ , \ h_\varphi = r \ sin(\theta) = \rho \ , \ h_r = 1 \ . \tag{4.22}$$

and equations (4.17) will then yield the results for the (u_i) coordinates, which we will refer to as modified spherical coordinates. Thus

$$u_1 = \alpha_0 \exp\left[\int_{\theta_0}^{\theta} \frac{d\theta'}{sin(\theta')}\right] cos(\varphi) \ .$$

and so

$$u_1 = \beta_0 \frac{cos(\varphi)}{cot(\theta)+csc(\theta)} = \beta_0 tan(\frac{\theta}{2}) cos(\varphi)$$

where $\beta_0 = \alpha_0[cot(\theta_0)+csc(\theta_0)]$.

The coordinate u_2 is calculated in a similar way, while the coordinate u_3 is found by calculating $\int_{r_0}^{r} h_r dr'$. Thus the results for the modified spherical coordinates are

$$\begin{aligned} u_1 &= \beta_0 tan(\tfrac{\theta}{2}) cos(\varphi) \\ u_2 &= \beta_0 tan(\tfrac{\theta}{2}) sin(\varphi) \\ u_3 &= r - r_0 \end{aligned} \tag{4.23}$$

and consequently the scale factors are

$$\begin{aligned} h = h_1 = h_2 &= r[1+cos(\theta)]/\beta_0 \\ h_3 &= 1 \end{aligned} \tag{4.24}$$

for the modified spherical coordinates. The constitutive parameters, given by equations (3.7), then satisfy the conditions

$$\frac{\epsilon}{\epsilon'} = \frac{\mu}{\mu'} = 1 \qquad (4.25)$$

since $h_3 = 1$. Thus one might choose ϵ' and μ' as ϵ_0 and μ_0 with the result that ϵ and μ are also ϵ_0 and μ_0 and the medium is free space. The structure defined by perfect conductors satisfying $f(u_1, u_2) = 0$ is called a conical transmission line. The transformation specified by equations (4.23) is the well known transformation for finding the TEM waves on such a conical structure (see [6]). Thus, while this example is a relatively simple one, it is illustrative of the procedure used to construct the (u_i) coordinate system. Moreover, the conical transmission line is used later in our design procedure.

We next derive some formulas for the radius and center of the spheres corresponding to surfaces of constant v in the case where the second fundamental form coefficients L and N are non-zero. Several approaches may be used and perhaps the simplest one would be to note that in our case we have the second fundamental form II expressed as

$$\mathit{II} = \lambda (ds)^2 .$$

One may then show (see [5] for example) that the radius R of a sphere of constant v is $R = \frac{1}{|\lambda|}$, which may also be written as $R = K^{-1/2}$, where K is the Gaussian curvature given by $K = \frac{LN}{EG}$. Direct substitution from our previous formulas for L, N, E, and G yield the formulas

$$K = \frac{1}{h_u^4}\left(\frac{\partial}{\partial v} h_u\right)^2 \qquad (4.26)$$

and

$$R = \left| f(u,v) h_u(u,v) / \frac{\partial g}{\partial u} \right| . \qquad (4.27)$$

The quantities K and R depend only upon v which is assumed to be constant. The center of each sphere is located on the z axis at a point z_0 (depending on v alone) given by

$$z_0 = (f\frac{\partial f}{\partial u}+g\frac{\partial g}{\partial u})/\frac{\partial g}{\partial u} \ . \tag{4.28}$$

This result may be checked easily from substitution into the equation $x^2 + y^2 + (z-z_0)^2 = R^2$.

5. THREE-DIMENSIONAL TEM LENSES

We now consider the design of lenses for transporting TEM waves of the type considered in Section 3. These TEM waves propagate on transmission lines with independent perfectly conducting boundaries described by

$$f(u_1,u_2) = 0 \tag{5.1}$$

That is, the boundaries do not involve the u_3 coordinate. These lenses will be used to transition TEM waves between conical and/or cylindrical transmission lines.

We begin with a given coordinate system similar to that given by equation (4.2), namely

$$\begin{aligned} x &= af(u,v)\cos(\varphi) \\ y &= af(u,v)\sin(\varphi) \\ z &= ag(u,v) \end{aligned} \tag{5.2}$$

where f , g are the real and imaginary parts of an analytic function of u and v , and a is a positive constant to be specified later. The (u_1,u_2,u_3) coordinates are specified by equations (4.17), i.e., by

$$u_1 = \alpha_0 \exp\left(\int_{b_0}^{u} \frac{h_u(u',v)}{f(u',v)}du'\right)(\cos(\varphi))$$

$$u_2 = \alpha_0 exp(\int_{b_0}^{u} \frac{h_u(u',v)}{f(u',v)} du')(sin(\varphi)) \tag{5.3}$$

$$u_3 = \int_0^v h_u(u_0,v')dv' \equiv F(v) ,$$

where h_u (given as in equation (4.4)) is the scale factor which appears in

$$(ds)^2 = h_u^2(du)^2 + f^2(d\varphi)^2 + h_u^2(dv)^2 . \tag{5.4}$$

and α_0 and b_0 positive constants to be specified. The line element, $(ds)^2$, may then be written in terms of the (u_1,u_2,u_3) coordinates as

$$(ds)^2 = h^2[(du_1)^2 + (du_2)^2] + h_3^2(du_3)^2 , \tag{5.5}$$

where

$$h^2 = [f^2 exp(-2\int_{b_0}^{u} \frac{h_u(u',v)}{f(u',v)} du')](\frac{a}{\alpha_0})^2 \tag{5.6}$$

and

$$h_3^2 = \frac{h_u^2}{F'(v)^2} = \frac{h_u^2(u,v)}{h_u^2(u_0,v)} = \frac{(\frac{\partial f}{\partial u}(u,v))^2 + (\frac{\partial f}{\partial v}(u,v))^2}{(\frac{\partial f}{\partial u}(u_0,v))^2 + (\frac{\partial f}{\partial v}(u_0,v))^2} . \tag{5.7}$$

