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James D. Quinn* and Carl E. Baum

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Keywords: dipole antennas, transmission lines

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# 1 Introduction

One aspect of interest in producing various EM environments is the development of local illuminators. As with any simulator, attention needs to be given to providing appropriate spatial and frequency characteristics. The PARTES concept [1] has established the possibility of utilizing a set of magnetic dipoles to approximate a free space plane wave over some limited volume of space. Each member of such a set of magnetic dipoles might be realized by a set of loops which would be selectively sized, located and driven to efficiently produce a single magnetic dipole moment over a broadband of frequencies.

Implicit in the use of a combination of dipoles (magnetic or electric) to synthesize a given field is the assumption of linearity. In this case, it is assumed possible that a simple addition of sources with appropriate coefficients can form a linear combination which results in some desired field. The validity of this assumption for electromagnetic fields is based loosely on the fact that Maxwell's equations which describe their behavior are linear and the constitutive relations are often linear (approximately). Linear differential equations of this type have the desirable property that the sum of individual solutions is also a solution. The use of this property is commonly used in electronics applications. In transient applications, it has been employed commonly as a means of synthesizing a desired waveform from a linear combination of component frequencies. This is a temporal superposition. A spatial superposition might be regarded as one in which the incident field is simulated by a number of discrete sources, in the case of current interest, by a series of magnetic dipoles. The justification for using magnetic dipoles as the elementary sources as a spatial superposition process has been developed using an integral-equation representation of the scattering process [1].

The feasibility is examined of using conducting loops to emulate the effective dipole sources of this superposition theory. In this study, one effective dipole in a spatial sense, is synthesized by a series of loops each contributing to a specific frequency range. The value of each magnetic-dipole contribution is the product of the equivalent area of the loop and its circulating current. A desirable aspect of the frequency characteristics of the current in the loop is that its radian wavelength be much larger than the radius of the loop. A scheme has been devised whereby the loops, with their inherent area-dependent inductance, are included in a cascaded ladder network. The network is chosen to have a constant input resistance, and relatedly, behaves as a series of low-pass filters. This gives rise to a configuration of a combined electrical network-loop system in which the physical dimensions of each loop can be matched to the frequency characteristics of its circulating current. The loops close to the source of the system are made electrically small, for the highest frequency components, while subsequent loops can increase in size appropriate to the highest remaining frequency component.

The model proposed assumes that there is no interaction (mutual inductance) between the loops and ignores the production of higher moments. An associated paper will deal with schemes for minimizing interaction by locating successive loops in a manner such that the mutual inductance for loops is minimized.

## 2 Combining Magnetic-Dipole Moments from a Number of Loops

A field equivalence principle is a statement of the fact that an actual source of an electromagnetic field can be replaced by some set of equivalent sources. Starting with Maxwell's equations for the case where electric and magnetic currents and charges are present, an expression for Love's field equivalence principle has been developed which relates an incident field to electric and magnetic sheet current densities [1] as

$$\begin{aligned} \vec{J}_S^{(\text{inc})}(\vec{r}_S,t) &= -\vec{1}_S(\vec{r}_S) \times \vec{H}^{(\text{inc})}(\vec{r}_S,t) \\ \vec{J}_{S_h}^{(\text{inc})}(\vec{r}_S,t) &= \vec{1}_S(\vec{r}_S) \times \vec{E}^{(\text{inc})}(\vec{r}_S,t) \end{aligned} \tag{2.1}$$

where $\vec{1}_S$ is the outward pointing normal on the closed surface $S$. In the combined field form, the combined surface current density on the surface, $J_{S_q}$ is in this case related to its components as

$$\vec{J}_{s_q} = \vec{J}_S + \frac{qj}{Z_0}\vec{J}_{S_h} \tag{2.2}$$

where $q$ is a separation index ($\pm 1$) and $Z_0$ is the characteristic impedance of free space. In this interpretation of the field equivalence principle, a perfectly conducting enclosed surface $S_S$ inside the enclosed surface $S$ supports the combined surface current $\vec{J}_{S_q}$. For the perfectly conducting surface $S$ near $S_S$, the electric surface current $\vec{J}_S$ is equal and opposite to $\vec{J}_S^{(\text{inc})}$ and the magnetic surface current $\vec{J}_{S_h}^{(\text{inc})}(r_S,t)$ is the important part which excites the scatterer resonances. The incident field, except for a quasi-static term, can thus be duplicated using $\vec{J}_{S_h}$ alone on the surface $S$.

The area $A$ of the surface $S_S$ needs to be subdivided into a set of elementary zones of area $A_{n'}$ i.e.,

$$A = \sum_{n'=1}^{N_S} A_{n'} \tag{2.3}$$

The magnetic dipole moment, with centers at $\vec{r}_{S_{n'}}$, associated with each element can be shown to be

$$\tilde{\vec{m}}_{n'}(s) = \frac{A_{n'}}{s\mu_0}\tilde{\vec{J}}_{S_n}^{(\text{inc})}(\vec{r}_{S_{n'}}, s) \tag{2.4}$$

The tilde ($\sim$) indicates the (two-sided) Laplace transform, and $s$ is the complex frequency. The relation holds only for the situation where the radian wavelength is large compared to each related elemental area $A_{n'}$.

# 3 Effective Magnetic Dipole from an Array of Loops

It has been shown that an effective field can be produced by a sum of elemental magnetic dipole moments by using the principle of superposition. A means of implementing an individual dipole moment in the imaginary surface is, conceivably, by a set of loops at the location of the desired dipole moment. Each loop has dimensions which remain small with respect to the radian wavelength for frequencies at which the loop significantly contributes to the magnetic dipole moment. The equivalent areas of the individual loops are assumed to be parallel, so that the resulting magnetic dipole moment has a constant direction. The effective magnetic dipole moment $\vec{1}_z\tilde{m}_{\text{eff}}$ can thus be expressed as the sum of the individual products of elemental loop area and elemental circulating loop currents, i.e.,

$$\tilde{\vec{m}}_{\text{eff}} = \vec{1}_z\tilde{m}_{\text{eff}} = \sum_{n=1}^{N} \tilde{I}_n \vec{A}_{eq_n} = \vec{1}_z \sum_{n=1}^{N} \tilde{I}_n(s) A_{eq_n} \tag{3.1}$$

where

$$\begin{aligned} \vec{1}_z &\equiv \text{unit vector for all the magnetic dipoles} \\ \tilde{I}_n(s) &\equiv \text{frequency-dependent current circulating in loop } n \\ A_{eq_n} &\equiv \text{equivalent area of loop } n \\ N &\equiv \text{total number of loops} \\ \tilde{m}_n &\equiv \text{elemental dipole moment from loop } n \\ N &\equiv \text{number of loops to produce one magnetic dipole} \end{aligned} \tag{3.2}$$

Higher order moments will be present in the loop implementation of the magnetic dipole surface. The magnitudes of higher moment quasi-static fields fall off at rates that are higher than the corresponding dipole fields. As a rule, quadrupole and therefore higher order moments can be ignored if the point of consideration is further than a wavelength from the loop exciting it. For situations where space precludes the natural attenuation with distance, a means of suppressing these higher order moments will need to be considered.

