Sensor and Simulation Notes

Note 306

19 May 1987

FOCUSED APERTURE ANTENNAS

Carl E. Baum

Air Force Weapons Laboratory

ABSTRACT

The electromagnetic fields from aperture antennas can be represented as integrals over the aperture electric field. Maximizing the fields at an observer defines a focused aperture. In this case, the integrals simplify and the spatial and frequency parts conveniently separate. This makes the results also conveniently expressible in time domain.

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aperture antennas

I. Introduction

A previous note [2] has considered planar distributed sources for radiating transient pulses. Under suitable approximations, this can be analyzed as what is referred to as an aperture antenna [4]. By assuming an appropriate distribution of the tangential electric field on the aperture the fields throughout space are expressible by integrals over the aperture.

In order to simplify the results and maximize the fields one can restrict consideration to what is called a focused aperture [4]. This separates the dependence of the fields at the observer into separate spatial and frequency terms. This also allows for a convenient representation in time domain.

After developing the general theory, explicit integrals are given for the spatial coefficients for the two frequency terms for the electric field and the three frequency terms for the magnetic field. Then the case of circular aperture, focused on the axis of symmetry, with a uniform tangential electric field is considered, resulting in closed-form expressions valid at any distance from the aperture.

II. Electromagnetic Fields from Arbitrary Aperture Fields in a Plane

The basic equations for electromagnetic fields from those on a half plane $(z = 0)$ are well known [3, 4]. In particular let us consider that the tangential electric field on the source plane S' (i.e., $z' = 0$) is specified. Let the $\vec{r}\,'$ coordinates be those on S', i.e.,

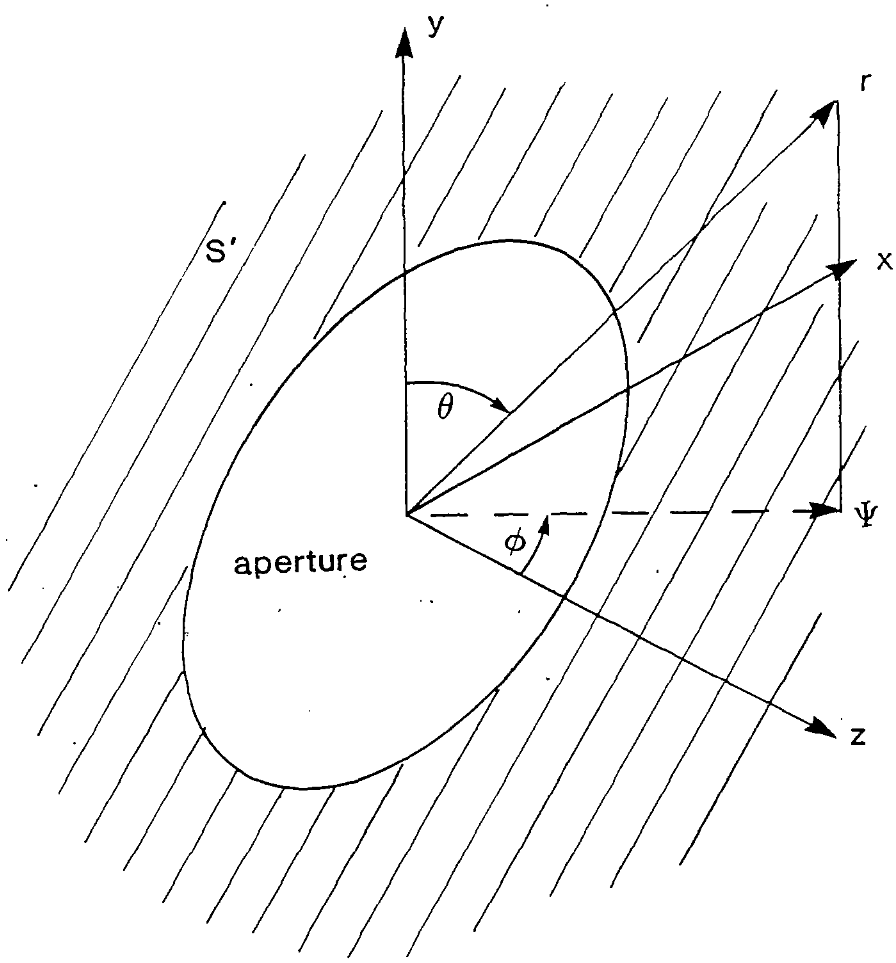
$$\vec{r}\,' = (x', y', 0) \tag{2.1}$$

Furthermore, let $\vec{r}$ be the coordinates of the fields away from S' as

$$\vec{r} = (x, y, z) \tag{2.2}$$

Then as in Figure 2.1 we have some assumed tangential field distribution on S' as

$$\begin{aligned}\vec{E}_t\,(x', y', 0; t) &= (E_{x'}(x', y', 0; t), E_{y'}(x', y', 0; t), 0)\\ &\equiv \vec{E}_t(x', y'; t)\end{aligned} \tag{2.3}$$



S' is defined
by $\vec{r}'=(x',y',0)$.

Figure 2.1. Electromagnetic fields from a source plane S'.

In addition, there is a z component of the electric field on S' which, however, does not appear as a source in the traditional equations.