Note that on the one hand if u_0 is a positive maximum for u in the range of interest and if $\frac{\partial h_u}{\partial u} > 0$ so that for fixed v, $h_u(u,v)$ is a monotonically increasing function of u for $0 \leq u \leq u_0$, then $h_3 \leq 1$ and $h_3 = 1$ on $u = u_0$. Since h_3 is also related to the constitutive parameters from the equations (3.7), i.e., from

$$\frac{\epsilon}{\epsilon'} = \frac{\mu}{\mu'} = \frac{1}{h_3} \tag{5.8}$$

one obtains the same condition on h_3 by setting $\epsilon' = \epsilon_0$, $\mu' = \mu_0$ and restricting $\epsilon \geq \epsilon_0$ and $\mu \geq \mu_0$.

On the other hand, if $\mu_0 = 0$ and $\frac{\partial h_u}{\partial u} < 0$ for $u \geq 0$ then for fixed v, h_u will be a monotonically decreasing function of u for $u \geq 0$ and once again $h_3 \leq 1$ and $h_3 = 1$ when $u = 0$. The analysis which follows is based on these two observations, which we'll separate into two cases. Thus, we have:

Case I: The main assumptions in this case are:

(a) for fixed v, $\frac{\partial h_u}{\partial u} > 0$ in a range $0 \leq u \leq u^*$.

(b) $R^2 < z_0^2$, where R and z_0 (see equations (4.27) and (4.28)) denote the radius and center of spheres corresponding to surfaces of constant V ,

(c) $\frac{\partial f}{\partial v}(u,v)\Big|_{v=0} = 0$, and

(d) $g(u,v)$ for fixed u is an odd function in v .

Case II: The main assumptions in this case are:

(a) for fixed v , $\frac{\partial h_u}{\partial u} < 0$ in a range $0 \leq u \leq u^*$

(b) $R^2 > z_0^2$ where R and z_0 are as above, and

(c) $\frac{\partial f}{\partial v}(u,v)\Big|_{v=0} = 0$, and

(d) $g(u,v)$ for fixed u is an odd function in v .

The analysis for Case I will correspond to a convergent lens, while that for Case II will correspond to a divergent lens.

Case I Analysis: (Convergent Lens).

Now that the u_i coordinates and scale factors h_i are known via equations (5.3), (5.6), and (5.7) let us consider the problem of joining a lens to a cylindrical transmission line. On the plane $z = 0$ on which $v = 0$ and $u_3 = 0$ (the condition $g(u,0) = 0$ will guarantee this) we have

$$u_1 = \alpha_0 exp\left(\int_{b_0}^{u} \frac{\frac{\partial f}{\partial u}(u',0)}{f(u',0)} du'\right)(cos(\varphi))$$

and so $u_1 = \alpha_0 exp(\ln f(u,0))cos(\varphi) = \alpha_0 f(u,0)cos(\varphi)$ while $x = af(u,0)cos(\varphi)$ from equations (5.1). Thus, if $\alpha_0 = a$ we will have

$$u_1 = x$$

and also $u_2 = y$ by a similar calculation. Now if the lens material specifying μ and ϵ is present in the region where $u_3 < 0$ (corresponding to $v < 0$, $z < 0$) and if for $z > 0$ the medium is free space, and if there are two or more perfect conductors forming a transmission line described by

$$f(x,y)\big|_{z \geq 0} = 0 \text{ , and } f(u_1,u_2)\big|_{z<0} = 0$$

then clearly the conductors are continuous through the interface. One may then conclude that the tangential components of $\vec{E}$ and $\vec{H}$ are continuous across $z = 0$ and the TEM waves are matched at the interface, and hence a TEM wave in the inhomogeneous lens (where coordinates are (u_1,u_2,u_3) and fields are $\vec{E}'$ and $\vec{H}'$) will propagate into free space with no reflection. This TEM wave has the form

$$\vec{E}' = G(t-u_3/c')\nabla_e{}'(u_1,u_2) \tag{5.9}$$

$$\vec{H}' = G(t-u_3/c')\nabla_h{}'(u_1,u_2)$$

We next introduce a second interface at $v = v_0 < 0$. The surface is a sphere described by

$$x^2 + y^2 + (z-z_0)^2 = R^2$$

where z_0 and R are given by equations (4.26) and (4.27). The region inside the sphere is assumed to be free space and in this region a conical transmission line with conductors matching those in the lens is placed. In order to center the apex of the conical line at the center of the sphere we choose $z_1 = z_0$. The coordinates u_1 and u_2 for the conical line are as given in equations (4.23), namely,

$$u_1 = \beta_0 \tan(\frac{\theta}{2}) \cos(\varphi)$$

$$u_2 = \beta_0 \tan(\frac{\theta}{2}) \sin(\varphi)$$

while for the lens the u_1 and u_2 coordinates are

$$u_1 = \alpha_0 (\exp \int_{b_0}^{u} \frac{h_u(u',v_0)}{f(u',v_0)} du')(\cos(\varphi))$$

$$u_2 = \alpha_0 (\exp \int_{b_0}^{u} \frac{h_u(u',v_0)}{f(u',v_0)} du')(\sin(\varphi)) \ .$$

We seek conditions which will guarantee continuity of the u_i across the sphere surface corresponding to $v = v_0$. On this surface the cylindrical coordinate ρ is given by

$$\rho = R \ \sin(\theta) \tag{5.10}$$

for the conical line, while for the lens we have

$$\rho = f(u,v_0) \ . \tag{5.11}$$

Since R, which depends only on v (a constant), is given by

$$R = \left| f(u,v_0) h_u(u,v_0) / \frac{\partial g}{\partial u} \right| \tag{5.12}$$

from equation (4.26), we must have

$$\sin(\theta) = \frac{1}{h_u(u,v_0)} \left| \frac{\partial g}{\partial u} \right| . \tag{5.13}$$