# 4 Cascaded RLC Constant-Resistance Low-Pass Network

An array of current-carrying loops is proposed as a means of approximating the magnetic dipole moment as described in the previous sections. The inductance inherent in the loops can be included in an RLC network, shown in Figure 4.1. A constant resistance network of this type is analyzed below to show its frequency-dependent current characteristics. The input admittance of a single stage can be written as,

$$\begin{aligned} \tilde{Y}_{\text{in}_n} &= \frac{1}{\frac{1}{sC_n} + R_n} + \frac{1}{sL_n + \tilde{Z}} \\ \tilde{Z} = \tilde{Z}_{\text{in}_{n+1}} &\equiv \text{input impedance of the next stage.} \end{aligned} \tag{4.1}$$

This relation can be rearranged to

$$\tilde{Y}_{\text{in}_n} = \frac{s^2 + s\left(\frac{\tilde{Z}}{L_n} + \frac{R_n}{L_n}\right) + \frac{1}{L_n C_n}}{R_n\left[s^2 + s\left(\frac{\tilde{Z}}{L_n} + \frac{1}{R_n C_n}\right) + \frac{\tilde{Z}}{R_n L_n C_n}\right]} \tag{4.2}$$

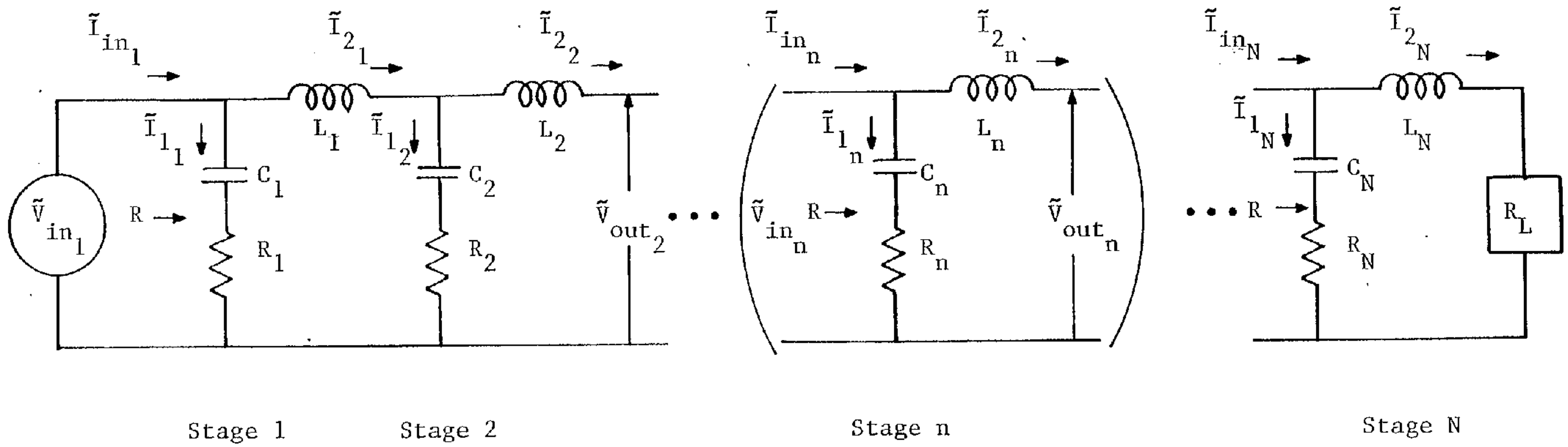
Inspection of the expression for $\tilde{Y}_{\text{in}}$ in this form shows that if

$$\begin{aligned} \tilde{Z} &= R_n \\ \frac{R_n}{L_n} &= \frac{1}{R_n C_n} \qquad \left(\text{i.e., } R_n = \sqrt{\frac{L_n}{C_n}}\right) \end{aligned} \tag{4.3}$$

Then

$$\tilde{Y}_{\text{in}} = \frac{1}{R_n} = \frac{1}{R} \qquad \text{for all } n \tag{4.4}$$

If these two conditions are satisfied together with the load $R_L = R$, then it can be seen that the constant resistance network can be cascaded and maintain the same input resistance for each stage. The transfer function for a given stage $n$, can be written in terms of the voltage out, $\tilde{V}_{\text{out}_n}$ and the voltage in, $\tilde{V}_{\text{in}_n}$, using the currents $\tilde{I}_{\text{in}_n}$ and $\tilde{I}_{2_n}$




$$R = \sqrt{L_n / C_n}$$

Figure 4.1: Cascaded RLC Constant-Resistance Network

$$
\begin{aligned}
\frac{\tilde{V}_{\text{out}_n}}{\tilde{V}_{\text{in}_n}} &= \frac{R}{sL_n + R} = \left[s\frac{L_n}{R} + 1\right]^{-1} \\
&= [s\sqrt{L_n C_n} + 1]^{-1} \\
R^2 &= \frac{L_n}{C_n} \\
\frac{\tilde{V}_{\text{out}_n}}{\tilde{V}_{\text{in}_n}} &= \frac{\tilde{I}_{2_n}}{\tilde{I}_{\text{in}_n}} \rightarrow 1 \qquad \text{for} \qquad s \rightarrow 0 \\
\frac{\tilde{V}_{\text{out}_n}}{\tilde{V}_{\text{in}_n}} &= \frac{\tilde{I}_{2_n}}{\tilde{I}_{\text{in}_n}} \rightarrow 0 \qquad \text{for} \qquad s \rightarrow \infty
\end{aligned}
\tag{4.5}
$$

Thus each stage acts as a low pass filter.

The transfer function for the cascaded network can be written in terms of the stage one input voltage $\tilde{V}_{\text{in}_1}$ and input current $\tilde{I}_{\text{in}_1}$ as

$$
\begin{aligned}
\tilde{I}_{2_n} &= \tilde{I}_{\text{in}_1} \prod_{\ell=1}^{n} [s\sqrt{L_\ell C_\ell} + 1]^{-1} \\
\tilde{V}_{\text{out}_n} &= \tilde{V}_{\text{in}_1} \prod_{\ell=1}^{n} [s\sqrt{L_\ell C_\ell} + 1]^{-1}
\end{aligned}
\tag{4.6}
$$

A special case occurs if all the stages have equal values of $C_n = C$ and $L_n = L$ as well as $R$ then the transfer function becomes

$$
\begin{aligned}
\tilde{I}_{2_n} &= \tilde{I}_{\text{in}_1}[s\sqrt{LC} + 1]^{-n} \\
\text{and} \qquad \tilde{V}_{\text{out}_n} &= \tilde{V}_{\text{in}_1}[s\sqrt{LC} + 1]^{-n}
\end{aligned}
\tag{4.7}
$$

# 5 Cascaded Low-Pass Symmetrical Constant-Resistance Filter

Another example of a low-pass filter with constant resistance can be regarded as a special form of a two-terminal symmetrical lattice. A constant-resistance symmetrical lattice can be realized in the Π, T or bridged-T circuit with equivalent results. Thus in addition to the ladder network structure of the previous section, the constant-resistance bridged-T circuit (Figure 5.1) may be considered. In the form using impedances of Figure 5.1

$$\tilde{Z}_{\text{in}} = R \tag{5.1}$$

if

$$\tilde{Z}_1 = \tilde{Z}_3 = R \; , \; R^2 = \tilde{Z}_2 \tilde{Z}_4 \tag{5.2}$$

The impedance $\tilde{Z}_5$ has been added to the circuit although it is redundant in some cases. It is included in the circuit in practice at a value equal to $R$ to add stability and symmetry to the circuit. A more detailed analysis of filters based on the two terminal-symmetrical lattice can be found in a number of works on network synthesis [2,3,4].

The purpose of the circuit is to drive a loop. With

$$\begin{aligned} \tilde{Z}_1 = \tilde{Z}_5 = R \qquad & \text{(symmetry)} \\ \tilde{Z}_3 = R \qquad & \text{(to be input to next filter)} \end{aligned} \tag{5.3}$$

and with the loops of successive filters to be in series at low frequencies (to increase the low-frequency magnetic dipole moment) this leaves $\tilde{Z}_4$ for the loop, so we take

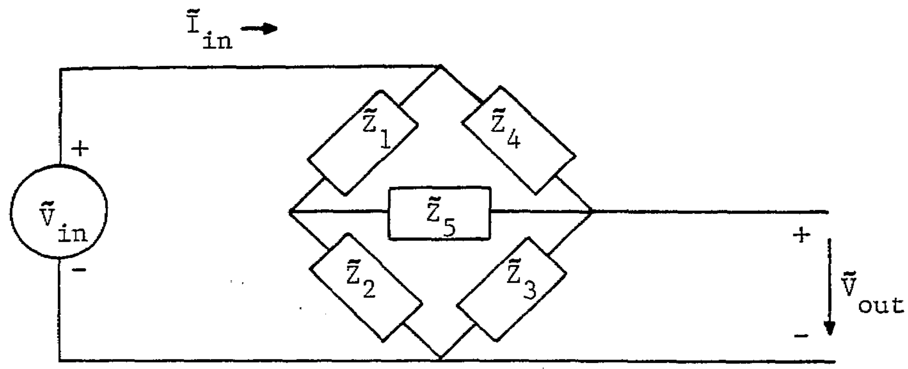
$$\tilde{Z}_4 = sL \tag{5.4}$$

which from (5.2) gives

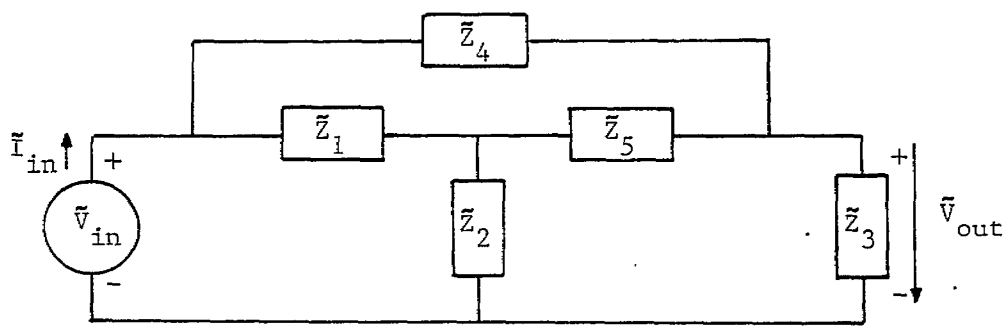
$$\begin{aligned} \tilde{Z}_2 &= \frac{R^2}{sL} \equiv \frac{1}{sC} \\ C &= \frac{L}{R^2} \end{aligned} \tag{5.5}$$