Now at some position $\vec{r}$ we wish to compute the fields. Define

$$R \equiv |\vec{r} - \vec{r}'| = [(x-x')^2 + (y-y')^2 + z^2]^{1/2}$$

$$\vec{1}_R \equiv \frac{x-x'}{R}\vec{1}_x + \frac{y-y'}{R}\vec{1}_y + \frac{z-z'}{R}\vec{1}_z$$

$s \equiv$ Laplace-transform (two-sided) variable

$\equiv$ complex frequency

$$\gamma \equiv \frac{s}{c} \equiv \text{free-space propagation constant} \tag{2.4}$$

$$c \equiv \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \equiv \text{speed of light}$$

$$Z_0 \equiv \sqrt{\frac{\mu_0}{\varepsilon_0}} \equiv \text{wave impedance of free space}$$

Our half space of interest ($z > 0$) is considered free space with zero conductivity, permittivity ε_0, and permeability μ_0.

The electromagnetic fields are computed as from an equivalent set of magnetic current sources on S' which give a vector potential [1, 3].

$$\tilde{\vec{A}}(\vec{r},s) = \frac{-1}{2\pi s}\int_{S'} \frac{\gamma R+1}{R^2}\,[[\vec{1}_z \times \tilde{\vec{E}}_t'(x',y';s)] \times \vec{1}_R]\, e^{-\gamma R} dS' \tag{2.5}$$

The scalar potential of a magnetic current distribution being zero, we have

$$\vec{E} = \frac{-\partial\vec{A}}{\partial t},\ \tilde{\vec{E}} = -s\tilde{\vec{A}}$$

$$\vec{B} = \mu_0\vec{H} = \nabla \times \vec{A},\ \ \tilde{\vec{B}} = \mu_0\tilde{\vec{H}} = \nabla \times \tilde{\vec{A}} \tag{2.6}$$

Then we have

$$\tilde{\vec{E}}(\vec{r},s) = \frac{1}{2\pi}\int_{S'} \frac{\gamma R+1}{R^2} [[\vec{1}_z \times \tilde{\vec{E}}_t'(x',y';s)] \times \vec{1}_R]\, e^{-\gamma R} dS'$$

$$\tilde{\vec{H}}(\vec{r},s) = \frac{-1}{2\pi\mu_o s} \{\nabla \times \int_{S'} \frac{\gamma R+1}{R^2} [[\vec{1}_z \times \tilde{\vec{E}}_t'(x',y';s)] \times \vec{1}_R]\, e^{-\gamma R} dS'\} \quad (2.7)$$

In component form, we have for the electric field

$$\tilde{E}_{\substack{x\\y}}(\vec{r},s) = \frac{1}{2\pi}\int_{S'} \frac{\gamma R+1}{R^2}\frac{z}{R} e^{-\gamma R}\, \tilde{E}'_{\substack{x\\y}}(x',y';s)\, dS'$$

$$\tilde{E}_z(\vec{r},s) = \frac{1}{2\pi}\int_{S'} \frac{\gamma R+1}{R^2} e^{-\gamma R} [\frac{x-x'}{R}\tilde{E}_x'(x',y';s) + \frac{y-y'}{R}\tilde{E}_y'(x',y';s)]\, dS' \quad (2.8)$$

The magnetic field components are somewhat more complicated as

$$\tilde{H}_x(\vec{r},s) = \frac{1}{2\pi\mu_o s}\int_{S'} \frac{e^{-\gamma R}}{R^3} \{\frac{-(x-x')(y-y')}{R^2}[(\gamma R)^2 + 3\gamma R + 3]\, \tilde{E}_x'(x',y';s)$$

$$+ [2\gamma R + 2 - \frac{(y-y')^2 + z^2}{R^2}[(\gamma R)^2 + 3\gamma R + 3]]\, \tilde{E}_y'(x',y';s)\}\, dS'$$

$$\tilde{H}_y(\vec{r},s) = \frac{1}{2\pi\mu_o s}\int_{S'} \frac{e^{-\gamma R}}{R^3} \{\frac{(x-x')(y-y')}{R^2}[(\gamma R)^2 + 3\gamma R + 3]\, \tilde{E}_y'(x',y';s)$$

$$- [2\gamma R + 2 - \frac{(x-x')^2 + z^2}{R^2}[(\gamma R)^2 + 3\gamma R + 3]\, \tilde{E}_x'(x',y';s)\}\, dS' \quad (2.9)$$

$$\tilde{H}_z(\vec{r},s) = \frac{1}{2\pi\mu_o s}\int_{S'} \frac{e^{-\gamma R}}{R^3} \{\frac{-(y-y')z}{R^2}[(\gamma R)^2 + 3\gamma R + 3]\, \tilde{E}_x'(x',y';s)$$

$$+ \frac{(x-x')z}{R^2}[(\gamma R)^2 + 3\gamma R + 3]\, \tilde{E}_y'(x',y';s)\}\, dS'$$

Note that one can think of the fields from an aperture as those from an array of magnetic dipoles. The electric field has R^{-1} and R^{-2} contributions and the magnetic field has R^{-1}, R^{-2}, and R^{-3} contributions.

In time domain one also has explicit formulas by making the association.