Thus

$$\beta_0 \tan(\frac{\theta}{2}) = \alpha_0 \exp \int_{b_0}^{u} \frac{h_u(u',v_0)}{f(u',v_0)} \, du'$$

and so

$$\tan^2(\frac{\theta}{2}) = (\frac{\alpha_0}{\beta_0})^2 \exp\left[2 \int_{b_0}^{u} \frac{h_u(u',v_0)}{f(u',v_0)} du' \right] . \tag{5.14}$$

If the trigonometric identity

$$\tan^2(\frac{\theta}{2}) = \frac{1-\cos(\theta)}{1+\cos(\theta)} = \frac{1-\sqrt{1-\sin^2(\theta)}}{1+\sqrt{1-\sin^2(\theta)}} , \; -\pi < \theta < \pi . \tag{5.15}$$

is used with $\sin(\theta)$ given by equation (5.13) and h_u given as in equation (4.4), one obtains

$$\tan^2(\frac{\theta}{2}) = \frac{h_u - \frac{\partial f}{\partial u}}{h_u + \frac{\partial f}{\partial u}} \tag{5.16}$$

and hence we must then have for $v = v_0$.

$$\left[\frac{\alpha_0}{\beta_0} \right]^2 = \left[\frac{h_u - \frac{\partial f}{\partial u}}{h_u + \frac{\partial f}{\partial u}} \right] \exp\left[-2 \int_{b_0}^{u} \frac{h_u(u',v_0)}{f(u',v_0)} du' \right] . \tag{5.17}$$

While α_0 and β_0 are constants it is not clear that the right hand side of equation (5.17) is independent of u . The fact that it does not depend on u is a consequence of the condition, given in equation (4.16), that the surfaces of constant v be spheres or planes. It is an easy exercise to check that the partial derivative of the right hand side of equation (5.17) with respect to u vanishes identically on surfaces of constant v. Thus u_1 and u_2 will be continuous across the surface corresponding to $v = v_0$. We would also like to have u_3 continuous across this surface. For the conical line we have

$$u_3 = R - r_0 \tag{5.18}$$

while for the lens

$$u_3 = \int_0^{v_0} h_u(u_0, v')dv' \tag{5.19}$$

and hence we choose r_0 by setting

$$r_0 = R - \int_0^{v_0} h_u(u_0, v')dv' \tag{5.20}$$

Since u_1 and u_2 are continuous across the surface $v = v_0$, the scale factor h is also continuous there. However, h_3 will have a step discontinuity at this surface. In any case our TEM wave will propagate without reflection through this surface.

To summarize the situation for Case I, we have the (u_i) coordinates inside the $v = v_0$ sphere given by equations (4.23) and the constitutive parameters are ϵ_0 and μ_0. In the lens bounded by $v = 0$, $v = v_0$, and $u = u_0$, the (u_i) coordinates are given by equations (5.3) while the constitutive parameters are determined from equations (5.7) and (5.8) with $\epsilon' = \epsilon_0$ and $\mu' = \mu_0$. Thus we will have

$$\frac{\epsilon}{\epsilon_0} = \frac{\mu}{\mu_0} = \frac{1}{h_3} = \frac{h_u(u_0, v)}{h_u(u, v)} \tag{5.21}$$

and $h_3 < 1$ within the lens, which will be classified as a convergent lens. In the next section we will give a specific example of such a lens by chosing a suitable orthogonal coordinate system of the form given by equations (4.2).

Case II Analysis: (Divergent Lens).

The analysis in this case is similar to that in Case I. Again we wish to join cylindrical and conical transmission lines to our lens. We take one boundary surface for the lens to be the plane $z = 0$ on which $v = 0$, and hence $u_3 = 0$. Note that this is guaranteed by equations (5.2) and (5.3) coupled with the assumption that $g(u,v)$ is an odd function in v . As in Case I we find that $u_1 = x$ and $u_2 = y$ when $\alpha_0 = a$. However, in this case we assume that the lens material is present only for $u_3 > 0$ (corresponding to $v > 0$ and $z > 0$) and that for $z < 0$ the medium is free space with constitutive parameters ϵ_0 and μ_0 , and (u_1, u_2, u_3) are cartesian coordinates (x, y, z) when $z \leq 0$, where we have a cylindrical transmission line. In the lens the conductors are curved to satisfy the expressions for u_1 and u_2 in equation (5.3) and the condition given in equation (5.1) which describes the conductor boundaries. Thus the u_i , h , and transmission line conductors are continuous through the plane $z = 0$ and tangential components of $\vec{E}$ and $\vec{H}$ will be continuous through this plane. However, for $z < 0$ we have $h_3 = h = 1$ and h_3 has a step discontinuity at $z = 0$.

The other bounding surface of the lens is introduced at $v = v_0 > 0$ and this surface is again assumed to be a sphere whose equation is

$$x^2 + y^2 + (z-z_0)^2 = R^2$$

where z_0 and R are given by equations (4.27) and (4.28), and $z_0 < R$. The region outside of this sphere is free space and it contains a conical transmission line whose conductors are matched to those in the lens. One may then calculate the u_i coordinates for the lens and compare them with the modified spherical coordinates for the conical line to obtain conditions which guarantee the continuity of u_1, u_2 and h across the surface $v = v_0$ so that the TEM wave will pass through this surface without reflection. The constitutive parameters for the lens are given by

$$\frac{\epsilon}{\epsilon_0} = \frac{\mu}{\mu_0} = \frac{1}{h_3} = \frac{h_u(0,v)}{h_u(u,v)} \tag{5.22}$$

where the lens is bounded by $v = 0$, $v = v_0$, and $u = u_0$, and $h_3 < 1$ within the lens, which is classified as a divergent lens.