The equivalence of this circuit to the earlier constant-resistance low-pass filter is noted with

$$\begin{aligned} \tilde{Z}_1 + \tilde{Z}_2 &= R + \frac{1}{sC} \\ \tilde{Z}_3 + \tilde{Z}_4 &= R + sL \end{aligned} \tag{5.6}$$




lattice




bridged-T

Figure 5.1: Equivalent lattice and bridged-T filters

The transfer characteristics for a symmetrical constant-resistance filter are identical to the RLC constant-impedance network. The transfer function reduces to

$$\frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{\tilde{I}_{\text{out}}}{\tilde{I}_{\text{in}}}\frac{R}{R} = \frac{\tilde{I}_{\text{out}}}{\tilde{I}_{\text{in}}} = \left(1 + \frac{\tilde{Z}_3}{\tilde{Z}_2}\right)^{-1} \tag{5.7}$$

But

$$\tilde{Z}_3 = R\ ,\ \tilde{Z}_2 = \frac{1}{sC} \tag{5.8}$$

hence

$$\frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \left[1 + s\sqrt{LC}\right]^{-1} \tag{5.9}$$

This is the result for a single section of a constant-resistance filter. The symmetrical constant-resistance low-pass filter can be cascaded, with each new section replacing $\tilde{Z}_3$, in an analogous way to the previous section. The results for the cascaded case are identical to the cascaded RLC constant-resistance low-pass network.

# 6 Dipole Moments From Loops in Cascaded Constant-Resistance RLC Networks and Filters

The model for producing dipole moments from an array of loops provides an expression for the effective magnetic dipole moment, $\tilde{m}_{\text{eff}}$ as

$$\tilde{m}_{\text{eff}} = \sum_{n=1}^{N} \tilde{I}_n A_{eq_n} \tag{6.1}$$

The current $\tilde{I}_n$ through each loop expressed in terms of the input current $\tilde{I}_{\text{in}}$ for a cascaded constant-resistance RLC network or filter is

$$\tilde{I}_n = \tilde{I}_{\text{in}} \prod_{n=1}^{N} \left[1 + s\sqrt{C_n L_n}\right]^{-1} \tag{6.2}$$

The effective magnetic dipole moment, $\tilde{m}_{\text{eff}}$, can be expressed in terms of the input current and the equivalent loop area of each section as

$$\begin{aligned}
\tilde{m}_{\text{eff}} &= \tilde{I}_{\text{in}} \sum_{n=1}^{N} A_{eq_n} \prod_{\ell=1}^{n} \left[1 + s\sqrt{C_\ell L_\ell}\right]^{-1} \\
&= \tilde{I}_{\text{in}} \{ \prod_{\ell=1}^{N} \left[1 + s\sqrt{C_\ell L_\ell}\right]^{-1} \} \sum_{n=1}^{N} A_{eq_n} \prod_{\ell'=n+1}^{N} \left(1 + s\sqrt{C_{\ell'} L_{\ell'}}\right)
\end{aligned} \tag{6.3}$$

## 6.1 A Special Case With Identical Sections

A variety of loop-constant-resistance cascaded-RLC networks or filters can be made with various relationships between the loop areas and the reactive components. One such scheme is for the simplest case of identical loops in which all loop areas are equal and all the $L$ and $C$ components are equal, then

$$\begin{aligned}
A_{eq_0} &= A_{eq_n} \quad \text{for all} \quad n \\
C_0 &= C_n \quad \text{for all} \quad n \quad (\text{consequently } L_0 = L_n)
\end{aligned} \tag{6.4}$$

A normalized magnetic dipole moment can be written as the sum of a geometric progression

$$\frac{\tilde{m}_{\text{eff}}}{\tilde{I}_{\text{in}} A_{eq_0}} = \sum_{n=1}^{N} \left[1 + s\sqrt{L_0 C_0}\right]^{-n} = \frac{1}{s\sqrt{L_0 C_0}} \left\{1 - \left[1 + s\sqrt{L_0 C_0}\right]^{-N}\right\} \tag{6.5}$$

For low frequencies

$$\begin{aligned}
\frac{\tilde{m}_{\text{eff}}}{\tilde{I}_{\text{in}} A_{eq_0}} &= \frac{1}{s\sqrt{L_0 C_0}} \left\{1 - \left(1 - Ns\sqrt{L_0 C_0} + O\left(\left(s\sqrt{L_0 C_0}\right)^2\right)\right)\right\} \\
&\to N \quad \text{as} \quad s\sqrt{L_0 C_0} \to 0
\end{aligned} \tag{6.6}$$

For high frequencies

$$\begin{aligned}
\frac{\tilde{m}_{\text{eff}}}{\tilde{I}_{\text{in}} A_{eq_0}} &= \frac{1}{s\sqrt{L_0 C_0}} \left\{1 + O\left(\left(s\sqrt{L_0 C_0}\right)^{-1}\right)\right\} \\
&\to \frac{1}{s\sqrt{L_0 C_0}} \quad \text{as} \quad s\sqrt{L_0 C_0} \to 0
\end{aligned} \tag{6.7}$$

The case for a large number of loops then reduces to

$$\frac{\tilde{m}_{\text{eff}}}{\tilde{I}_{\text{in}} A_{eq_0}} = \frac{1}{s\sqrt{L_0 C_0}} \qquad \text{for} \qquad N = \infty \tag{6.8}$$

The normalized magnetic dipole moment plotted against $\omega\sqrt{L_0C_0}$ ($s = j\omega$) for 1 to 100 cascaded loops is shown in Figure 6.1. For circuits with five or less loops, the low-frequency asymptotic magnitude, $N$, is maintained to a breakpoint where it decreases and eventually falls off in proportion to the reciprocal of $\omega\sqrt{L_0C_0}$. If there are more than five loops, notches become increasingly apparent, with pronounced minima occurring at points where $\omega\sqrt{L_0C_0} = \tan\left(\frac{2\pi P}{N}\right)$, $P$ being any integer, $P \geq 1$. An alternate form of the expression can be written to illustrate the phenomenon, i.e.,

$$\frac{|\tilde{m}_{\text{eff}}|}{A_{eq_0}\tilde{I}_{\text{in}}} = \frac{1}{\omega\sqrt{L_0C_0}}\{1-2(1+\omega^2L_0C_0)\cos(N\arctan(\omega\sqrt{L_0C_0}))+(1+\omega^2L_0C_0)^{-N}\}^{1/2} \tag{6.9}$$

## 6.2 A Case With Common Proportional Factor for Area and Inductance

A case of combined loop cascaded constant-resistance networks of symmetrical filters worthy of investigation is one in which the loop areas and loop inductances vary according to a geometrical scaling so that the geometrical scale factor $\beta$ is used to multiply the loop dimensions. The area and inductance for succeeding loop areas and inductances are related for this case by

$$\begin{aligned} A_{eq_{n+1}} &= \beta^2 A_{eq_n} \\ L_{n+1} &= \beta L_n \end{aligned} \tag{6.10}$$

The general expression for a normalized magnetic dipole then can be written as

$$\frac{\tilde{m}_{\text{eff}}}{\tilde{I}_{\text{in}} A_{eq_1}} = \sum_{n=1}^{N} \beta^{2(n-1)} \prod_{\ell=1}^{n} (1 + j\beta^{\ell-1}X)^{-1}$$

$$X = \omega\frac{L_1}{R} = \omega C_1 R = \omega\sqrt{L_1 C_1} \qquad (6.11)$$
$$s = j\omega$$