Laplace (complex frequency) Domain $\leftrightarrow$ Time Domain

$$e^{-\gamma R}\,\tilde{\vec{E}}_t'(x',y';s) \leftrightarrow \vec{E}_t'(x',y';\,t-\frac{R}{c}) \tag{2.10}$$

$$s \leftrightarrow \frac{\partial}{\partial t}$$

$$s^{-1} \leftrightarrow \int dt$$

Substituting the time-domain forms in (2.7) through (2.9) these equations are all converted to explicit time-domain formulae for the fields in terms of time-domain aperture fields.

III. Focused Aperture

As discussed in various texts [4], one can have a focused aperture. By this we mean that the phase of the tangential electric field (source) on S' is adjusted such that at some observer at $\vec{r} = \vec{r}_0$ the signals from each elementary position on S' all arrive with the same phase. Looking at (2.7) it is the factor $e^{-\gamma R}\,\tilde{\vec{E}}_t'$, which we need to control. While $\tilde{\vec{E}}_t'$ is only a function of the source coordinates, $e^{-\gamma R}$ is a function of both source and field coordinates.

Considering a fixed observer position at $\vec{r} = \vec{r}_0$ then we have

$$R \equiv |\vec{r}_0 - \vec{r}'|, \qquad R_0 \equiv |\vec{r}_0| \equiv r_0 \tag{3.1}$$

Let us now constrain at $\vec{r}_0$

$$e^{-\gamma R}\,\tilde{\vec{E}}_t'(x',y';s) \equiv e^{-\gamma R_0} E_0\,\tilde{f}(s)\,\vec{g}(x',y') \tag{3.2}$$

With this choice, then at $\vec{r}_0$ all sources (or equivalent elementary magnetic dipoles) have the same waveform f(t) arriving at the same time at $\vec{r}_0$. The variation of the source amplitude over S' is constrained by a separate factor

$\vec{g}(x',y')$ (real) with an overall scale factor E_0 (real).

Substituting (3.2) into (2.7) through (2.9) gives the fields at the focus $\vec{r}_0$. Note that in the magnetic field equation in (2.7) the curl operator is with respect to $\vec{r}$, not $\vec{r}_0$. In terms of the components the electric field at $\vec{r}_0$ is

$$\tilde{E}_{\substack{x\\y}}(\vec{r}_0,s) = \frac{e^{-\gamma R_0}}{2\pi} E_0 \tilde{f}(s) \int_{S'} \frac{\gamma R+1}{R^2} \frac{z_0}{R} g_{\substack{x\\y}}(x',y')\, dS' \tag{3.3}$$

$$\tilde{E}_z(\vec{r}_0,s) = \frac{e^{-\gamma R_0}}{2\pi} E_0 \tilde{f}(s) \int_{S'} \frac{\gamma R+1}{R^2} \left[\frac{x_0-x'}{R} g_x(x',y') + \frac{y_0-y'}{R} g_y(x',y')\right] dS'$$

Similarly, the magnetic field is

$$\tilde{H}_x(\vec{r}_0,s) = \frac{e^{-\gamma R_0}}{2\pi\mu_0 s} E_0 \tilde{f}(s) \int_{S'} \left\{\frac{-(x_0-x')(y_0-y')}{R^5} [(\gamma R)^2 + 3\gamma R + 3]\, g_x(x',y')\right.$$

$$\left. + \left[\frac{2\gamma R+2}{R^3} - \frac{(y_0-y')^2 + z_0^2}{R^5} [(\gamma R)^2 + 3\gamma R + 3]\right] g_y(x',y')\right\} dS'$$

$$\tilde{H}_y(\vec{r}_0,s) = \frac{e^{-\gamma R_0}}{2\pi\mu_0 s} E_0 \tilde{f}(s) \int_{S'} \left\{\frac{(x_0-x')(y_0-y')}{R^5} [(\gamma R)^2 + 3\gamma R + 3]\, g_y(x',y')\right.$$

$$\left. - \left[\frac{2\gamma R+2}{R^3} - \frac{(x_0-x')^2+z_0^2}{R^5} [(\gamma R)^2 + 3\gamma R + 3]\right] g_x(x',y')\right\} dS' \tag{3.4}$$

$$\tilde{H}_z(\vec{r}_0,s) = \frac{e^{-\gamma R_0}}{2\pi\mu_0 s} E_0 \tilde{f}(s) \int_{S'} \left\{\frac{-(y_0-y')z_0}{R^5} [(\gamma R)^2 + 3\gamma R + 3]\, g_x(x',y')\right.$$

$$\left. + \frac{(x_0-x')z_0}{R^5} [(\gamma R)^2 + 3\gamma R + 3]\, g_y(x',y')\right\} dS'$$