6. SOME EXAMPLES OF LENSES

The procedure described in Section 5 will be illustrated by the examples which appear in reference [2]. Our first example is that of an inhomogeneous lens. If we start with a complex analytic transformation in the plane

$$z = a \tan\left(\frac{w}{2}\right) = a\left[\frac{\sin(u)}{\cos(u) + \cosh(v)} + i\,\frac{\sinh(v)}{\cos(u) + \cosh(v)}\right] \tag{6.1}$$

where a is a positive constant and take

$$x = f(u,v) = \frac{a \sin(u)}{\cos(u) + \cosh(v)} \tag{6.2}$$

$$y = g(u,v) = \frac{a \sinh(v)}{\cos(u) + \cosh(v)}$$

and rotate the plane about the y axis, we obtain bispherical coordinates

$$x = \left[\frac{a \sin(u)}{\cos(u) + \cosh(v)}\right]\cos(\varphi)$$

$$y = \left[\frac{a \sin(u)}{\cos(u) + \cosh(v)}\right]\sin(\varphi) \tag{6.3}$$

$$z = \frac{a \sinh(v)}{\cos(u) + \cosh(v)}$$

with $0 \leq u \leq \pi$, $-\infty < v < +\infty$. Surfaces of constant v are spheres, while surfaces of constant u are either "apple-shaped" with dimples on the x axis (for $u < \frac{\pi}{2}$) or spindles (for $u > \frac{\pi}{2}$). The surfaces of constant u will intersect planes of constant φ in circles. The scale factors are

$$h_u = h_v = \frac{a}{\cos(u) + \cosh(v)} = \frac{\rho}{\sin(u)}, \tag{6.4}$$

$$h_\varphi = \frac{a \sin(u)}{\cos(u) + \cosh(v)} = f(u,v).$$

From equations (4.17) we calculate the modified u_i coordinates and obtain (using $b_0 = \pi/2$ and $\bar{u} = u_0$)

$$u_1 = \alpha_0 \tan(\frac{u}{2})\cos(\varphi)$$

$$u_2 = \alpha_0 \tan(\frac{u}{2})\sin(\varphi) \tag{6.5}$$

$$u_3 = \frac{2a}{\sin(u_0)}\arctan[\tanh(\frac{v}{2})\tan(\frac{u_0}{2})].$$

while the scale factors which are calculated from equations (4.19) are given by

$$h_1 = h_2 = \frac{a}{\alpha_0}\frac{1 + \cos(u)}{\cos(u) + \cosh(v)}$$

and

$$h_3 = \frac{\cos(u_0) + \cosh(v)}{\cos(u) + \cosh(v)} \tag{6.6}$$

where $0 \leq u < u_0$ in the inhomogeneous medium. We also note that since $h_u(u,v)/f(u,v) = 1/\sin(u)$, $h_u/f(u,v)$ is independent of v and so, as expected, the surfaces of constant v are spheres or planes. The radii and location of the centers of the spheres are calculated from equations (4.27) and (4.28), and we obtain

$$R = \left| f(u,v)h_u(u,v)/\frac{\partial g}{\partial u} \right| = a\,|\sinh(v)|^{-1} \tag{6.7}$$

and

$$z_0 = (f\frac{\partial f}{\partial u}+g\frac{\partial g}{\partial u})/\frac{\partial g}{\partial u} = a \coth(v) \ . \qquad (6.8)$$

Thus if we follow the analysis outlined in Case I of section 5 we find that on the plane $z = 0$ on which $v = 0$ and $u_3 = 0$ we will have $u_1 = x$ and $u_2 = y$ if $a = \alpha_0$. Hence the transmission line conductors are continuous through this interface. The medium is free space when $z > 0$ and the lens material is present in the region defined by $u_3 < 0$ (which corresponds to $v < 0$, $z < 0$). A second interface is then introduced at $v = v_0 < 0$ and the surface is a sphere whose equations is then

$$x^2 + y^2 + (z-a \coth(v_0))^2 = a^2/\sinh^2(v_0) \ . \qquad (6.9)$$

A cross section of this lens is illustrated in figure 1. At the second interface continuity of the u_1 and u_2 coordinates is guaranteed if $\alpha_0 \tan(\frac{u}{2}) = \beta_0 \tan(\frac{\theta}{2})$ on v_0, where α_0 and β_0 are chosen to satisfy the condition

$$\left[\frac{\alpha_0}{\beta_0}\right]^2 = \frac{\cosh(v_0) - 1}{\cosh(v_0) + 1} = \tanh^2(\frac{v_0}{2}) \qquad (6.10)$$

and so on $v = v_0$,

$$\tanh(\frac{v_0}{2})\tan(\frac{u}{2}) = \tan(\frac{\theta}{2}) \ . \qquad (6.11)$$

The constant r_0 is determined by

$$r_0 = \left|\frac{a}{\sinh(v_0)}\right| - \frac{2a}{\sin(u_0)}\arctan[\tanh(\frac{v_0}{2})\tan(\frac{u_0}{2})] \ . \qquad (6.12)$$

Moreover, the constitutive parameters for the lens material are given by

$$\frac{\epsilon}{\epsilon_0} = \frac{\mu}{\mu_0} = \frac{\cos(u) + \cosh(v)}{\cos(u_0) + \cosh(v)} = \frac{1}{h_3} \qquad (6.13)$$

Finally, we note that all of the assumptions made for Case I are satisfied in this example. That is, we have:

(a) for fixed $\frac{\partial h_u}{\partial v} = -\frac{a\sinh(v)}{(\cosh(v) + \cos(u))^2} > 0$ for fixed $v < 0$

(b) $R^2 < (z_0)^2$ where R, z_0 are given by equations (6.7) and (6.8)

(c) $\frac{\partial f}{\partial v}(u,v) = -\frac{a\ \sin(u)\ \sinh(v)}{(\cos(u) + \cosh(v))^2}\Big|_{v=0} = 0$ and

(d) $g(u,v) = \frac{a\ \sinh(v)}{\cos(u) + \cosh(v)}$ for fixed u is an odd function in v.