The low-frequency and high-frequency behavior of this expression can be obtained by manipulation of $X = \omega\sqrt{L_1C_1}$. At low frequencies

$$\frac{\tilde{m}_{\text{eff}}}{\tilde{I}_{\text{in}} A_{eq_1}} = \sum_{n=1}^{N} \beta^{2(n-1)} \prod_{\ell=1}^{n}[1 + O(jX)] = \{\sum_{n=1}^{N} \beta^{2(n-1)}\}[1 + O(jX)]$$
$$= \frac{\beta^{2N} - 1}{\beta^2 - 1}[1 + O(jX)] \quad \text{as} \quad X \to 0 \qquad (6.12)$$

At high frequencies

$$\frac{\tilde{m}_{\text{eff}}}{\tilde{I}_{\text{in}} A_{eq_1}} = \sum_{n}^{N} \beta^{2(n-1)} \prod_{\ell=1}^{n} \frac{-j}{\beta^{\ell-1}X}\left[1 + O\left(\frac{1}{jX}\right)\right]$$
$$= -\frac{j}{X}\left[1 + O\left(\frac{1}{jX}\right)\right] \quad \text{as} \quad X \to \infty \qquad (6.13)$$

The normalized magnetic moment is plotted against $\omega\sqrt{L_1C_1}$ for various numbers of cascaded sections with a common proportional factor and shown in Figures 6.2 and 6.3, for $\beta = \sqrt{2}$ and $\beta = 2$, respectively. The increasing efficiency as $\beta$ is made larger is obvious in the three graphs. The case for equal segments (Fig. 6.1) can be viewed as equivalent to the one for a common proportional factor of unity. Note that as $N$ is increased in Figs. 6.2 and 6.3 there is not a pronounced set of notches as in Fig. 6.1. Apparently, the geometric progression for $\beta > 1$ results in a smoother performance as a function of frequency.

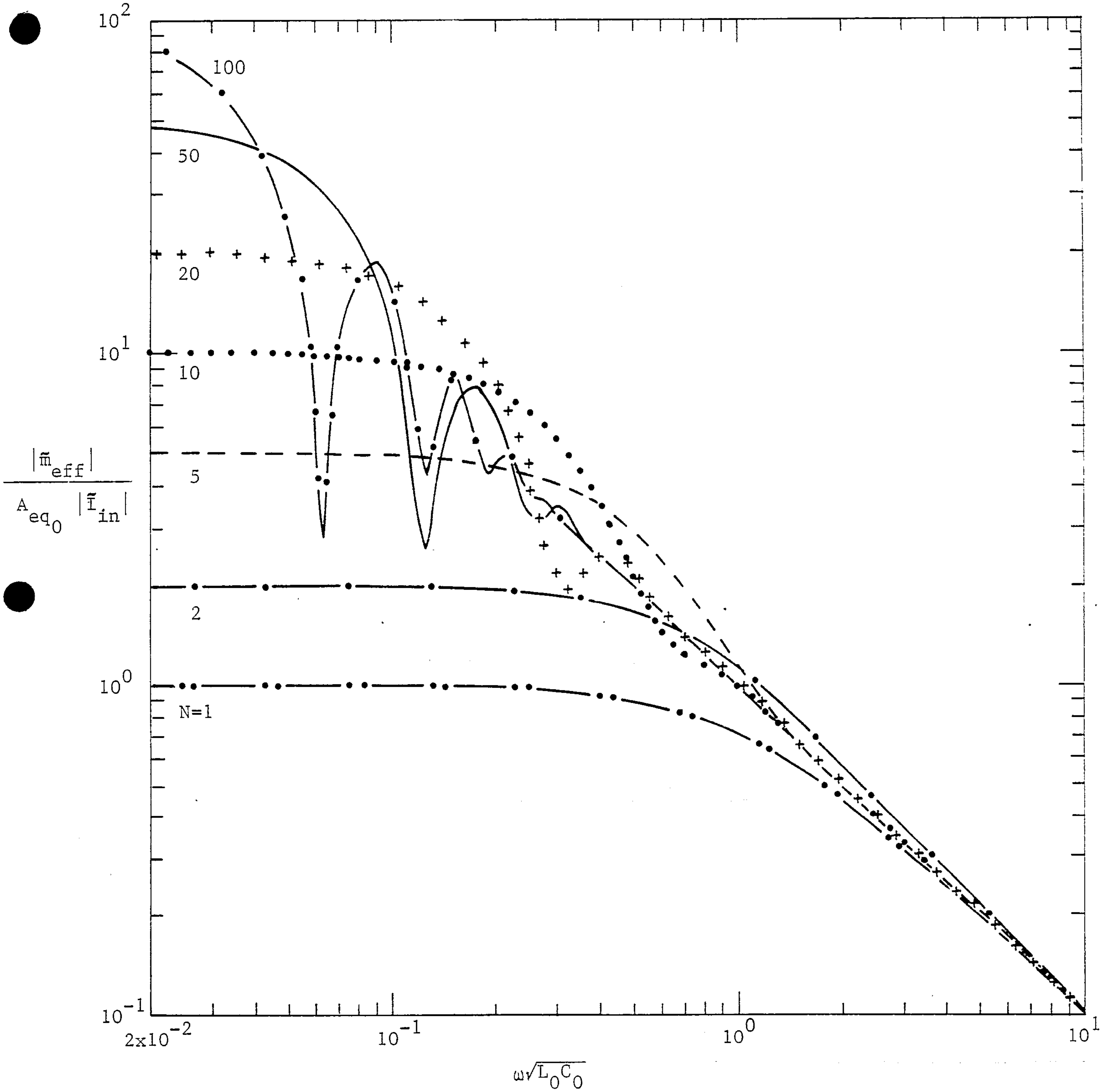



Figure 6.1: Magnetic Dipole Moment for Cascaded Identical Sections of Cascaded RLC Networks or Symmetrical Constant-Resistance Filters