After allowing for the delay term $e^{-\gamma R_0}$ and the source (tangential E) waveform $\tilde{f}(s)$, note that the electric field has terms proportional to s^1 and s^0, while the magnetic field has terms proportional to s^1, s^0, and s^{-1}. Then let us write for the electric field

$$\tilde{E}_x(\vec{r}_o,s) = E_o e^{-\gamma R_o} \tilde{f}(s) \{\alpha_{x,x}^{(1)} \frac{a}{R_o} \frac{as}{c} + \alpha_{x,x}^{(2)} \frac{a^2}{R_o^2}\}$$

$$\tilde{E}_y(\vec{r}_o,s) = E_o e^{-\gamma R_o} \tilde{f}(s) \{\alpha_{y,y}^{(1)} \frac{a}{R_o} \frac{as}{c} + \alpha_{y,y}^{(2)} \frac{a^2}{R_o^2}\}$$

$$\tilde{E}_z(\vec{r}_o,s) = E_o e^{-\gamma R_o} \tilde{f}(s) \{[\alpha_{z,x}^{(1)} + \alpha_{z,y}^{(1)}] \frac{a}{R_o} \frac{as}{c} + [\alpha_{z,x}^{(2)} + \alpha_{z,y}^{(2)}] \frac{a^2}{R_o^2}\} \tag{3.5}$$

Similarly, for the magnetic field

$$Z_o\tilde{H}_x(\vec{r}_o,s) = E_o e^{-\gamma R_o} \tilde{f}(s) \{\beta_{x,x}^{(1)} \frac{a}{R_o} \frac{as}{c} + \beta_{x,x}^{(2)} \frac{a^2}{R_o^2} + \beta_{x,x}^{(3)} \frac{a^3}{R_o^3} \frac{c}{as} + \beta_{x,y}^{(1)} \frac{a}{R_o} \frac{as}{c} + \beta_{x,y}^{(2)} \frac{a^2}{R_o^2} + \beta_{x,y}^{(3)} \frac{a^3}{R_o^3} \frac{c}{as}\}$$

$$Z_o\tilde{H}_y(\vec{r}_o,s) = E_o e^{-\gamma R_o} \tilde{f}(s) \{\beta_{y,x}^{(1)} \frac{a}{R_o} \frac{as}{c} + \beta_{y,x}^{(2)} \frac{a^2}{R_o^2} + \beta_{y,x}^{(3)} \frac{a^3}{R_o^3} \frac{c}{as} + \beta_{y,y}^{(1)} \frac{a}{R_o} \frac{as}{c} + \beta_{y,y}^{(2)} \frac{a^2}{R_o^2} + \beta_{y,y}^{(3)} \frac{a^3}{R_o^3} \frac{c}{as}\} \tag{3.6}$$

$$Z_o\tilde{H}_z(\vec{r}_o,s) = E_o e^{-\gamma R_o} \tilde{f}(s) \{\beta_{z,x}^{(1)} \frac{a}{R_o} \frac{as}{c} + \beta_{z,x}^{(2)} \frac{a^2}{R_o^2} + \beta_{z,x}^{(3)} \frac{a^3}{R_o^3} \frac{c}{as} + \beta_{z,y}^{(1)} \frac{a}{R_o} \frac{as}{c} + \beta_{z,y}^{(2)} \frac{a^2}{R_o^2} + \beta_{z,y}^{(3)} \frac{a^3}{R_o^3} \frac{c}{as}\}$$

where

$a \equiv$ some characteristic dimension of the aperture (3.7)

In this form the α and β coefficients are frequency independent and dimensionless,

The above results are also directly expressable in time domain for the electric field as

$$E_x(\vec{r}_o,t) = E_o\{\alpha^{(2)}_{x,x} \frac{a^2}{R_o c} \frac{\partial}{\partial t} f(t - \frac{R_o}{c}) + \alpha^{(2)}_{x,x} \frac{a^2}{R_o^2} f(t - \frac{R_o}{c})\}$$

$$E_y(\vec{r}_o,t) = E_o \{\alpha^{(1)}_{y,y} \frac{a^2}{R_o c} \frac{\partial}{\partial t} f(t - \frac{R_o}{c}) + \alpha^{(2)}_{y,y} \frac{a^2}{R_o^2} f(t - \frac{R_o}{c})\}$$

$$E_z(\vec{r}_o,t) = E_o \{[\alpha^{(1)}_{z,x} + \alpha^{(1)}_{z,y}] \frac{a^2}{R_o c} \frac{\partial}{\partial t} f(t - \frac{R_o}{c})$$

$$+ [\alpha^{(2)}_{z,x} + \alpha^{(2)}_{z,x}] \frac{a^2}{R_o^2} f(t - \frac{R_o}{c})\} \qquad (3.8)$$

In time domain the magnetic field is

$$Z_o H_x(\vec{r}_o,t) = E_o \{[\beta^{(1)}_{x,x} + \beta^{(1)}_{x,y}] \frac{a^2}{R_o c} \frac{\partial}{\partial t} f(t - \frac{R_o}{c})$$

$$+ [\beta^{(2)}_{x,x} + \beta^{(2)}_{x,y}] \frac{a^2}{R_o^2} f(t - \frac{R_o}{c}) + [\beta^{(3)}_{x,x} + \beta^{(3)}_{x,y}] \frac{a^2 c}{R_o^3} \int_{-\infty}^{t} f(t' - \frac{R_o}{c})\, dt'\} \qquad (3.9)$$

$$Z_o H_y(\vec{r}_o,t) = E_o\{[\beta^{(1)}_{y,x} + \beta^{(1)}_{y,y}] \frac{a^2}{R_o c} \frac{\partial}{\partial t} f(t - \frac{R_o}{c})$$