The lens given in this example is that of a convergent lens.

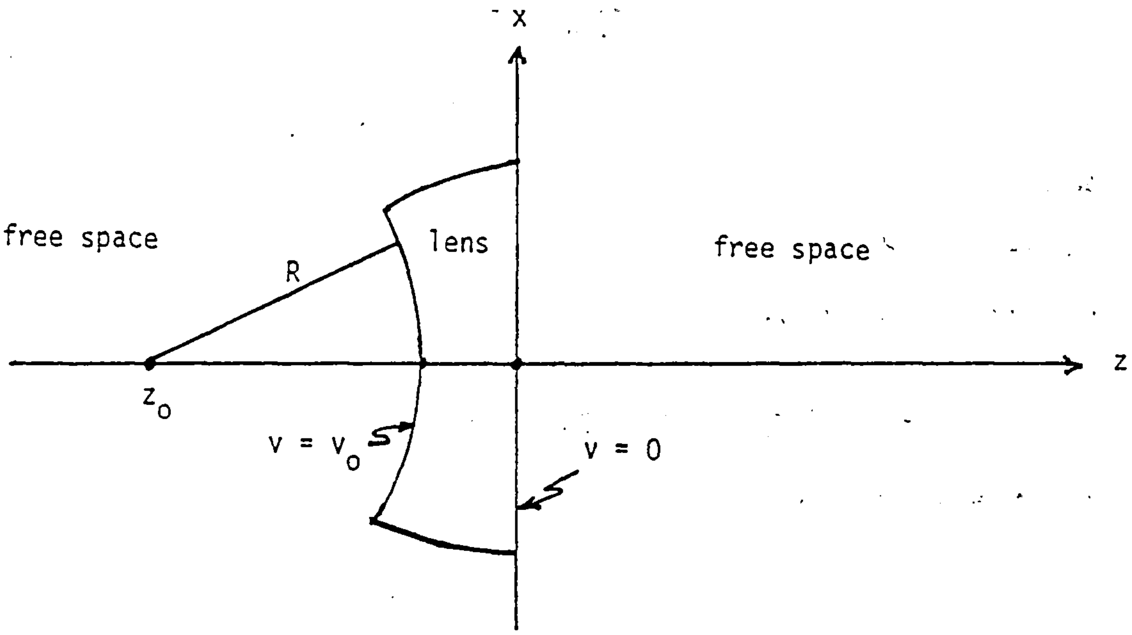

Figure 1

Our second example is that of an inhomogeneous divergent lens. We start, as in our first example, with a complex analytic transformation in the plane. This time we consider

$$z = a\ \tanh(\frac{w}{2}) = a[\frac{\sinh(u)}{\cosh(u) + \cos(v)} + i\frac{\sin(v)}{\cosh(u) + \cos(v)}] \qquad (6.14)$$

where a is a positive constant, and take

$$x = f(u,v) = \frac{a \; \sinh(u)}{\cosh(u) + \cos(v)}$$

$$(6.15)$$

$$y = g(u,v) = \frac{a \; \sin(v)}{\cosh(u) + \cos(v)} \; .$$

A rotation of the plane about the y axis yields a toroidal coordinate system

$$\begin{aligned} x &= \left[\frac{a \; \sinh(u)}{\cosh(u) + \cos(v)}\right]\cos(\varphi) \\ y &= \left[\frac{a \; \sinh(u)}{\cosh(u) + \cos(v)}\right]\sin(\varphi) \\ z &= \frac{a \; \sin(v)}{\cosh(u) + \cos(v)} \end{aligned} \tag{6.16}$$

where $0 \leq u < +\infty$, $-\pi < v \leq \pi$. Surfaces of constant v are spheres, surfaces of constant φ are half planes through the z axis, and surfaces of constant u are toroids. Since the line element is

$$(ds)^2 = \left(\frac{a}{\cosh(u) + \cos(v)}\right)^2 [(du)^2 + \sinh^2(u)(d\varphi)^2 + (dv)^2] \tag{6.17}$$

the scale factors are given by

$$h_u = h_v = \frac{a}{\cosh(u) + \cos(v)} \tag{6.18}$$

and

$$h_\varphi = \frac{a \; \sin(u)}{\cosh(u) + \cos(v)} = f(u,v) \; . \tag{6.19}$$

The modified coordinates u_i are calculated from equations (4.17), with $b_0 = \infty$ and $\bar{u} = 0$, and we obtain

$$\begin{aligned} u_1 &= \alpha_0 \tanh\left(\frac{u}{2}\right)\cos(\varphi) \\ u_2 &= \alpha_0 \tanh\left(\frac{u}{2}\right)\sin(\varphi) \\ u_3 &= a \; \tan\left(\frac{v}{2}\right) \; . \end{aligned} \tag{6.20}$$

The scale factors for the u_l coordinates are then

$$h = h_1 = h_2 = \frac{a}{\alpha_0} \frac{\cosh(u) + 1}{\cosh(u) + \cos(v)} \tag{6.21}$$

while

$$h_3 = \frac{1 + \cos(v)}{\cosh(u) + \cos(v)} \tag{6.22}$$

and clearly $h_3 = 1$ on $u = 0$ and for fixed v, $-\pi < v < \pi$, h_3 is a monotonically decreasing function of u. The surfaces of constant v are spheres (the ratio h_u/f is of course independent of v) whose radii and centers are calculated from equations (4.27) and (4.28) as

$$R = \left| f(u,v) h_u(u,v) / \frac{\partial g}{\partial u} \right| = a \left| \sin(v) \right|^{-1} \tag{6.23}$$

and

$$z_0 = \left(f \frac{\partial f}{\partial u} + g \frac{\partial g}{\partial u}\right) / \frac{\partial g}{\partial u} = -a \cot(v) \ . \tag{6.24}$$