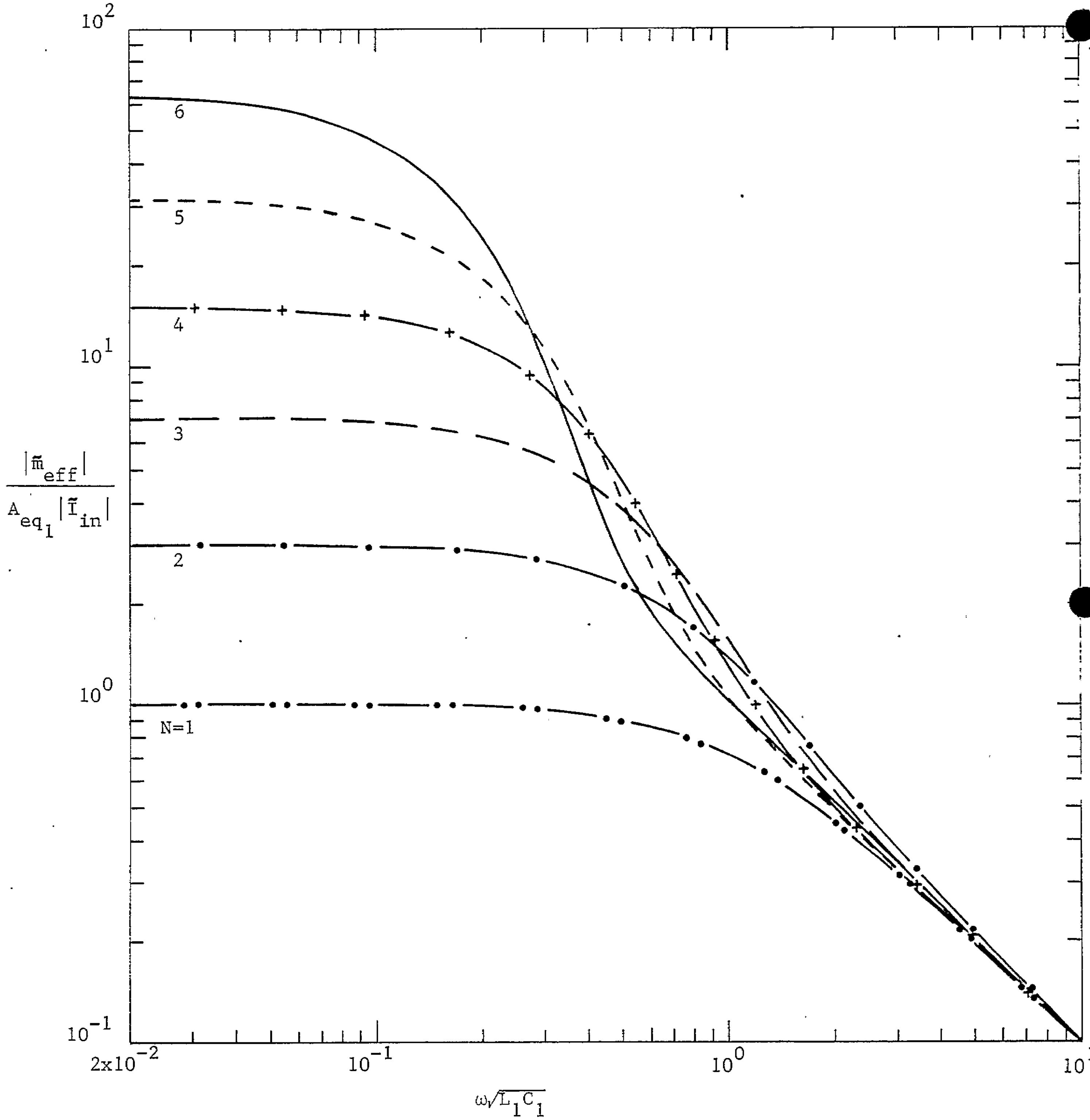



Figure 6.2: Magnetic Dipole Moment for Cascaded Proportional Sections of Cascaded RLC Networks or Symmetrical Constant-Resistance Filters $\beta = \sqrt{2}$

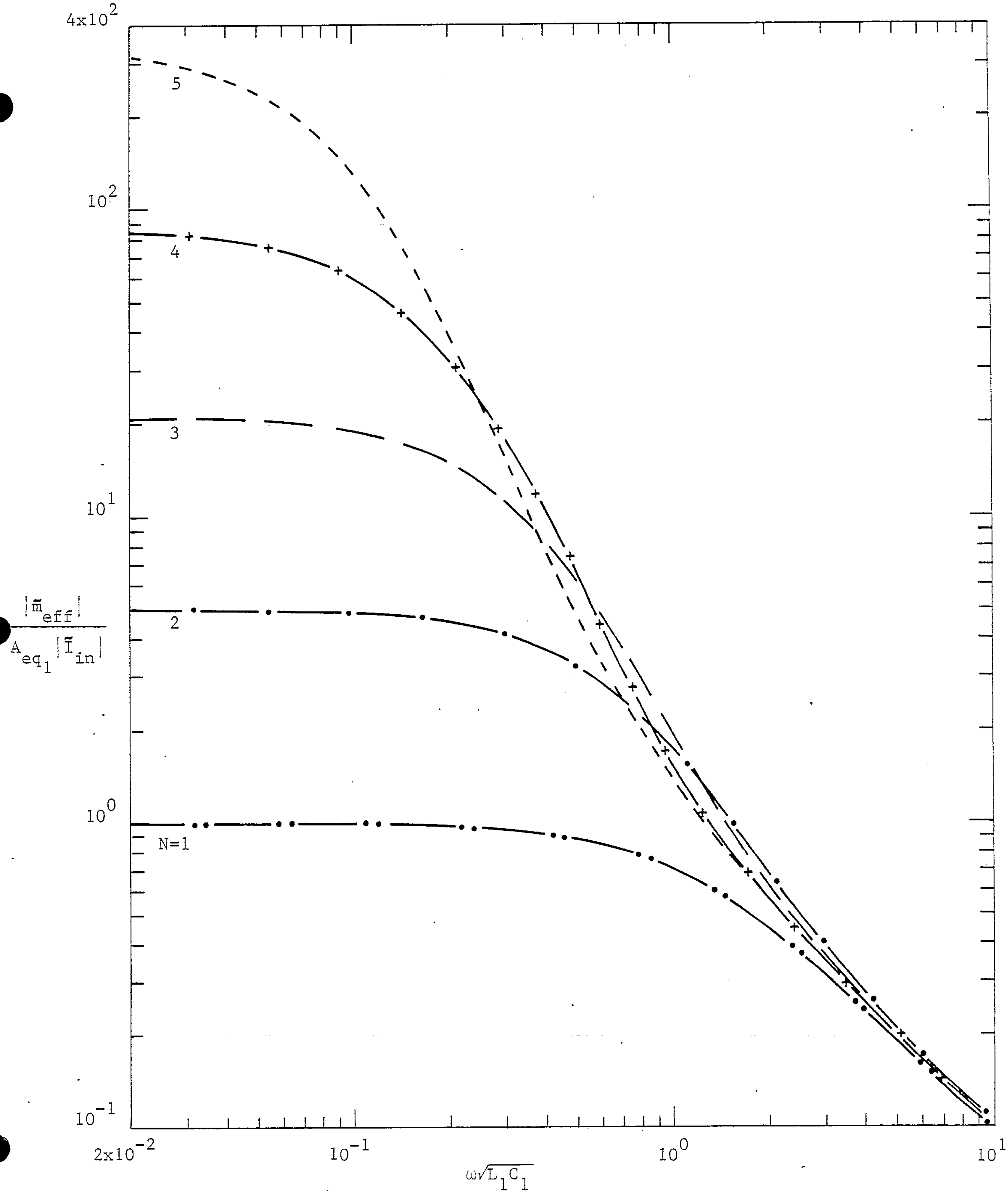



Figure 6.3: Magnetic Dipole Moment for Cascaded Proportional Sections of Cascaded RLC Networks or Symmetrical Constant-Resistance Filters; $\beta=2$

# 7 Dipole Moments Produced in a Continuous Loop-Transmission-Line Model

Relationships have been derived for an array of loops included in cascaded networks. The behavior of a cascaded constant-resistance network and a cascaded symmetrical filter are essentially identical, and also bear a close relationship to a transmission-line. It is therefore of interest to examine a model for producing magnetic dipole moments using a transmission-line approach.

In this model the radiating loops are considered to form an integral part of a transmission-line. The effective area of the loop is continuously increasing at a rate with respect to distance $x$, expressed by $\beta^{2x/x_0}$, where $x_0$ is an arbitrary constant. The associated inductance per unit length, $L'$, varies at a rate with respect to $x$, expressed by $\beta^{x/x_0}$. In this respect, the transmission-line model is similar to the general cascaded lumped-element case, but substitutes discrete sections with continuously varying functions related to transmission-line analysis.

Assuming that the radiating loop area can be included in a transmission line that has no series loss term the transmission-line equations can be written as

$$\begin{aligned} \frac{d\tilde{V}}{dx} &= -sL'\tilde{I} \\ \frac{d\tilde{I}}{dx} &= -[G' + sC']\tilde{V} \end{aligned} \qquad (7.1)$$

The transmission-line is given an added characteristic that the inductance per unit length $L'$ varies proportionate to $\beta^{x/x_0}$, thus an added relation is

$$L' = L'_0\beta^{\frac{x}{x_0}} \equiv L'_0 e^{\frac{x}{x_1}}\ ,\ e^{\frac{x_0}{x_1}} = \beta \qquad (7.2)$$

Assuming that the impedance of this particular type of transmission-line is independent of $x$, as per the usual case, then

$$\frac{\tilde{V}}{\tilde{I}} = \tilde{Z} \neq f(x) \tag{7.3}$$

This enables the first transmission-line equation to be expressed in terms of $\tilde{V}$ as

$$\frac{d\tilde{V}}{dx} = -sL'\frac{\tilde{V}}{\tilde{Z}} \tag{7.4}$$

The solution for $\tilde{V}$ then is

$$\begin{aligned} \tilde{V} &= \tilde{V}_{0'} e^{\frac{-sL'_0 x_1}{\tilde{Z}} e^{\frac{x}{x_1}}}, \\ \tilde{V}_{0'} &= \tilde{V}_0 e^{\frac{sL_0 x_1}{\tilde{Z}} e^{\frac{x_0}{x_1}}} \\ \tilde{I} &= \frac{\tilde{V}}{\tilde{Z}} = \frac{\tilde{V}_{0'}}{\tilde{Z}} e^{-\frac{sL'_0 x_1}{\tilde{Z}} e^{\frac{x}{x_1}}} \end{aligned} \tag{7.5}$$

On substituting for $\tilde{V}$ and $\tilde{I}$ in the second form of the original transmission-line equation,

$$\frac{d\tilde{I}}{dx} = -[G' + sC']\tilde{V} \tag{7.6}$$

we obtain

$$\begin{aligned} \frac{V_{0'}}{\tilde{Z}} e^{-\frac{sL'_0 x_1}{\tilde{Z}} e^{\frac{x_1}{x}}} \left(-\frac{sL'_0}{\tilde{Z}} e^{\frac{x}{x_1}}\right) &= -(G' + sC'_0 e^{\frac{x}{x_1}})\tilde{V}_{0'} e^{-\frac{sL_0 x_1}{\tilde{Z}} e^{\frac{x}{x_1}}} \\ \frac{sL'_0}{\tilde{Z}^2} e^{\frac{x}{x_1}} &= G' + sC'_0 e^{\frac{x}{x_0}} \end{aligned} \tag{7.7}$$

i.e., a condition for this equation is that

$$G' = 0\,,\ \frac{L'_0}{C'_0} = \tilde{Z}^2 \tag{7.8}$$

Thus the assumption that the impedance be independent of $x$ is valid so long as these conditions are satisfied.