$$+ [\beta^{(2)}_{y,x} + \beta^{(2)}_{y,y}] \frac{a^2}{R_o^2} f(t - \frac{R_o}{c}) + [\beta^{(3)}_{y,x} + \beta^{(3)}_{y,y}] \frac{a^2 c}{R_o^3} \int_{-\infty}^{t} f(t' - \frac{R_o}{c})\, dt'\}$$

$$Z_o H_z(\vec{r}_o,t) = E_o \{[\beta^{(1)}_{z,x} + \beta^{(1)}_{z,y}] \frac{a^2}{R_o c} \frac{\partial}{\partial t} f(t - \frac{R_o}{c})$$

$$+ [\beta^{(2)}_{z,x} + \beta^{(2)}_{z,y}] \frac{a^2}{R_o^2} f(t - \frac{R_o}{c}) + [\beta^{(3)}_{z,x} + \beta^{(3)}_{z,y}] \frac{a^2 c}{R_o^3} \int_{-\infty}^{t} f(t' - \frac{R_o}{c})\, dt'\}$$

Note for the time integrals the excitation waveform is assumed to be identically zero before some turn-on time. In (3.8) and (3.9) the utility of focusing at $\vec{r}_o$ is indicated by the simple form the results take in time domain. If the aperture sources did not all reach $\vec{r}_o$ at the same time the three time-domain terms (derivative waveform, waveform, and integral waveform) would be smeared out. This is another way of considering that the fields are maximized at $\vec{r}_o$ by focusing.

Now we have the coefficients which depend on $\vec{r}_o$ and the spatial distribution on the source. The coefficients of the derivative terms for the electric field are

$$\alpha^{(1)}_{x,x} = \frac{1}{2\pi}\frac{R_o z_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-1} g_x(x',y')\ dS'$$

$$\alpha^{(1)}_{y,y} = \frac{1}{2\pi}\frac{R_o z_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-1} g_y(x',y')\ dS' \qquad (3.10)$$

$$\alpha^{(1)}_{z,x} = \frac{1}{2\pi}\frac{R_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-1}(x_o-x')\ g_x(x',y')\ dS'$$

$$\alpha^{(1)}_{z,y} = \frac{1}{2\pi}\frac{R_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-1}(y_o-y')\ g_y(x',y')\ dS'$$

The coefficients of the terms proportional to the source waveform for the electric field are

$$\alpha^{(2)}_{x,x} = \frac{1}{2\pi}\frac{R_o^2 z_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-3/2} g_x(x',y')\ dS'$$

$$\alpha^{(2)}_{y,y} = \frac{1}{2\pi}\frac{R_o^2 z_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-3/2} g_y(x',y')\ dS' \qquad (3.11)$$

$$\alpha^{(2)}_{z,x} = \frac{1}{2\pi}\frac{R_o^2}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-3/2}(x_o-x')\ g_x(x',y')\ dS'$$

$$\alpha^{(2)}_{z,y} = \frac{1}{2\pi}\frac{R_o^2}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-3/2}(y_o-y')\ g_y(x',y')\ dS'$$

The coefficients of the derivative terms for the magnetic field are

$$\beta^{(1)}_{x,x} = \frac{-1}{2\pi}\frac{R_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-3/2}(x_o-x')(y_o-y')\ g_x(x',y')\ dS'$$

$$\beta^{(1)}_{x,y} = \frac{-1}{2\pi}\frac{R_o}{a^2}\int_{S'}[(x_o-x')^2+(y_o-y')^2+z_o^2]^{-3/2}[(y_o-y')^2+z_o^2]\ g_y(x',y')\ dS'$$

$$\beta^{(1)}_{y,x} = \frac{1}{2\pi}\frac{R_o}{a^2}\int_{S'}[(x_o-x')^2+(y_o-y')^2+z_o^2]^{-3/2}[(x_o-x')^2+z_o^2]\ g_x(x',y')\ dS' \tag{3.12}$$

$$\beta^{(1)}_{y,y} = \frac{1}{2\pi}\frac{R_o}{a^2}\int_{S'}[(x_o-x')^2+(y_o-y')^2+z_o^2]^{-3/2}(x_o-x')(y_o-y')\ g_y(x',y')\ dS'$$

$$\beta^{(1)}_{z,x} = \frac{-1}{2\pi}\frac{R_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-3/2}\ (y_o-y')z_o\ g_x(x',y')\ dS'$$

$$\beta^{(1)}_{z,y} = \frac{1}{2\pi}\frac{R_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-3/2}\ (x_o-x')\ z_o\ g_y(x',y')\ dS'$$

