Sensor and Simulation Notes

Note 286

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Measurement of the Surface Curl of the Surface Current Density

Carl E. Baum
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It is commonplace to measure the surface divergence of the surface current density on a perfectly conducting object. This is related to the surface charge density, or the normal electric field. This paper explores the measurement of a complementary quantity, the surface curl of the surface current density which can be related to an equivalent magnetic charge density, or the normal derivative of the normal magnetic field. Techniques and difficulties are discussed.

current density, surface curl, measurement, electric fields, SEM (singularity expansion method)

I. Introduction

After many years of trying to understand the various ramifications of electromagnetic theory, there has occurred to me an interesting apparent discrepancy. We are generally aware of the fact that a vector field can be considered as comprised of a solenoidal (zero-divergence) and irrotational (zero-curl) part. Considering the current density (such as might exist on a scatterer or antenna) this can be explicitly decomposed into two such parts [1]. It is, furthermore, well known that the interior or exterior problem of a perfectly conducting sphere can be represented in terms of eigenmodes of the Maxwell equations in spherical coordinates which are E (TM) or H (TE) in character and that the resulting surface-current-density modes can be divided into these same categories [2,3,8]. Here the surface-current-density modes have either zero curl (E modes) or zero divergence (H modes), but not both (otherwise being identically zero). One can speculate that this property may extend to the eigenmodes of more generally shaped perfectly conducting objects as may be defined from appropriate integral equations [5,7,11]. From this vantage point such a property can be readily extended to natural modes as used in the singularity expansion method (SEM).

Actually, we are quite familiar with the divergence of the surface current density on a perfectly conducting object which is proportional to the surface charge density or normal electric field; this is, in fact, commonly measured. Perhaps one should also similarly consider the curl of the surface current density on a perfectly conducting object. Can this also be measured? This paper investigates this and related questions.

II. Considerations from the Maxwell Equations

As a first perspective consider the Maxwell equations

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} - \vec{J}_m \\ \nabla \times \vec{H} &= \frac{\partial \vec{D}}{\partial t} + \vec{J} \end{aligned} \tag{2.1}$$

where both electric and magnetic current densities have been included in the usual way for generality. Taking the divergence of these equations and assuming zero initial conditions gives

$$\begin{aligned} \nabla \cdot \vec{J}_m &\equiv -\frac{\partial}{\partial t}\rho_m \ , \ \nabla \cdot \vec{B} = \rho_m \\ \nabla \cdot \vec{J} &\equiv -\frac{\partial}{\partial t}\rho \ , \ \nabla \cdot \vec{D} = \rho \end{aligned} \tag{2.2}$$

With constitutive parameters we have

$$\tilde{\vec{D}} = \tilde{\overleftrightarrow{\varepsilon}} \cdot \tilde{\vec{E}} \quad , \quad \tilde{\vec{B}} = \tilde{\overleftrightarrow{\mu}} \cdot \tilde{\vec{H}} \tag{2.3}$$

where the tilde, ~, above a quantity indicates the two-sided Laplace transform making the quantities functions of the complex frequency, s.

Restricting our attention to free space, we have the constants

$$\begin{aligned} c &= (\mu_o \varepsilon_o)^{1/2} \quad \text{(propagation speed)} \\ Z_o &= \left(\frac{\mu_o}{\varepsilon_o}\right)^{1/2} \quad \text{(wave impedance)} \end{aligned} \tag{2.4}$$

and a separation index

$$q = \pm 1 \tag{2.5}$$

We have the combined-field form of the foregoing quantities [4,6]

$$\begin{aligned} \vec{E}_q &\equiv \vec{E} + qjZ_o\vec{H} \\ \vec{J}_q &\equiv \vec{J} + \frac{qj}{Z_o}\vec{J}_m \\ \rho_q &\equiv \rho + \frac{qj}{Z_o}\rho_m \end{aligned} \tag{2.6}$$

Then the Maxwell equations reduce to

$$\nabla \times - \frac{qj}{c} \frac{\partial}{\partial t} \vec{E}_q = qjZ_o \vec{J}_q \tag{2.7}$$

Applying the divergence gives (with zero initial conditions)

$$\nabla \cdot \vec{J}_q = - \frac{\partial}{\partial t} \rho_q \ , \ \nabla \cdot \vec{E}_q = \frac{1}{\varepsilon_o} \rho_q \tag{2.8}$$

expressing the equations in compact form which incorporates the symmetry between electric and magnetic parameters.

While it is not our purpose to introduce magnetic charge per se, still it is a useful concept for our development. Note that we have the quantities

$$\begin{aligned}
&\text{electric charge: } Q \text{ (coulombs, C)} \\
&\text{magnetic charge: } Q_m \text{ (teslas, T)} \\
&\text{electric charge density: } \rho \ (\text{C/m}^3) \\
&\text{magnetic charge density: } \rho_m \ (\text{T/m}^3) \\
&\text{electric surface charge density: } \rho_s \ (\text{C/m}^2) \\
&\text{magnetic surface charge density: } \rho_{s_m} \ (\text{T/m}^2)
\end{aligned} \tag{2.9}$$

The units of these quantities provide additional insight, including some indication concerning measurements.

III. Boundary Conditions on Surface

Consider some general scatterer such as depicted in fig. 3.1 and characterized by some surface S which we often take as perfectly conducting, but not necessarily so, in which case the interior volume V and whatever it may contain are significant. In any event, we have a surface current density $\vec{J}_s$ (electric) on S.