If we proceed as in the analysis outlined in Case II of section 5, and join the conical and cylindrical transmission lines to the lens, then one boundary surface for the lens is taken as the plane $z = 0$ on which $v = 0$, $u_3 = 0$. The lens material is assumed to be present only for $u_3 > 0$ (which corresponds to $v > 0$, $z > 0$ as in figure 2). For $z < 0$ the medium is free space and for $z \leqslant 0$ the u_l coordinates are rectangular cartesian coordinates. In the lens the conductors are curved so as to satisfy equations (5.1) and (6.20) with $a = \alpha_0$. For the toroidal coordinates defined by equations (6.16) we confine our attention to those u satisfying $0 \leqslant u \leqslant u^* < +\infty$. Thus $u = u^*$ will be a boundary for the lens material, and the u_l, h, and transmission line conductors are continuous through the plane $z = 0$.

A second lens surface is introduced at $v = v_0$, where $0 < v_0 < \pi$. This surface is the surface of a sphere whose equation is

$$x^2 + y^2 + (z+a \cot(v_0))^2 = a^2/\sin^2(v_0) \tag{6.25}$$

Thus this sphere is centered on the z axis at $z_0 = -a \cot(v_0)$ and it has a radius equal to $a\left|\sin(v_0)\right|^{-1}$. Figure 2 shows a lens cross section in the xz plane. The region outside of this sphere is taken to be free space which contains a conical transmission line with conductors matched to those in the lens. The apex of the conical line is centered at z_0 by choosing $z_1 = -a \cot(v_0)$ in equation (4.20). The u_l coordinates at $v = v_0$ are matched by comparing the equations describing the modified spherical coordinates with those describing the modified toroidal coordinates (see equations (4.23) and (6.20). The resulting requirement is that we must have

$$\alpha_0 \tanh(\frac{u}{2}) = \beta_0 \tan(\frac{\theta}{2}) \tag{6.26}$$

on $v = v_0$. Moreover equation (5.16) requires

$$\left[\frac{\alpha_0}{\beta_0}\right]^2 = \frac{1 - \cos(v_0)}{1 + \cos(v_0)} = \tan^2(\frac{v_0}{2}) \tag{6.27}$$

and so

$$\tan(\frac{v_0}{2})\tanh(\frac{u}{2}) = \tan(\frac{\theta}{2}) \quad , \tag{6.28}$$

which gives a relation between the θ and u coordinates on $v = v_0$. Since the u_3 coordinate is specified for the lens by equation (6.20) and for the conical line by equation (4.23), we must have on $v = v_0$ the relation

$$a \tan(\frac{v_0}{2}) = a\left|\sin(v_0)\right|^{-1} - r_0 \tag{6.29}$$

and hence we must have r_0 defined by

$$\frac{r_0}{a} = \frac{1}{\sin(v_0)} - \tan(\frac{v_0}{2}) = -\cot(v_0) \ . \tag{6.30}$$

Thus u_1 and u_2 will be continuous across $v = v_0$ and hence h, given by equation (6.21), will also be continuous on this surface. However h_3, given in equation (6.22), has a step discontinuity on $v = v_0$. In any event, our TEM wave will pass through this surface with no reflection. In the lens, which is bounded by $v = 0$, $v = v_0$, $u = u^*$, the modified coordinates are given by equations (6.20), and the constitutive parameters are given by equations (5.21) and (6.22), where ϵ' and μ' assume their free space values. Thus we have

$$\frac{\epsilon}{\epsilon_0} = \frac{\mu}{\mu_0} = \frac{1}{h_3} = \frac{\cosh(u) + \cos(v)}{1 + \cos(v)} \tag{6.31}$$

for the lens. This lens, which is based on a toroidal coordinate system, is classified as a divergent lens. For this lens, all of the assumptions which were made for Case II are satisfied. That is,

(a) $\frac{\partial h_u}{\partial u} = - \frac{a \sinh(u)}{(\cosh(u) + \cos(v))^2} < 0$ for $u > 0$,

(b) $R^2 > z_0^2$ where R and z_0 are as given in equations (6.23) and (6.24),

(c) $\frac{\partial f}{\partial v}(u,v)\Big|_{v=0} = \frac{a \sinh(u) \sin(v)}{(\cosh(u) + \cos(v))^2}\Big|_{v=0} = 0$, and

(d) $g(u,v) = \frac{a \sin(v)}{\cosh(u) + \cos(v)}$ for fixed u is an odd function of v.

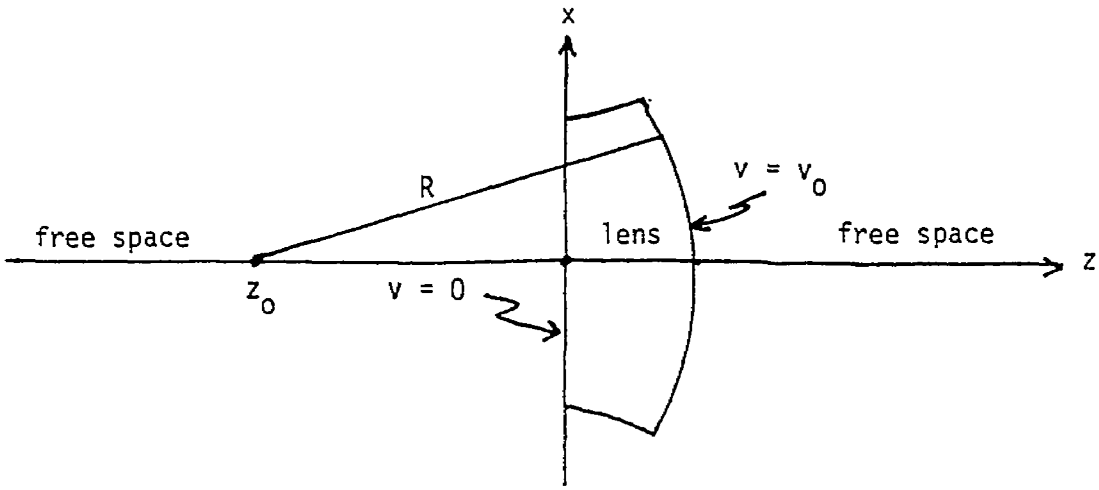

Figure 2

Thus in this section examples of lenses which provide a perfect matching section between conical and cylindrical coaxial waveguides have been given. In order to obtain a class of solutions for a lens design it should be clear that the condition given in equation (4.16), namely,