The coefficients of the terms proportional to the source waveform for the magnetic field are

$$\beta^{(2)}_{x,x} = \frac{-3}{2\pi}\frac{R_o^2}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-2}\ (x_o-x')(y_o-y')\ g_x(x',y')\ dS'$$

$$\beta^{(2)}_{x,y} = \frac{1}{2\pi}\frac{R_o^2}{a^2}\int_{S'}\{2\ [(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-1}$$

$$- 3[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-2}[(y_o-y')^2 + z_o^2]\}\ g_y(x',y')\ dS' \tag{3.13}$$

$$\beta^{(2)}_{y,x} = \frac{1}{2\pi}\frac{R_o^2}{a^2}\int_{S'}\{-2[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-1}$$

$$+ 3[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-2}[(x_o-x')^2 + z_o^2]\} \; g_x(x',y') \; dS' \tag{3.13 (cont'd.)}$$

$$\beta^{(2)}_{y,y} = \frac{3}{2\pi}\frac{R_o^2}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-2} \; (x_o-x')(y_o-y') \; g_y(x',y') \; dS'$$

$$\beta^{(2)}_{z,x} = \frac{-3}{2\pi}\frac{R_o^2 z_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-2}(y_o-y') \; g_x(x',y') \; dS'$$

$$\beta^{(2)}_{z,y} = \frac{3}{2\pi}\frac{R_o^2 z_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-2}(x_o-x') \; g_y(x',y') \; dS'$$

The coefficients of the integral terms for the magnetic field are

$$\beta^{(3)}_{x,x} = \frac{-3}{2\pi}\frac{R_o^3}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-5/2}(x_o-x')(y_o-y') \; g_x(x',y') \; dS'$$

$$\beta^{(3)}_{x,y} = \frac{1}{2\pi}\frac{R_o^3}{a^2}\int_{S'}\{2\,[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-3/2}$$

$$- 3[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-5/2}[(y_o-y')^2 + z_o^2]\} \; g_y(x',y') \; dS' \tag{3.14}$$

$$\beta^{(3)}_{y,x} = \frac{1}{2\pi}\frac{R_o^2}{a^2}\int_{S'}\{-2\,[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-3/2}$$

$$+ 3[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-5/2}[(x_o-x')^2 + z_o^2]\} \; g_x(x',y') \; dS'$$

$$\beta^{(3)}_{y,y} = \frac{3}{2\pi}\frac{R_o^3}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-5/2}(x_o-x')(y_o-y') \; g_y(x',y') \; dS'$$

$$\beta^{(3)}_{z,x} = \frac{-3}{2\pi}\frac{R_o^3 z_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-5/2}(y_o-y') \; g_x(x',y') \; dS'$$

$$\beta^{(3)}_{z,y} = \frac{3}{2\pi}\frac{R_o^3 Z_o}{a^2}\int_{S'}[(x_o-x')^2 + (y_o-y')^2 + z_o^2]^{-5/2}(x_o-x') \; g_y(x',y') \; dS'$$

IV. Case of Focus at Large Distance from Aperture

Now let $R_0/a \to \infty$ where a is some characteristic dimension of the aperture. Specifically let all source fields be zero outside of some radius on S' given by some constant times a. For this purpose we have coordinates (cartesian, cylindrical, spherical) as

$$x = \Psi \cos(\phi), \qquad y = \Psi \sin(\phi)$$

$$z = r \cos(\theta), \qquad \Psi = r \sin(\theta)$$

$$x = r \sin(\theta) \cos(\phi), \qquad y = r \sin(\theta) \sin(\phi) \tag{4.1}$$

These are subscripted with 0 for the observer at $\vec{r} = \vec{r}_0$.

Considering that θ_0 and ϕ_0 are fixed as $R_0 \to \infty$, we have to order $(a/R_0)^{-1}$ for the electric coefficients in (3.10) and (3.11).

$$\alpha^{(1)}_{x,x} = \alpha^{(2)}_{x,x} = \frac{1}{2\pi} \cos(\theta_0) \frac{1}{a^2} \int_{S'} g_x(x',y') \, dS'$$

$$\alpha^{(1)}_{y,y} = \alpha^{(2)}_{x,x} = \frac{1}{2\pi} \cos(\theta_0) \frac{1}{a^2} \int_{S'} g_y(x',y') \, dS'$$

$$\alpha^{(1)}_{z,x} = \alpha^{(2)}_{z,x} = \frac{1}{2\pi} \sin(\theta_0) \cos(\phi_0) \frac{1}{a^2} \int_{S'} g_x(x',y') \, dS' \tag{4.2}$$

$$\alpha^{(1)}_{z,y} = \alpha^{(2)}_{z,x} = \frac{1}{2\pi} \sin(\theta_0) \sin(\phi_0) \frac{1}{a^2} \int_{S'} g_y(x',y') \, dS'$$

as $\frac{a}{R_0} \to 0$ to $O(\frac{a}{R_0})$

Similarly for the magnetic coefficients in (3.12) through (3.14)

$$\beta^{(1)}_{x,x} = \frac{1}{3} \beta^{(2)}_{x,x} = \frac{1}{3} \beta^{(3)}_{x,x} = \frac{-1}{2\pi} \sin^2(\theta_0) \cos(\phi_0) \sin(\phi_0) \frac{1}{a^2} \int_{S'} g_x(x',y') \, dS'$$

$$\beta^{(1)}_{x,y} = \frac{-1}{2\pi} [\cos^2(\theta_0) + \sin^2(\theta_0) \sin^2(\phi_0)] \frac{1}{a^2} \int_{S'} g_y(x',y') \, dS' \tag{4.3}$$

$$\beta^{(2)}_{x,y} = \frac{1}{2\pi} [2-3[\cos^2(\theta_o) + \sin^2(\theta_o) \sin^2(\phi_o)]] \frac{1}{a^2} \int_{S'} g_y(x',y') \, dS'$$