For convenience, let us establish a coordinate system, in general an orthogonal curvilinear system, on this surface. In a u_1, u_2, u_3 coordinate system let S be a surface of constant u_3. As indicated in fig. 3.2 consider some patch of S on which there is indicated a right-handed set of unit vectors

$$\vec{1}_1 \times \vec{1}_2 = \vec{1}_3 \ , \ \vec{1}_3 \times \vec{1}_1 = \vec{1}_2 \ , \ \vec{1}_2 \times \vec{1}_3 = \vec{1}_1 \tag{3.1}$$

corresponding to the coordinate system. Note that

$$\vec{1}_1 // S \ , \ \vec{1}_2 // S \ , \ \vec{1}_3 \perp S \tag{3.2}$$

If S is a closed surface we can choose $\vec{1}_3$ as outward pointing. If S is not closed then our choice of $\vec{1}_3$ orientation is arbitrary. Let us designate the two sides of S by $S^{(+)}$ and $S^{(-)}$ and have $\vec{1}_3$ as the outward normal to $S^{(+)}$. In effect $\vec{1}_3$ as in fig. 3.2 defines $S^{(+)}$.

The surface current density on a surface (infinitessimally thick) serves as a boundary condition for the tangential components of the magnetic field as

$$\vec{1}_3 \times [\vec{H}^{(+)} - \vec{H}^{(-)}] = \vec{J}_s \tag{3.3}$$

Here we have assumed that there is only an electric surface current density on S as in fig. 3.3A. More general conditions are found in [6]. A boundary condition for the normal components of the electric field is

$$\epsilon_0 \vec{1}_3 \cdot [\vec{E}^{(+)} - \vec{E}^{(-)}] = \rho_s \tag{3.4}$$

with the surface charge density related to the surface current density as in (2.2).