The accumulated effective magnetic dipole over a length of such a radiating loop-transmission-line extending from 0 to $x$ can be written as

$$\tilde{m}_{\text{eff}} = \int_0^x \tilde{I} A' dx \tag{7.9}$$

where $I'$ is a current per unit length, and $A'$ is an effective area per unit length.

From an examination of the properties of the proposed transmission-line, the relation for current as a function of $x$ has been derived as

$$\tilde{I} = \frac{\tilde{V}_{0'}}{\tilde{Z}} e^{-\frac{sL_0 x_1}{\tilde{Z}} e^{\frac{x}{x_1}}} \tag{7.10}$$

The relationship of effective area $A$, as a function of $x$, is defined as

$$A' = A'_0 e^{\frac{2x}{x_1}} \tag{7.11}$$

To approximate the proportional section case, we can look at the effect of a number, $n$, of sections each with an equal length. For convenience, the length can be taken to be $x_0$. Thus, the upper limit of integration becomes $nx_0$. On substituting for $\tilde{I}$ and $A'$ in the original equation for $\tilde{m}_{\text{eff}}$, and taking the upper limit of integration as $nx_0$, the integral becomes

$$\tilde{m} = A'_0 \int_0^{nx_0} \frac{\tilde{V}_{0'}}{\tilde{Z}} e^{-\frac{sL'_0 x_1}{\tilde{Z}} e^{\frac{x}{x_1}}} e^{\frac{2x}{x_1}} \, dx \tag{7.12}$$

The solution for this equation can be written as

$$\tilde{m}_{\text{eff}} = \frac{A'_0 \tilde{V}_{0'} x_1}{\tilde{Z}} e^{-\frac{sL'_0 x_1}{\tilde{Z}} e^{\frac{x}{x_0}}} \left[ -e^{\frac{x}{x_0}} \frac{\tilde{Z}}{sL'_0 x_1} - \left( \frac{\tilde{Z}}{sL'_0 x_1} \right)^2 \right] \Bigg|_{x=0}^{x=nx_0} \tag{7.13}$$

Substituting for $\tilde{V}'_0$ this equation reduces to

$$\tilde{m}_{\text{eff}} = \frac{A'_0 \tilde{V}_0 x_1}{\tilde{Z}} \left\{ \left[ \frac{\tilde{Z}}{sL'_0 x_1} + \left( \frac{\tilde{Z}}{sL'_0 x_1} \right)^2 \right] - e^{\frac{sL'_0 x_1}{\tilde{Z}}(1-\beta^n)} \left[ \frac{\beta^n \tilde{Z}}{sL'_0 x_1} + \left( \frac{\tilde{Z}}{sL'_0 x_1} \right)^2 \right] \right\} \tag{7.14}$$

The coefficients for equivalent area per unit length, $A'_0$, and for inductance per unit length, $L'_0$, can be replaced by new values $A_1$ and $L_1$ to scale the expression to the cascaded model. Thus, for the first section of the transmission-line model (from $n = 0$ to $n = 1$) the effective area and inductance is made equal to $A_1$ and $L_1$. For equivalent area first

$$A_1 = A'_0 \int_0^{x_0} e^{\frac{2x_0}{x_1}} \, dx = \frac{A_0 x_1}{2} (e^{\frac{2x_0}{x_1}} - e^0) = \frac{A'_0 x_1}{2} (\beta^2 - 1) \tag{7.15}$$

This gives a scaling relation

$$A_0' = \frac{2A_1}{x_1(\beta^2 - 1)} \tag{7.16}$$

A similar relationship can be obtained for $L_1$ by

$$L_1 = L_0' \int_0^{x_0} e^{\frac{x}{x_1}} dx = L_0' x_1(\beta - 1) \tag{7.17}$$

This gives a scaling relation for $L_0'$

$$L_0' = \frac{L_1}{x_1(\beta^2 - 1)} \tag{7.18}$$

Substituting for $A_0'$ and $L_0'$ into the equation for the magnetic dipole moment gives

$$\tilde{m}_{\text{eff}} = \frac{\tilde{V}_0}{\tilde{Z}} \frac{2A_1}{1} \frac{1}{\beta^2 - 1} \left\{ \left[ \frac{\tilde{Z}}{s} \frac{(\beta - 1)}{L_1} + \left( \frac{\tilde{Z}}{s} \frac{(\beta - 1)}{L_1} \right)^2 \right] - e^{\frac{sL_1}{\tilde{Z}} \left( \frac{1 - \beta^n}{\beta - 1} \right)} \left[ \frac{\tilde{Z}\beta^n(\beta - 1)}{sL_1} + \left( \frac{\tilde{Z}(\beta - 1)}{sL_1} \right)^2 \right] \right. \tag{7.19}$$

The impedance term can be written in terms of $L_1$ and $C_1$ as $\tilde{Z} = \sqrt{\frac{L_1}{C_1}}$. Further, if $s = j\omega$, then the term $\frac{\tilde{Z}}{sL_1}$ can be substituted by $\frac{1}{j\omega\sqrt{L_1C_1}}$. For simplification $\omega\sqrt{L_1C_1}$ has been replaced by $k$. The equation for the magnetic dipole moment can be written as

$$\begin{aligned}
\tilde{m}_{\text{eff}} &= -\frac{\tilde{V}_0}{\tilde{Z}} \frac{2A_1}{k^2} \frac{\beta - 1}{\beta + 1} \left\{ \left[ 1 + \frac{jk}{\beta - 1} \right] - e^{jk\frac{1 - \beta^n}{\beta - 1}} \left[ 1 + j\frac{\beta^n k}{\beta - 1} \right] \right\} \\
&= -\frac{\tilde{V}_0}{\tilde{Z}} \cdot \frac{2A_1}{k^2} \frac{\beta - 1}{\beta + 1} \left\{ 1 - \cos\left( k \left( \frac{1 - \beta^n}{\beta - 1} \right) \right) + \beta^n \frac{k}{\beta - 1} \sin\left( k \left( \frac{1 - \beta^n}{\beta - 1} \right) \right) \right. \\
&\quad \left. + j \left[ \frac{k}{\beta - 1} - \sin\left( k \left( \frac{1 - \beta^n}{\beta - 1} \right) \right) - \beta^n \frac{k}{\beta - 1} \cos\left( k \left( \frac{1 - \beta^n}{\beta - 1} \right) \right) \right] \right\}
\end{aligned} \tag{7.20}$$

Plots of the magnitude of the normalized magnetic dipole moment $\frac{\tilde{m}_{\text{eff}}\tilde{Z}}{A_1V_0}$ versus $\omega\sqrt{L_1C_1}$ are shown in Fig. 7.1 for three values of $n$, the number of transmission-line sections. Figure 7.1 shows $\left|\frac{\tilde{m}_{\text{eff}}\tilde{Z}}{A_1V_0}\right|$ relating to $n = 1, 2$, and 5, for $\beta = \sqrt{2}$ and Figure 7.2 that for $\beta = 2$.