$$f\left(\frac{\partial f}{\partial u}\frac{\partial^2 g}{\partial u^2} - \frac{\partial g}{\partial u}\frac{\partial^2 f}{\partial u^2}\right) = \frac{\partial g}{\partial u} h_u^2 \ ,$$

is one that must be satisfied. The functions f and g must come from the real and imaginary parts of an analytic function of a complex variable. In the next section we will find that a class of solutions to the design problem can be described.

7. A CLASS OF SOLUTIONS TO THE LENS DESIGN PROBLEM

In Section 5 the conditions necessary to obtain a satisfactory design for a lens section matching a conical transmission line to a cylindrical transmission line were stated. The main result of this section is that the solution of a Riccati differential equation

$$\frac{dp}{dq} = \frac{1}{2}(1+p^2) \qquad (7.1)$$

where $p = z$ and $q = w$ or $p = iz$ and $q = iw$, $i = \sqrt{-1}$, will satisfy both equation (4.16) and conditions (a), (b), (c), and (d) of Case I or Case II for either a convergent or a divergent lens. For simplicity we will assume that $a = 1$, where a is the constant which appears in equations (5.2). If $a \neq 1$, the form of (7.1) becomes

$$\frac{dp}{dq} = \frac{a}{2}\left(1+\left(\frac{p}{a}\right)^2\right) , \tag{7.2}$$

and thus our analysis when $a \neq 1$ is basically unaltered.

We consider first the case in which $p = z$ and $q = w$ in equation (7.1), and we investigate the initial value problem

$$\frac{dz}{dw} = \frac{1}{2}(1+z^2) \tag{7.3}$$

$$z(0) = x_0 .$$

The solution of this problem will be

$$z = \frac{[(1-x_0^2)\sin(u) + 2x_0\cos(u)] + i[(1+x_0^2)\sinh(v)]}{(1-x_0^2)\cos(v) + (1+x_0^2)\cosh(v) - 2x_0\sin(u)} \tag{7.4}$$

where x_0 is a real number, and $w = u + iv$. This solution is obtained in a standard way by first noting that $z = \tan\left(\frac{w+c}{2}\right)$, with c an arbitrary constant, is a solution of the differential equation. If one uses various trigonometric and hyperbolic identities, the result (7.4) will then be obtained, with the initial value x_0 satisfying $\tan\left(\frac{c}{2}\right) = x_0$. We next note that if we write

$$z = \mathbf{F}(w) = f(u,v) + ig(u,v) , \tag{7.5}$$

then

$$\frac{dz}{dw} = \frac{\partial f}{\partial u} + i\frac{\partial g}{\partial u} = \frac{1}{2}(1+z^2) \tag{7.6}$$

and so

$$\frac{dz}{dw} = \frac{1}{2}[(1+f^2(u,v)-g^2(u,v)) + i2f(u,v)g(u,v)] \quad . \tag{7.7}$$

Hence we obtain

$$\frac{\partial f}{\partial u} = \frac{1}{2}[1+f^2-g^2] \tag{7.8}$$

and

$$\frac{\partial g}{\partial u} = fg \quad . \tag{7.9}$$

Since we also have

$$\frac{d^2z}{dw^2} = z\frac{dz}{dw} = \frac{\partial^2 f}{\partial u^2} + i\frac{\partial^2 g}{\partial u^2} \tag{7.10}$$

and

$$\frac{d^2z}{dw^2} = (f+ig)(\frac{\partial f}{\partial u}+i\frac{\partial g}{\partial u}) \tag{7.11}$$

and

$$\frac{d^2z}{dw^2} = (f\frac{\partial f}{\partial u} - g\frac{\partial g}{\partial u}) + i(f\frac{\partial g}{\partial u} + g\frac{\partial f}{\partial u}) \tag{7.12}$$

it follows that

$$\frac{\partial^2 f}{\partial u^2} = f\frac{\partial f}{\partial u} - g\frac{\partial g}{\partial u} \tag{7.13}$$

$$\frac{\partial^2 g}{\partial u^2} = f\frac{\partial g}{\partial u} + g\frac{\partial f}{\partial u} \quad . \tag{7.14}$$

Equations (7.13) and (7.14) can also be obtained directly by taking partial derivatives with respect to u in equations (7.8) and (7.9). If now equation (7.13) is multiplied by $\frac{\partial g}{\partial u}$, and equation (7.14) is multiplied by $\frac{\partial f}{\partial u}$, and we subtract the resulting equations, we obtain

$$g\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial u}\right)^2\right] = \frac{\partial f}{\partial u}\frac{\partial^2 g}{\partial u^2} - \frac{\partial g}{\partial u}\frac{\partial^2 f}{\partial u^2} \quad . \tag{7.15}$$

and since $h_u^2 = \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial u}\right)^2$, we obtain

$$gh_u^2 = \frac{\partial f}{\partial u}\frac{\partial^2 g}{\partial u^2} - \frac{\partial g}{\partial u}\frac{\partial^2 f}{\partial u^2} \; . \tag{7.16}$$

Thus, since $\frac{\partial g}{\partial u} = fg$ from equation (7.9), we obtain the necessary and sufficient condition that surfaces of constant v be spheres or planes, namely

$$f\left[\frac{\partial f}{\partial u}\frac{\partial^2 g}{\partial u^2} - \frac{\partial g}{\partial u}\frac{\partial^2 f}{\partial u^2}\right] = \frac{\partial g}{\partial u} h_u^2 \tag{7.17}$$

which agrees with equation (4.16).