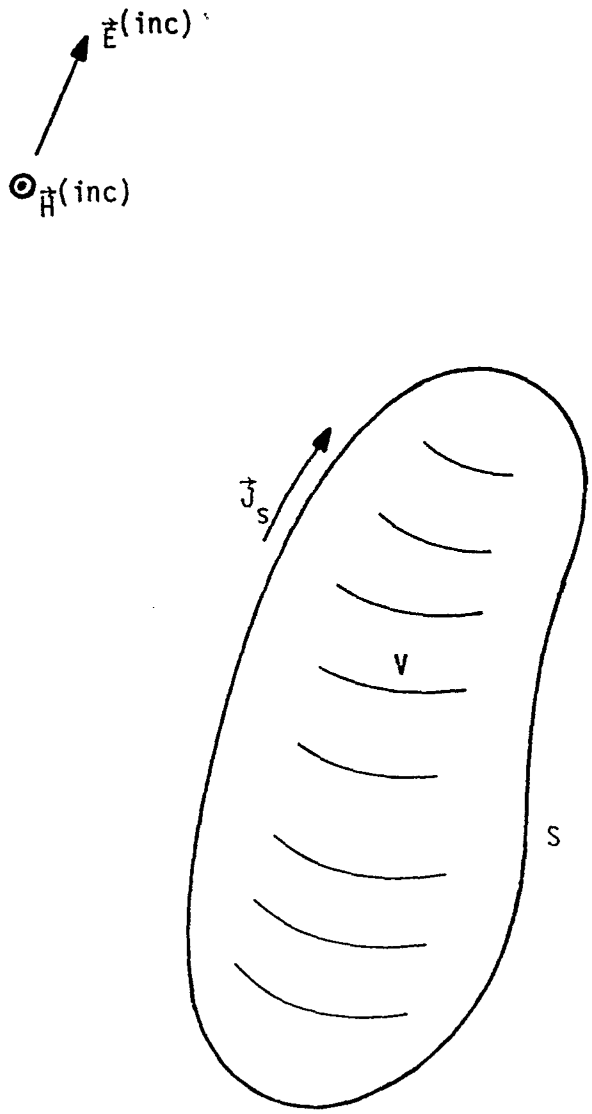


Fig. 3.1. A General Scatterer

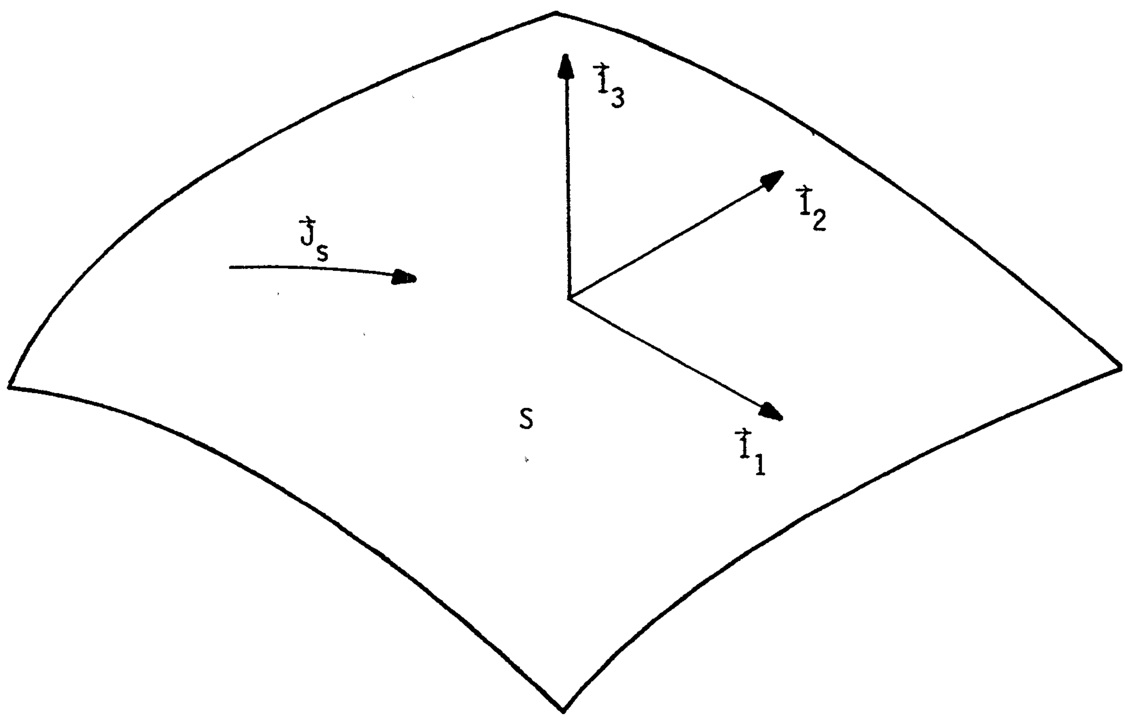
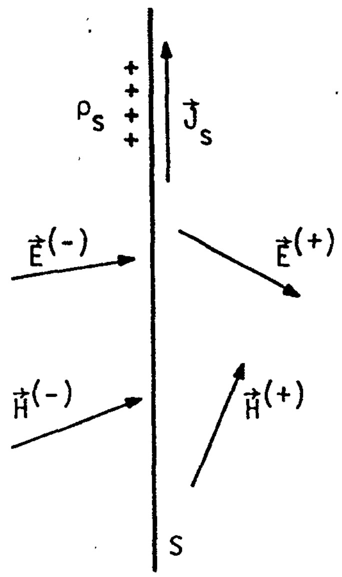
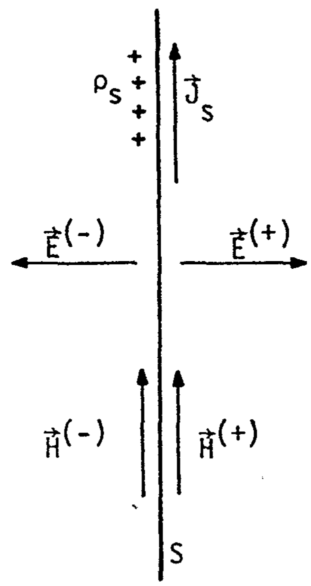


Fig. 3.2. Patch of General Surface with Local Coordinates



A. General Surface (Sheet)



B. Perfectly Conducting Surface

Fig. 3.3. Boundary Conditions

Suppose now that S is perfectly conducting. Then the tangential electric and normal magnetic fields are well known to be zero and, as in fig. 3.3B, we have

$$
\begin{aligned}
&\overleftrightarrow{1}_t \cdot \vec{E}^{(+)} = \overleftrightarrow{1}_t \cdot \vec{E}^{(-)} = \vec{0} \\
&\vec{1}_3 \cdot \vec{H}^{(+)} = \vec{1}_3 \cdot \vec{H}^{(-)} = 0 \\
&\overleftrightarrow{1}_t \equiv \vec{1}_1\vec{1}_1 + \vec{1}_2\vec{1}_2 = \overleftrightarrow{1} - \vec{1}_3\vec{1}_3 \equiv \text{transverse dyad} \\
&\overleftrightarrow{1} \equiv \vec{1}_1\vec{1}_1 + \vec{1}_2\vec{1}_2 + \vec{1}_3\vec{1}_3 \equiv \text{identity dyad} \\
&\overleftrightarrow{1}_t \cdot \quad = -\vec{1}_3 \times [\vec{1}_3 \times \quad]
\end{aligned}
\tag{3.5}
$$

as well as the conditions in (3.3) and (3.4).

An important case is that in which S is not only perfectly conducting, but also closed. Then the internal fields are all zero and the previous conditions reduce to

$$
\begin{aligned}
&\vec{E}^{(-)} = \vec{0} \ , \ \vec{H}^{(-)} = \vec{0} \\
&\overleftrightarrow{1}_t \cdot \vec{E}^{(+)} = \vec{0} \ , \ \vec{1}_3 \cdot \vec{H}^{(+)} = 0 \\
&\vec{1}_3 \times \vec{H}^{(+)} = \vec{J}_s \\
&\epsilon_0 \vec{1}_3 \cdot \vec{E}^{(+)} = \rho_s
\end{aligned}
\tag{3.6}
$$

With this summary of the boundary conditions on S let us go on to some additional considerations.

IV. Surface Divergence and Curl of Surface Current Density

One can look at the divergence of the magnetic field as in (2.2) to define a magnetic charge density, but that is not our purpose here. Let us look at what can be considered analogous to this by looking at the conventional (electric) surface current density on a perfectly conducting surface. In particular, let us consider the divergence and curl of the surface current density and the electromagnetic quantities related to these. Let us also regard the surface as closed so that the electromagnetic fields are only nonzero on one side (the exterior). The results are readily generalized to open surfaces provided the surface current density is divided into two parts, one for each side as defined by the surface magnetic field on each side. The u_1, u_2, u_3 coordinate system is taken as an orthogonal curvilinear system.