Inspection of the equation indicates that a new constant $k_{\text{eff}} = \frac{k}{\beta - 1}$ can be substituted. This gives

$$\begin{aligned}
\tilde{m}_{\text{eff}} &= -\frac{V_0}{\tilde{Z}} \cdot \frac{2A_1}{k_{\text{eff}}^2} \cdot \frac{1}{(\beta^2 - 1)} \{[1 - \cos(k_{\text{eff}}(1 - \beta^n)) + \beta^n k_{\text{eff}} \sin(k_{\text{eff}}(1 - \beta^n))] \\
&\quad + j[k_{\text{eff}} - \sin(k_{\text{eff}}(1 - \beta^n)) - \beta^n k_{\text{eff}} \cos(k_{\text{eff}}(1 - \beta^n))]\}
\end{aligned} \tag{7.21}$$

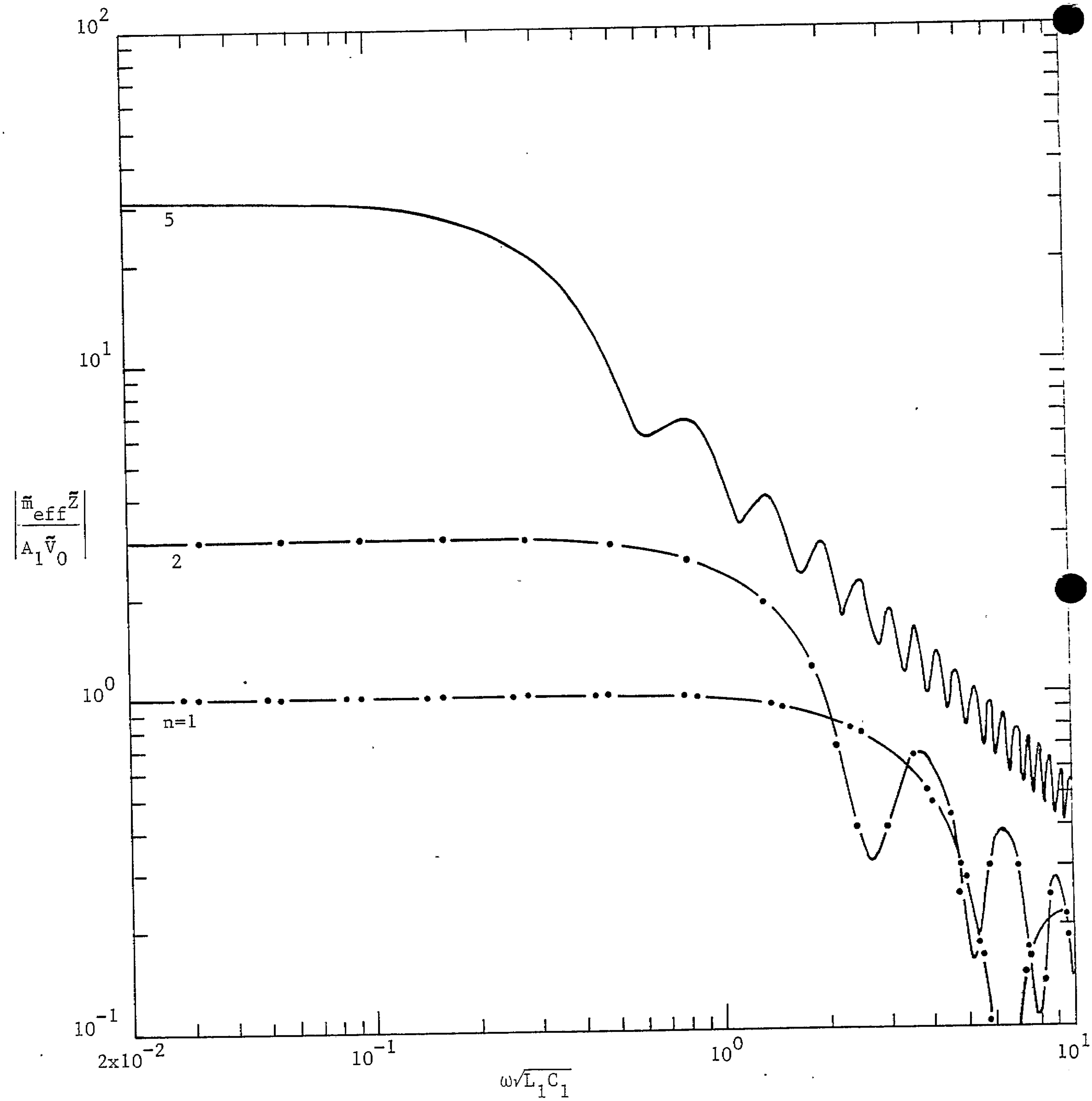



Figure 7.1: Magnetic Dipole Moment for Transmission-Line Model with an Effective Proportional Factor $\beta = \sqrt{2}$

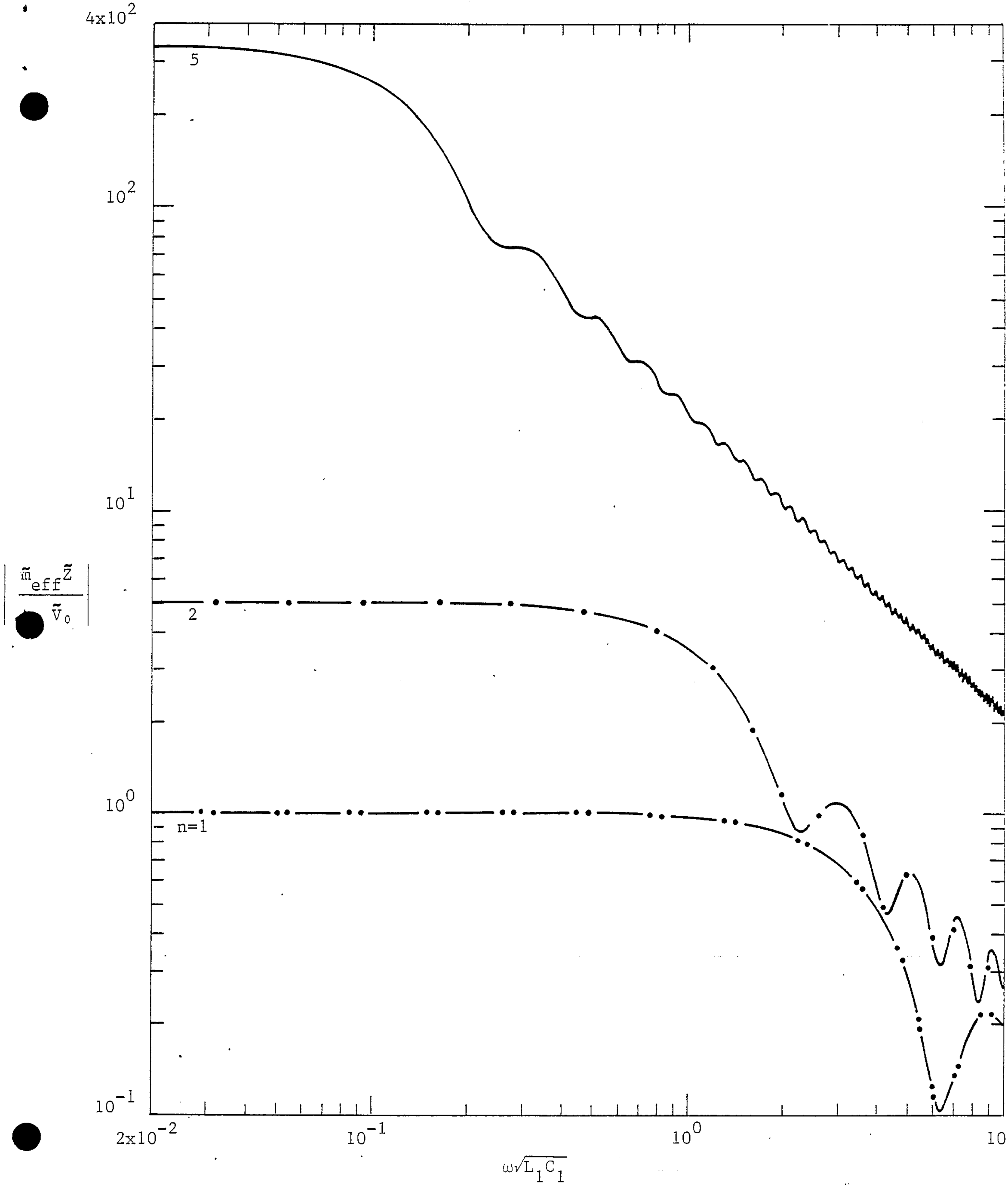



Figure 7.2: Magnetic Dipole Moment for Transmission-Line Model with an Effective Proportional Factor β = 2