$$\beta^{(3)}_{x,y} = \frac{1}{2\pi} [2-3[\cos^2(\theta_o) + \sin^2(\theta_o) \sin^2(\phi_o)]] \frac{1}{a^2} \int_{S'} g_y(x',y') \, dS'$$

$$\beta^{(1)}_{y,x} = \frac{1}{2\pi} [\cos^2(\theta_o) + \sin^2(\theta_o) \cos^2(\phi_o)] \frac{1}{a^2} \int_{S'} g_x(x',y') \, dS' \qquad (4.3) \text{ (cont'd.)}$$

$$\beta^{(2)}_{y,x} = \frac{1}{2\pi} [-2+3[\cos^2(\theta_o) + \sin^2(\theta_o) \cos^2(\phi_o)]] \frac{1}{a^2} \int_{S'} g_x(x',y') \, dS'$$

$$\beta^{(3)}_{y,x} = \frac{-1}{2\pi} [-2+3 [\cos^2(\theta_o) + \sin^2(\theta_o) \cos^2(\phi_o)]] \frac{1}{a^2} \int_{S'} g_x(x',y') \, dS'$$

$$\beta^{(1)}_{z,x} = \frac{1}{3} \beta^{(2)}_{z,x} = \frac{1}{3} \beta^{(3)}_{z,x} = \frac{-1}{2\pi} \cos(\theta_o) \sin(\theta_o) \sin(\phi_o) \frac{1}{a^2} \int_{S'} g_x(x',y') \, dS'$$

$$\beta^{(1)}_{z,y} = \frac{1}{3} \beta^{(2)}_{z,y} = \frac{1}{3} \beta^{(3)}_{z,y} = \frac{1}{2\pi} \cos(\theta_o) \sin(\theta_o) \cos(\phi_o) \frac{1}{a^2} \int_{S'} g_y(x',y') \, dS' .$$

as $\frac{a}{R_o} \to 0$ to $O(\frac{a}{R_o})$

Note the common factors in the above. If we choose as a special case a uniform illumination (after focusing) of a circular aperture of radius a, with a single polarization in the x direction, we have

$$g_x(x',y') \equiv \begin{cases} 1 & \text{for } 0 \leq \Psi' < a \\ 0 & \text{for } \Psi' > a \end{cases}$$

$$g_y(x',y') \equiv 0$$

$$\frac{1}{a^2} \int_{S'} g_x(x',y') \, dS' = \pi$$

$$\frac{1}{a^2} \int_{S'} g_y(x',y') \, dS' = 0 \qquad (4.4)$$

These can be substituted in (4.2) and (4.3) to give the usual results for a uniformly illuminated circular aperture focused at ∞.

In a more general sense, let the aperture be of area A with uniform illumination g_x. Then

$$\int_{S'} g_x(x',y')\, dS' = A \tag{4.5}$$

which is a well-known result for uniformly illuminated apertures focused at infinity.

V. Circular Aperture with Uniform Tangential Electric Field Focused on Axis

Now consider the special case of a circular aperture with uniform illumination (tangential electric field)

$$g_x(x',y') \equiv \begin{cases} 1 & \text{for } 0 < \Psi' < a \\ 0 & \text{for } \Psi' > a \end{cases}$$

$$g_y(x',y') \equiv 0 \tag{5.1}$$

Furthermore, specialize the case to an observer at a distance z_0 on the z axis.

The non-zero electric-field coefficients are

$$\begin{aligned}
\alpha_{x,x}^{(1)} &= \frac{1}{2\pi}\frac{z_o^2}{a^2}\int_{S'}[x'^2 + y'^2 + z_o^2]^{-1}\, g_x(x',y')\, dS' \\
&= \frac{1}{2\pi}\frac{z_o^2}{a^2}\int_o^a\int_o^{2\pi}[\Psi'^2 + z_o^2]^{-1}\,\Psi' d\phi' d\Psi' \\
&= \frac{z_o^2}{a^2}\int_o^a[\Psi'^2 + z_o^2]^{-1}\,\Psi' d\Psi' \qquad (5.2) \\
&= \frac{z_o^2}{2a^2}\int_o^{a^2}[\nu + z_o^2]^{-1}\, d\nu
\end{aligned}$$

$$= \frac{z_o^2}{2a^2} \ln\left[1 + \frac{a^2}{z_o^2}\right] \tag{5.2 (cont'd.)}$$

$$\alpha^{(2)}_{x,x} = \frac{1}{2\pi} \frac{z_o^3}{a^2} \int_{S'} [x'^2 + y'^2 + z_o^2]^{-3/2} g_x(x',y')\, dS'$$

$$= \frac{1}{2\pi} \frac{z_o^3}{a^2} \int_0^a \int_0^{2\pi} [\Psi'^2 + z_o^2]^{-3/2}\, \Psi' d\phi' d\Psi'$$