First consider the surface divergence of a vector field which is given as [9]

$$\begin{aligned} \nabla_s \cdot \vec{P} &= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(h_2 P_1) + \frac{\partial}{\partial u_2}(h_1 P_2)\right] - \left[\frac{1}{R_1} + \frac{1}{R_2}\right] P_3 \\ &= \nabla_s \cdot \vec{P}_t - \left[\frac{1}{R_1} + \frac{1}{R_3}\right] P_3 \end{aligned} \tag{4.1}$$

$$h_n = \left|\frac{\partial \vec{r}}{\partial u_n}\right| \quad \text{for } n = 1,2,3 \quad \text{(metrical coefficients)}$$

$$R_1, R_2 \equiv \text{principal radii of curvature}$$

$$\nabla \cdot \vec{P} = \nabla_s \cdot \vec{P} + \frac{\partial P_3}{\partial n}$$

where $\frac{\partial}{\partial n}$ indicates the normal derivative, i.e., in the u_3 direction, but with respect to the $\vec{r}$ coordinates instead of u_3. Applying this to the surface current density we have

$$\begin{aligned} \nabla \cdot \vec{J}_s &= \nabla_s \cdot \vec{J}_s \\ J_3 &\equiv 0 \end{aligned} \tag{4.2}$$

Next for simplicity let the u_1, u_2 coordinates be taken along lines of curvature of S. Then the surface curl is given as [9]

$$\nabla_s \times \vec{P} = [\nabla_s P_3] \times \vec{1}_3 + \frac{P_2}{R_2}\vec{1}_1 - \frac{P_1}{R_1}\vec{1}_2 + \vec{1}_3 \nabla_s \cdot [\vec{P} \times \vec{1}_3] \tag{4.3}$$

where the surface gradient is simply

$$\nabla_s f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\vec{1}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\vec{1}_2 \tag{4.4}$$

Applying the surface curl to the surface current density gives

$$\nabla_s \times \vec{J}_s = \frac{J_{s_2}}{R_2}\vec{1}_1 - \frac{J_{s_1}}{R_1}\vec{1}_2 + \vec{1}_3 \nabla_s \cdot [\vec{J}_s \times \vec{1}_s] \tag{4.5}$$

Carrying the development further consider the important normal component of the surface curl as

$$\vec{1}_3 \cdot [\nabla_s \times \vec{P}] = \nabla_s \cdot [\vec{P} \times \vec{1}_3] \tag{4.6}$$

the deleted terms in (4.3) all being vectors tangential to S. Applying this to the surface current density gives

$$\vec{1}_3 \cdot [\nabla_s \times \vec{J}_s] = \nabla_s \cdot [\vec{J}_s \times \vec{1}_3] \tag{4.7}$$

Now apply the above formulas to the magnetic field on the outside (+ side) of S, noting that the normal magnetic field is zero at a perfectly conducting surface. Note first that

$$\nabla \cdot \vec{H} = \nabla \cdot \vec{B} = 0 \tag{4.8}$$

Writing

$$\vec{H} = \vec{H}_s + H_3\vec{1}_3 \tag{4.9}$$

then

$$\begin{aligned} \nabla_s \cdot \vec{H} &= \nabla \cdot \vec{H} - \frac{\partial H_3}{\partial n} = - \frac{\partial H_3}{\partial n} \\ \nabla_s \cdot \vec{B} &= - \frac{\partial B_3}{\partial n} \end{aligned} \tag{4.10}$$

So while the divergence of the magnetic field is zero, the surface divergence need not be so.

Another relationship of interest is

$$\nabla_s \cdot [\vec{P} \times \vec{1}_3] = \vec{1}_3 \cdot [\nabla_s \times \vec{P}] \tag{4.11}$$

Recall from section 3

$$\vec{1}_3 \times \vec{H}^{(+)} = \vec{J}_s \ , \ \vec{1}_3 \times \vec{J}_s = -\vec{H}_t^{(+)} \tag{4.12}$$

again assuming nonzero magnetic field only on the plus side.

Combining some of the foregoing results

$$\begin{aligned} \vec{1}_3 \cdot [\nabla_s \times \vec{J}_s] &= \nabla_s \cdot [\vec{J}_s \times \vec{1}_3] \\ &= \nabla_s \cdot \vec{H}_t^{(+)} \\ &= -\frac{\partial H_3^{(+)}}{\partial n} \end{aligned} \tag{4.13}$$

and similarly

$$\begin{aligned} \vec{1}_3 \cdot [\nabla_s \times \vec{H}_t^{(+)}] &= \nabla_s \cdot [\vec{H}_t^{(+)} \times \vec{1}_3] \\ &= -\nabla_s \cdot \vec{J}_s \end{aligned} \tag{4.14}$$

So the normal component of the surface curl of the surface current density is proportional to the surface divergence of the tangential magnetic field, and conversely.

V. Interpretation of the Surface Divergence and Curl of the Surface Current Density

Noting from (4.2) that the divergence and surface divergence of the surface current density are the same thing, we have from (2.2), (3.6), and (4.14)

$$\nabla_s \cdot \vec{J}_s = -\frac{\partial}{\partial t}\rho_s = -\frac{\partial}{\partial t}\left[\epsilon_o E_3^{(+)}\right] = -\frac{\partial}{\partial t} D_3^{(+)}$$

$$= -\vec{1}_3 \cdot \left[\nabla_s \times \vec{H}_t^{(+)}\right] \tag{5.1}$$

Thus we can think of $\nabla_s \cdot \vec{J}_s$ in terms of the surface charge density, the normal electric field or displacement, and the normal component of $\nabla_s \times \vec{H}_t^{(+)}$.