The low-frequency behavior of this expression can be obtained by examining it for $k_{\text{eff}}$ at a very small value. The real part of the expression can be expanded for the cosine and sine terms, excluding the multiplying factor

$$\begin{aligned}
\{\text{Real Part}\} &= 1 - \cos k_{\text{eff}}(1-\beta^n) + \beta^n k_{\text{eff}} \sin k_{\text{eff}}(1-\beta^n) \\
&= 1 - 1 + \frac{k_{\text{eff}}^2(1-\beta^n)^2}{2} + O(k_{\text{eff}}^4) \\
&\quad + \beta^n k_{\text{eff}}^2(1-\beta^n) - \frac{\beta^n k_{\text{eff}}^4(1-\beta^n)^3}{3!} + O(k_{\text{eff}}^6) \\
&= k_{\text{eff}}^2\left(\frac{(1-\beta^n)^2}{2} + \beta^n(1-\beta^n)\right) + O(k_{\text{eff}}^4) \\
&= k_{\text{eff}}^2\left(\frac{1}{2} - \beta^n + \frac{1}{2}\beta^{2n} - \beta^{2n} + \beta^n\right) + O(k_{\text{eff}}^4) \\
&= k_{\text{eff}}^2\frac{1-\beta^{2n}}{2} + O(k_{\text{eff}}^4) \quad \text{as} \quad k_{\text{eff}} \to 0 \qquad (7.22) \\
\{\text{Imag Part}\} &= k_{\text{eff}} - \sin(k_{\text{eff}}(1-\beta^n)) - \beta^n k_{\text{eff}} \cos(k_{\text{eff}}(1-\beta^n)) \\
&= k_{\text{eff}} - k_{\text{eff}}(1-\beta^n) + k_{\text{eff}}^3\frac{(1-\beta^n)^3}{3!} - O((k_{\text{eff}})^5) \\
&\quad - \beta^n k_{\text{eff}} + \beta^n k_{\text{eff}}^3\frac{(1-\beta^n)^2}{2!} - O((k_{\text{eff}})^5) \\
&= k_{\text{eff}}^3\left(\frac{(1-\beta^n)^3}{3!} + \frac{\beta^n(1-\beta^n)^2}{2!}\right) + O((k_{\text{eff}})^5) \quad \text{as} \quad k_{\text{eff}} \to 0
\end{aligned}$$

i.e., for $k_{\text{eff}}$ small, the imaginary part may be ignored.

The magnetic dipole moment for low frequencies can thus be written as

$$\begin{aligned}
\tilde{m}_{\text{eff}} &= -\frac{\tilde{V}_0}{\tilde{Z}} \cdot \frac{2A_1}{k_{\text{eff}}^2}\frac{1}{\beta^2-1} \times \frac{k_{\text{eff}}^2(1-\beta^{2n})}{2} \quad \text{as} \quad k_{\text{eff}} \to 0 \\
&= -\frac{\tilde{V}_0 A_1}{\tilde{Z}}\frac{(1-\beta^{2n})}{\beta^2-1} \quad \text{as} \quad k_{\text{eff}} \to 0 \\
&= \frac{\tilde{V}_0 A_1}{\tilde{Z}} \cdot \frac{\beta^{2n}-1}{\beta^2-1} \quad \text{as} \quad k_{\text{eff}} \to 0 \qquad (7.23)
\end{aligned}$$

The magnitude of magnetic dipole for the low-frequency case for the transmission-line is identical to the cascaded constant-resistance RLC network and symmetrical filter cases.

The magnitude of the magnetic dipole moment for large values of $k$ can be evaluated from the expression for $\tilde{m}_{\text{eff}}$ in the form

$$
\begin{aligned}
|\tilde{m}_{\text{eff}}| &= \frac{\tilde{V}_0}{\tilde{Z}} \frac{2A_1}{k_{\text{eff}}^2} \frac{1}{(\beta^2-1)} [2 + k_{\text{eff}}^2 + \beta^{2n} k_{\text{eff}}^2 - 2(1+\beta^n k_{\text{eff}}^2)\cos(k_{\text{eff}}(1-\beta^n)) \\
&\quad + 2k_{\text{eff}}(\beta^n - 1)\sin(k_{\text{eff}}(1-\beta^n))]^{1/2}
\end{aligned} \tag{7.24}
$$

For $k$ large, this equation can be simplified to

$$
\begin{aligned}
|\tilde{m}_{\text{eff}}| &= \frac{\tilde{V}_0}{\tilde{Z}} \frac{2A_1}{k_{\text{eff}}} \frac{1}{\beta^2-1} [1 + \beta^{2n} - 2\beta^n \cos(k_{\text{eff}}(1-\beta^n)) + O(k^{-1})]^{1/2} \\
&= \frac{\tilde{V}_0}{\tilde{Z}} \frac{2A_1}{k_{\text{eff}}} \frac{1}{\beta^2-1} [1 + \beta^{2n} - 2\beta^n \cos(k_{\text{eff}}(1-\beta^n))]^{1/2} + O(k^{-2})
\end{aligned} \tag{7.25}
$$

Thus the relationship for high-frequency conditions differs from the cascaded high-frequency case by being dependent upon $\beta$ as well as the inverse of $k$. The relationship shows a strongly oscillating nature for $\beta$ approximately equal to (but greater than) one, but this can be ignored if $n$ is sufficiently large so that $\beta^n \gg 2$, in which case the high-frequency behavior of the transmission-line model can be written as

$$
\begin{aligned}
|\tilde{m}_{\text{eff}}| &= \frac{\tilde{V}_0}{\tilde{Z}} \frac{2A_1}{k_{\text{eff}}} \frac{\beta^n}{\beta^2-1} [1 + O(\beta^{-n}) + O(k^{-1})] \\
&\quad \text{as} \quad \beta^n \to \infty \qquad \text{and} \qquad k \to \infty
\end{aligned} \tag{7.26}
$$

Thus the high frequencies behavior of the transmission-line model is similar to the cascaded constant-resistance network and cascaded symmetrical constant-resistance filter in its dependence on the reciprocal of $k_{\text{eff}}$, but differs in its dependence on $\beta$ and $n$. The transmission-line model introduces a factor $2\beta^n/(\beta^2-1)$ in the relationship for large $k_{\text{eff}}$.

# 8 Summary

The feasibility is established in this paper of using a combined array of parallel loops and an effective low-pass network to produce controlled accumulative magnetic dipole moments. Models for describing the effective magnetic dipole moment have been developed as a function of frequency and circuit parameters for cascaded constant-resistance ladder networks, symmetrical constant-resistance filters and for a continuous case based on a lossless transmission line. The models for cascaded constant-resistance ladder networks and symmetrical constant resistance filters produce identical results. For this case, configurations with a common proportional constant for loop area and inductance variation between sequential cascaded sections produce magnetic dipole moments with flat characteristics up to a predictable break frequency. At this point the behavior is proportional to $(\text{frequency})^{-1}$. This gives rise to the possibility of providing magnetic dipole moments over a frequency range determined by network parameters and input current characteristics.

A model developed for the continuous lossless transmission line case with an equivalent proportional constant produces identical results to the cascaded constant-resistance circuits for low-frequency conditions. The general nature of the intermediate and high-frequency behavior is similar in nature but exhibits an extra factor in its asymptotes and further contains a significant oscillatory component. The significance of the oscillating component diminishes as the equivalent proportional constant and the number of equivalent sections are increased. The oscillations are likely to be further diminished if some loss is added to the transmission line in the model, but that has not been considered here.

The models relate a composite magnetic dipole moment over the complete frequency range for a given input current and circuit configuration. This can satisfy the frequency aspect of approximating an electromagnetic field by an array of magnetic dipoles. To synthesize an electromagnetic field over an extended volume of space, a number of such arrays (or one array used at successive locations) need to be located such that they are effectively radiating as dipoles from a surface. Implementing the field equivalence principle by an array of loops is then feasible in both the frequency and spatial sense. The current paper has addressed the relationship of the electrical parameters of a combined driver-radiating configuration to the frequency components of a combined electromagnetic field. The interaction between loops is an essential aspect not considered here. A number of approaches are available for addressing this problem, such as locating loops where interaction is absent or alternatively adding the mutual interaction component to the model. The interaction of loops are to be studied in an associated note.

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