Similar algebraic manipulations applied to the equations (7.13) and (7.14) yield another condition which is

$$g\left[\frac{\partial f}{\partial u}\frac{\partial^2 f}{\partial u^2} + \frac{\partial g}{\partial u}\frac{\partial^2 g}{\partial u^2}\right] = \frac{\partial g}{\partial u} h_u^2 \; . \tag{7.18}$$

Equations (7.17) and (7.18) may now be used to verify that the conditions (a), (b), (c) and (d) of Case I in section 5 are satisfied. Thus

$$\frac{\partial h_u}{\partial u} = \left(\frac{\partial f}{\partial u}\frac{\partial^2 f}{\partial u^2} + \frac{\partial g}{\partial u}\frac{\partial^2 g}{\partial u^2}\right)/h_u$$

$$= \left(\frac{\partial g}{\partial u}h_u\right)/g = fh_u > 0$$

if $f(u,v) > 0$. Since

$$f(u,v) = \frac{(1-x_0^2)\sin(u) + 2x_0\cos(u)}{(1-x_0^2)\cos(v) + (1+x_0^2)\cosh(u) - 2x_0\sin(u)} \tag{7.19}$$

we see that if $0 < x_0 < 1$ and $0 < u < \frac{\pi}{2}$ we will have $f > 0$, and h_u will be positive in a suitable range of u, and condition (a) must hold.

To verify that condition (b), namely that $R^2 < z_0^2$, where R and z_0 are given in equations (4.27) and (4.28), is also satisfied, we use equations (7.8) and (7.9) in combination with the expressions for R and z_0 and find that

$$z_0^2 - R^2 = 1 \tag{7.20}$$

and hence condition (b) is verified. Finally, since $f(u,v)$ and $g(u,v)$ are given as the real and imaginary parts of z in equation (7.4) it is clear that

$$\frac{\partial f}{\partial v}(u,v)|_{v=0} \tag{7.21}$$

and $g(u,v)$ is an odd function in v, and hence conditions (c) and (d) are also satisfied. Thus we have verified that all solutions of the initial value problem (7.3) satisfy the conditions for a design of a convergent lens, if appropriate restrictions are placed on the range of u, and the initial value x_0 is suitably restricted. The case x_0 corresponds exactly to the case of bispherical coordinates given in equations (6.3), with $a = 1$.

If we now set $p = iw$ in equation (7.1) and consider the initial value problem

$$\frac{dz}{dw} = \frac{1}{2}(1-z^2), \quad z(0) = x_0 . \tag{7.22}$$

where x_0 is a real number, we find that its solutions are given by

$$z = \frac{[(1+x_0^2)sinh(u) + 2x_0 cosh(u)] + i[1-x_0^2]sin(v)}{(1+x_0^2)cosh(u) + (1-x_0^2)cos(v) + 2x_0 sinh(u)} . \tag{7.23}$$

This form of the solution is obtained from the general solution $z = tanh(\frac{w+c}{2})$, where c is an arbitrary constant. One may then verify that all solutions of this initial value problem, with x_0 and u suitably restricted, satisfy the condition of equation (4.16) as well as conditions (a), (b), (c), and (d) for the case of a divergent lens. The case $x_0 = 0$ corresponds exactly to the case of toroidal coordinates given in equations (6.16) with $a = 1$.

Thus we have established the result that a class of solutions to our lens design problem is obtainable from the solutions to a Riccati differential equation.

8. CONCLUSION

A question that arises naturally from our analysis in section 7 is whether or not all solutions to the design problem for a lens matching a conical to a cylindrical transmission line have been obtained. Only isotropic inhomogeneous media for such lenses have been considered, and in order to extend this work one might permit the medium to be anisotropic. The main condition to be satisfied by an analytic function $z = F(w) = f(u,v) + jg(u,v)$ is given by equation (7.17). One may impose some additional requirements on $F(w)$ and find that the differential equation (7.1) is satisfied, but the question is are these natural? For example, if we require that f and g satisfy both equations (7.17) and (7.18) then it is a straight forward exercise to verify that

$$\frac{\frac{d^2z}{dw^2}}{\frac{dz}{dw}} = z\frac{\partial g}{\partial u}/fg \quad .$$

Hence if $fg = \frac{\partial g}{\partial u}$ as in equation (7.8), then

$$\frac{d^2z}{dw^2} = z\frac{dz}{dw}$$

and if initial conditions

$$z(0) = 0$$

$$z'(0) = \frac{1}{2}$$

are specified, then one recovers the differential equation (7.13), namely

$$\frac{dz}{dw} = \frac{1}{2}(1+z^2) \ .$$

Thus one is led to ask the obvious question concerning uniqueness. That is, if the condition given in equation (7.16), which is necessary and sufficient for surfaces of constant v to be spheres or planes, and the conditions given by (a), (b), (c), and (d) of either case I or case II, are all satisfied, then does $z = f + ig$ satisfy a Riccati equation of the type described by equation (7.1)?

In addition to the question of uniqueness, there are several other problems which should be addressed. For example, if the medium is allowed to be anisotropic as well as inhomogeneous, or if the orthogonality condition on our coordinate systems is removed, how should the lens design be specified? The determination of a symmetry group for which the lens problem considered here is a subgroup may provide further insight on these and related questions. The techniques of differential geometry, if properly applied, can be one of possibly several fruitful ways in which modern mathematics can further contribute to the development of transient and broad-band electromagnetics.

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