Turning to the surface curl of the surface current density (4.13) gives

$$\vec{1}_3 \cdot \left[\nabla_s \times \vec{J}_s\right] = -\frac{\partial}{\partial n} H_3^{(+)} = -\frac{1}{\mu_o}\frac{\partial}{\partial n} B_3^{(+)}$$

$$= \nabla_s \cdot \vec{H}_t^{(+)} = \frac{1}{\mu_o}\nabla_s \cdot \vec{B}_t^{(+)} \tag{5.2}$$

From (2.2) this can be then thought of as a magnetic charge density. Define

$$\nabla_s \cdot \vec{B}_t^{(+)} = k \tag{5.3}$$

similar to

$$\nabla \cdot \vec{B} = \rho_m$$

Dimensionally these are the same (T/m^3 or magnetic charge per unit volume). However, even if we let the magnetic charge density ρ_m be zero, the equivalent magnetic charge density k is not in general zero. Note that k is a volume rather than surface magnetic charge density. Since we can write

$$k = -\frac{\partial}{\partial n} B_3^{(+)} \tag{5.5}$$

then k is not given by a discontinuity in the normal component of the magnetic flux density, but by a discontinuity in the normal derivative of this component. Note that since k "exists" only on the surface S, a volume integral of k over a finite volume containing some portion of S within it will give zero net "magnetic charge" on any portion of S.

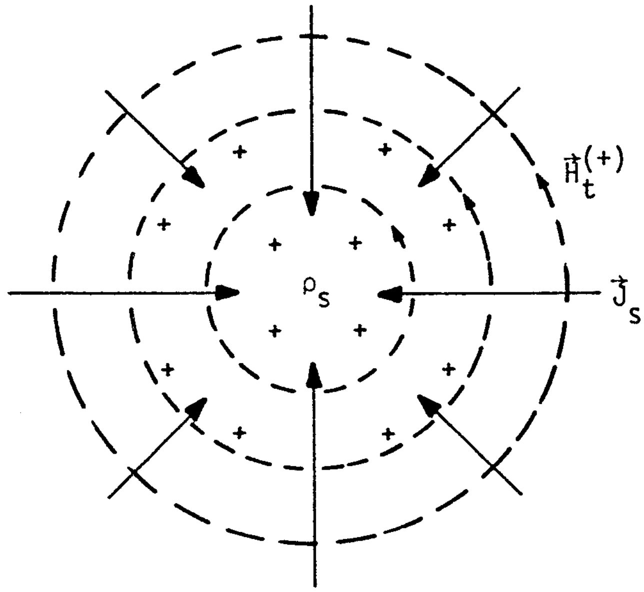
Turning the equations around we have

$$\rho_s(\vec{r}_s,t) = - \int_{-\infty}^{t} \nabla_s \cdot \vec{J}_s(\vec{r}_s,t')\, dt'$$

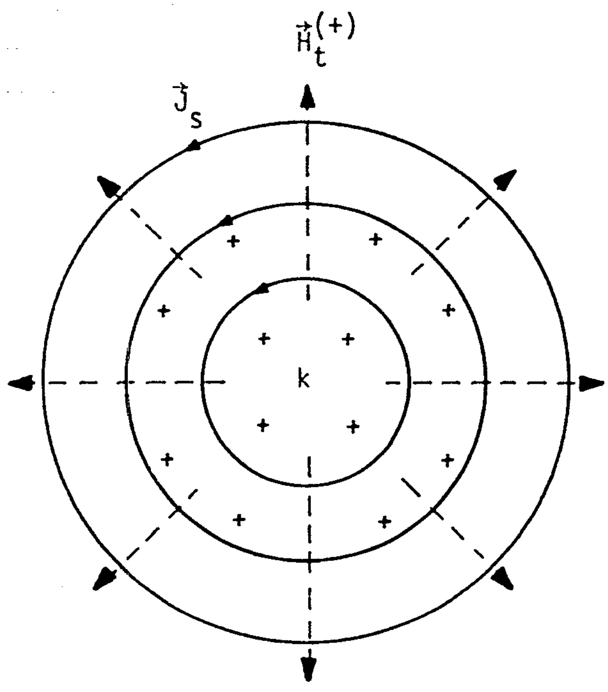
$$k(\vec{r}_s,t) = \mu_0 \vec{1}_3 \cdot [\nabla_s \times \vec{J}_s] \tag{5.6}$$

$$\vec{r}_s \equiv \vec{r} \text{ on } S$$

illustrating the decomposition of the surface current density into two terms, the first associated with the surface charge density and the second associated with the equivalent magnetic current density. To help in understanding this decomposition consider the diagrams in fig. 5.1. Figure 5.1A shows a case of a surface current density with no curl (locally) but a local maximum in the surface charge density. Figure 5.1B shows a case of a surface current density with no divergence (locally) but a local maximum in the equivalent surface magnetic charge density.



A. Curlless (Irrotational) Surface Current Density



B. Divergenceless (Solenoidal) Surface Current Density

Fig. 5.1. Types of Surface Current Density: View Normal to S from + Side

VI. Measurement of the Equivalent Magnetic Charge Density

Having defined an equivalent magnetic charge density associated with the curl of the surface current density, let us now consider how to measure it. From (5.5), and assuming a nonzero first normal derivative of the normal magnetic field, we can sample the normal magnetic field at a distance h from S giving

$$k = -\frac{\partial}{\partial n} B_3^{(+)} \simeq -\frac{1}{h} B_3 \Big|_{\vec{r}=\vec{r}_s+h\vec{1}_3}$$

$$\vec{r} \equiv \text{position of measurement of magnetic field} \tag{6.1}$$

$$\vec{r}_s \equiv \text{position of interest on S}$$

noting that

$$B_3^{(+)} \Big|_{\vec{r}=\vec{r}_s} = 0 \tag{6.2}$$

So our problem is how to measure the normal magnetic field at some distance away from, but near, S. Since at S the normal magnetic field is zero, then at small h there will be a small normal magnetic field, leading to a possible signal-to-noise problem. In this case the "noise" may come from not-so-small other electromagnetic-field components.

VII. A Type of Sensor for the Equivalent Magnetic Charge Density

Let us now consider some location on S for which

$$h \ll |R_1|, \ |R_2| \text{ (both principal radii of curvature)} \tag{7.1}$$

so that the local geometry of S near the $\vec{r}_s$ of interest can be considered approximately flat.

Consider a special magnetic field distribution near $\vec{r}_s$ like that in fig. 5.1B. At $\vec{r}_s$ place the origin of a cartesian coordinate system (x',y',z') with z' normal to S in the + direction as indicated in fig. 7.1. Here is also the magnetic-field distribution of interest in which the tangential magnetic field (as well as the normal magnetic field) is zero. Furthermore, the magnetic field and associated surface current density are assumed to have axial (or rotational) symmetry about the local z' axis. Define cylindrical (Ψ',ϕ',z') and spherical (r',θ',ϕ') coordinate systems as indicated in fig. 7.1 and related to the local cartesian system by

$$\begin{aligned} x' &= \Psi' \cos(\phi') = r' \sin(\theta') \cos(\phi') \\ y' &= \Psi' \sin(\phi') = r' \sin(\theta') \sin(\phi') \\ z' &= r' \cos(\theta') \end{aligned} \tag{7.2}$$

Let us then seek a quasi-static surface-current-density distribution of the form

$$\vec{J}_s = J_{s_{\phi'}}(\Psi') \, \vec{1}_{\phi'} \tag{7.3}$$

and an asociated surface-magnetic-field distribution of the form

$$\begin{aligned} \vec{H}_t^{(+)} &= H_{t_{\Psi'}}(\Psi') \, \vec{1}_{\Psi'} \\ H_{t_{\Psi'}}(\Psi') &= J_{s_{\phi'}}(\Psi') \end{aligned} \tag{7.4}$$

with

$$J_{s_{\phi'}}(0) = 0 \tag{7.5}$$

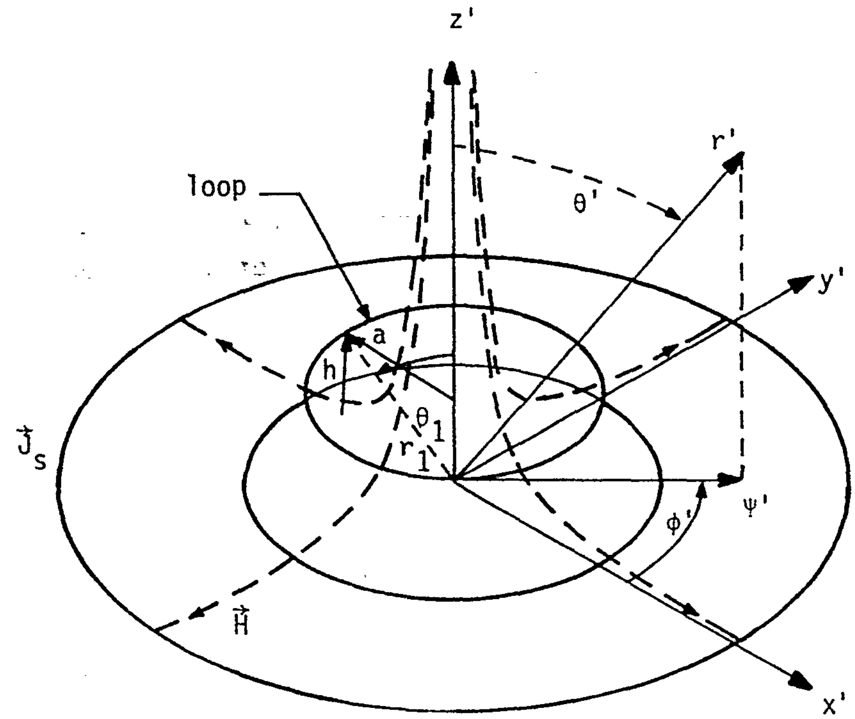


Fig. 7.1. Local Coordinates, Magnetic-Field Distribution, and Sensing Loop Near $\vec{r}_s$

The magnetic field can be found from a magnetic potential which is expanded in spherical coordinates using terms of the form [10]

$$\Xi_{n,m} = \left\{ \begin{matrix} r'^{n} \\ r'^{-n-1} \end{matrix} \right\} P_n^{(m)} \; (\cos(\theta')) \left\{ \begin{matrix} \cos(m\phi') \\ \sin(m\phi') \end{matrix} \right\} \tag{7.6}$$

$n > 0$

with all possible combinations of terms in the braces allowed. First we rule out negative powers of r' since the desired magnetic field is zero at the origin. Second we choose $m = 0$ for the desired symmetry in the magnetic field. This leaves terms of the form

$$\Xi_n = r'^{n} P_n(\cos(\theta')) \tag{7.7}$$

The Legendre functions are calculable from [3]

$$P_n(\xi) \equiv P_n^{(0)}(\xi) \equiv \frac{1}{2^n n!} \frac{d^n}{d\xi^n} (\xi^2 - 1)^n$$

$$P_n^{(m)}(\xi) \equiv (-1)^m (1 - \xi^2)^{m/2} \frac{d^m}{d\xi^m} P_n(\xi) \tag{7.8}$$

giving the first few as

$$P_0(\xi) = 1 \; , \; P_0(\cos(\theta')) = 1$$

$$P_1(\xi) = \xi \; , \; P_1(\cos(\theta')) = \cos(\theta') \tag{7.9}$$

$$P_2(\xi) = \frac{3\xi^2}{2} - \frac{1}{2} \; , \; P_2(\cos(\theta')) = \frac{3 \cos^2(\theta')}{2} - \frac{1}{2}$$

The $n = 0$ term is a constant giving no magnetic field. The $n = 1$ term is $r'\cos(\theta')$ or just z'; but this is not allowed by the boundary condition at S that allows no normal magnetic field. The $n = 2$ term is the one of interest giving a magnetic potential

$$\Xi = - \frac{H_o r'^2}{2r_o} P_2(\cos(\theta')) \tag{7.10}$$

with constants H_0 a convenient magnetic field and r_0 a convenient distance; the coefficient of -1/2 is for later convenience.

In spherical coordinates we can now find the magnetic field (for $z' > 0$) from

$$\vec{H} = \nabla\Xi = \frac{\partial\Xi}{\partial r'} \vec{1}_{r'} + \frac{1}{r'} \frac{\partial\Xi}{\partial\theta'} \vec{1}_{\theta'} + \frac{1}{r' \sin(\theta')} \frac{\partial\Xi}{\partial\phi'} \vec{1}_{\theta'} \tag{7.11}$$

which gives

$$\vec{H} = -\frac{H_o}{2r_o} r'[3 \cos^2(\theta') - 1] \vec{1}_{r'} + \frac{3}{2} \frac{H_o}{r_o} r' \cos(\theta') \sin(\theta') \vec{1}_{\theta'} \tag{7.12}$$

Note that on S (corresponding to $\theta' = \pi/2$) the normal magnetic field ($H_{\theta'}$) is zero as required. This result can also be cast in cylindrical coordinates as

$$\begin{aligned} H_{\Psi'} &= H_{r'} \sin(\theta') + H_{\theta'} \cos(\theta') \\ &= \frac{H_o}{r_o} \frac{r' \sin(\theta')}{2} = \frac{H_o}{r_o} \frac{\Psi'}{2} \\ H_{z'} &= H_{r'} \cos(\theta') - H_{\theta'} \sin(\theta') \\ &= -\frac{H_o}{r_o} r' \cos(\theta') = -\frac{H_o}{r_o} z' \end{aligned} \tag{7.13}$$

which is quite simple in form.

From (5.5) we have

$$\begin{aligned} k &= -\frac{\partial}{\partial n} B_3^{(+)} = -\mu_o \frac{\partial}{\partial z'} H_{z'} \\ &= \mu_o \frac{H_o}{r_o} \end{aligned} \tag{7.14}$$

and (7.13) can be written in the form

$$\begin{aligned} B_{\Psi'} &= k \frac{\Psi'}{2} \\ B_{z'} &= -kz' \end{aligned} \tag{7.15}$$

This expresses our special magnetic field distribution in terms of the equivalent magnetic charge density (which is conveniently uniform on our local part of S).

As in fig. 7.1, the loop of radius a and height h from S has a flux through it

$$\Phi = -\pi a^2 B_{z'} = \pi a^2 hk \tag{7.16}$$

which allows us to define an equivalent volume

$$V_{eq} \equiv \frac{\Phi}{k} = \pi a^2 h \tag{7.17}$$

as the sensitivity parameter for this type of sensor. The open-circuit voltage from such a sensor is

$$V_{o.c.} = \frac{d\Phi}{dt} = V_{eq} \frac{dk}{dt} \tag{7.18}$$

This type of sensor has properties similar to a regular magnetic-field sensor (loop). Its Thevenin equivalent circuit (assuming the sensor is electrically small) is the series combination of a voltage source from (7.18) and an inductance, L (at least in the case of conducting loops without impedance loading). There are various possible specific sensor designs involving various numbers of loop turns, spatial distribution of loops, and specific loop geometries. Such designs might be considered in future papers.

In optimizing the sensor design one may wish to maximize V_{eq} subject to other constraints, such as electrical or physical size, or upper bandwidth. Suppose, for example, one wished to place the sensor in a hemisphere of radius r_1 centered on $\vec{r}' = \vec{0}$. Then for a single loop lying on the hemisphere surface at some $\theta' = \theta_1$ we have

$$\begin{aligned} a &= r_1 \sin(\theta_1) \ , \ h = r_1 \cos(\theta_1) \\ V_{eq} &= \pi a^2 h = \pi r_1^3 \sin^2(\theta_1) \cos(\theta_1) \end{aligned} \tag{7.19}$$

With r_1 assumed fixed, this is maximized by setting

$$\left.\frac{dV_{eq}}{d\theta'}\right|_{\theta'=\theta_1} = 0 = \pi r_1^3 \left[2 \sin(\theta_1) \cos^2(\theta_1) - \sin^3(\theta_1)\right] \tag{7.20}$$

giving

$$\tan(\theta_1) = \sqrt{2}\ , \ \sin(\theta_1) = \sqrt{\frac{2}{3}}\ , \ \cos(\theta_1) = \sqrt{\frac{1}{3}}$$

$$\frac{h}{a} = \cot(\theta_1) = \frac{1}{\sqrt{2}} \tag{7.21}$$

One should note, however, that this is not necessarily the only kind of optimization condition.

VIII. Improvement of the Sensor by Reduction of Unwanted Field Components

A good sensor design considers the response of the sensor to other electromagnetic field components and the amelioration of this possible problem. It may be desirable to construct some kind of structure which suppresses any normal electrical field, $E_{z'}$, and any approximately uniform tangential magnetic field, $H_{x'}$ and $H_{y'}$ in the vicinity of $\vec{r}' = \vec{0}$. In addition, this structure should not significantly interfere with the magnetic-field distribution discussed in section 7, or at least leave some magnetic-field distribution related to it and which can be sensed effectively.

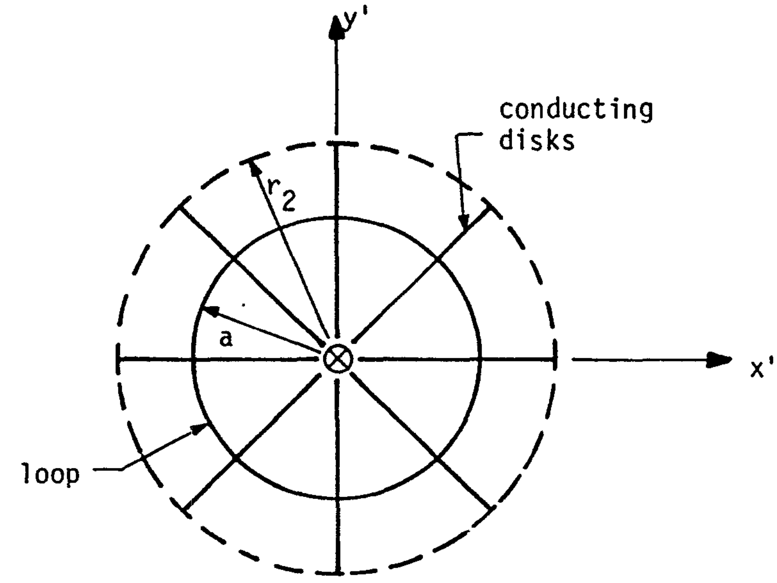
Noting that the magnetic-field distribution of interest is characterized by $H_{\phi'} = 0$, then conducting sheets on planes of constant ϕ' do not interfere with this kind of field. If these sheets extend above the sensing loop they can form a shield against the electric field, $E_{z'}$. The electric field can be terminated at the upper portions of the sheets, reducing the associated electric field near the sensing loop.

A tangential magnetic-field distribution is distorted by such sheets, pushing the field away from S to go "above" and "around" the sheets. An important consideration is that the distortion of this unwanted magnetic field not couple to the sensing loop. Distorting the uniform part of the magnetic field around the sensor can produce a z' component. It is essential that this z' component not couple to the sensing loop (or loops). In effect, the net magnetic flux from this distorted field should be made to be zero. This requires that the magnetic flux linking the loop in the z' direction be balanced by the magnetic flux linking the loop in the -z' direction. Now a perfectly conducting sheet on a plane of constant ϕ' and $\phi' + \pi$ will produce a distortion of this field with the desired property provided the loop is symmetric with respect to this plane. Since the undesired (incident) uniform tangential magnetic field has in general two components (x' and y' components) then two such perfectly conducting sheets can be used to distort both. So that the combination of these two (or more) sheets does not collectively distort the uniform tangential magnetic field in a manner which couples into the sensing loop, let us constrain the set of perfectly conducting sheets to have at least two planes of symmetry, all of which contain the z' axis. A circular loop (or multiple loops) coaxial with the z' axis of course has these same planes of symmetry.

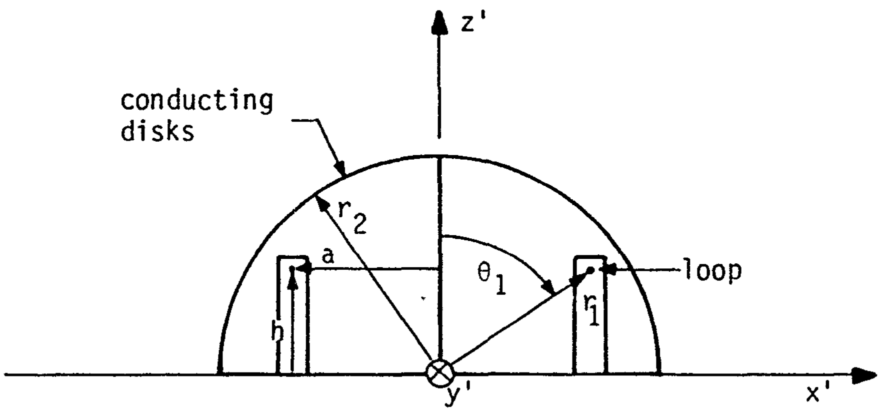
Figure 8.1 illustrates a possible realization of this concept. The ideally perfectly conducting sheets are assumed to all have the same shape and be uniformly spaced with respect to ϕ'. Each sheet is then also symmetrical with respect to the z' axis. For convenience each disk might be assumed to be half of a circular disk, of radius r_2, terminated in a perfectly conducting surface, the $z' = 0$ plane, or S. However, other shapes, such as rectangular, are also possible. (Even sheets that do not continue through the z' axis are allowed provided the two symmetry planes for the ensemble are maintained.)

Figure 8.1 illustrates this concept for the case of many perfectly conducting half circular disks. There may be various portions of the disk(s) cut out for the passage of the sensing loop. However, even these cutouts and/or connections of the sensing loop(s) with respect to these disks should still maintain the two symmetry planes through the z' axis for the ensemble.

Note that the disk array described above also suppresses the electric field (incident) normal to S. This suppression is enhanced by the use of many disks so that the edges of the disks are close together, approximating a surface (e.g., hemispherical) "covering" the sensing loop(s).



A. Top View



B. Side View

Fig. 8.1. Equivalent Magnetic-Charge-Density Sensor with Suppression for Other Field Components

IX. Summary

Having elucidated a way of measuring the surface curl of the surface current density, it is apparent that there is much design detail to be considered. The definition of the equivalent magnetic current density is quite clear, but its measurement may present some practical difficulties which may require some time to overcome.

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