$$= \frac{z_o^3}{a^2} \int_0^a [\Psi'^2 + z_o^2]^{-3/2}\, \Psi' d\Psi'$$

$$= \frac{z_o^3}{2a^2} \int_0^{a^2} [\nu + z_o^2]^{-3/2} d\nu$$

$$= \frac{-z_o^3}{a^2} \{[a^2 + z_o^2]^{-1/2} - z_o^{-1}\}$$

$$= \frac{z_o^2}{a^2} \{1 - [1 + \frac{a^2}{z_o^2}]^{-1/2}\} \quad .$$

Substituting these results in (3.8) we have

$$E_x(z_o \mathbf{1}_z, t) = E_o\{\frac{z_o}{2c} \ln[1 + \frac{a^2}{z_o^2}] \frac{\partial}{\partial t} f(t - \frac{z_o}{c})$$

$$+ \{1 - [1 + \frac{a^2}{z_o^2}]^{-1/2}\} f(t - \frac{z_o}{c})\} \tag{5.3}$$

which is a quite compact result valid for $z_o > 0$. This can be expanded for small a^2/z_o^2 to give

$$E_x(z_o \mathbf{1}_z, t) = E_o\{\frac{a^2}{2cz_o} [1 + O(\frac{a^2}{z_o^2})] \frac{\partial}{\partial t} f(t - \frac{z_o}{c}) \tag{5.4}$$

$$+ \frac{1}{2} \frac{a^2}{z_o^2} [1 + O(\frac{a^2}{z_o^2})] f(t - \frac{z_o}{c})\}$$

$$\text{as } \frac{a^2}{z_0^2} \to 0 \tag{5.4 cont'd.}$$

This exhibits the usual far-field results (derivative term) proportional to (aperture area)/z_0. In addition for the term proportional to the excitation waveform there is a leading term proportional to (aperture area)/z_0^2.

The non-zero magnetic-field coefficients are

$$\beta_{y,x}^{(1)} = \frac{1}{2\pi}\frac{z_0}{a^2}\int_{S'}[x'^2 + y'^2 + z_0^2]^{-3/2}[x'^2 + z_0^2]\; g_x(x',y')\; dS'$$

$$= \frac{1}{2\pi}\frac{z_0}{a^2}\int_0^a\int_0^{2\pi}[\Psi'^2 + z_0^2]^{-3/2}[\Psi'^2\cos^2(\phi') + z_0^2]\;\Psi' d\phi' d\Psi'$$

$$= \frac{1}{2}\frac{z_0}{a^2}\int_0^a[\Psi'^2 + z_0^2]^{-3/2}[\Psi'^2 + 2z_0^2]\;\Psi' d\Psi'$$

$$= \frac{1}{4}\frac{z_0}{a^2}\int_0^{a^2}[\nu + z_0^2]^{-3/2}[\nu + 2z_0^2]\; d\nu$$

$$= \frac{1}{4}\frac{z_0}{a^2}\{2\;\{[a^2 + z_0^2]^{1/2} + z_0^2\;[a^2 + z_0^2]^{-1/2} - 2z_0\}$$

$$- 4z_0^2\;\{[a^2 + z_0^2]^{-1/2} - z_0^{-1}\}\}$$

$$= \frac{1}{2}\frac{z_0^2}{a^2}\{[1 + \frac{a^2}{z_0^2}]^{1/2} - [1 + \frac{a^2}{z_0^2}]^{-1/2}\} \tag{5.5}$$

$$\beta_{y,x}^{(2)} = \frac{1}{2\pi}\frac{z_0^2}{a^2}\int_{S'}\{-2[x'^2+y'^2+z_0^2]^{-1} + 3[x'^2+y'^2+z_0^2]^{-2}[x'^2+z_0^2]\}\; g_x(x',y')\; dS'$$

$$= \frac{1}{2\pi}\frac{z_0^2}{a^2}\int_0^a\int_0^{2\pi}\{-2[\Psi'^2+z_0^2]^{-1} + 3[\Psi'^2 + z_0^2]^{-2}[\Psi'^2\cos^2(\phi')+z_0^2]\}\;\Psi' d\phi' d\Psi'$$

$$= \frac{1}{2}\frac{z_o^2}{a^2}\int_o^a \{-4[\Psi'^2+z_o^2]^{-1} + 3[\Psi'^2+z_o^2]^{-2}[\Psi'^2+2z_o^2]\}\ \Psi' d\Psi'$$

$$= \frac{1}{4}\frac{z_o^2}{a^2}\int_o^{a^2} \{-4[\nu+z_o^2]^{-1} + 3[\nu+z_o^2]^{-2}[\nu+2z_o^2]\}\ d\nu\ .$$

$$= \frac{1}{4}\frac{z_o^2}{a^2}\{-4\ell n[1 + \frac{a^2}{z_o^2}] + 3\{\ell n[1 + \frac{a^2}{z_o^2}] + z_o^2[[a^2+z_o^2]^{-1} - z_o^{-2}]\}$$

$$- 6z_o^2[[a^2 + z_o^2]^{-1} - z_o^{-2}]\}$$

$$= \frac{z_o^2}{a^2}\{\frac{-1}{4}\ell n[1 + \frac{a^2}{z_o^2}] - \frac{3}{4}[[1 + \frac{a^2}{z_o^2}]^{-1} - 1]\} \qquad (5.5)\ \text{(cont'd.)}$$

$$\beta_{y,x}^{(3)} = \frac{1}{2\pi}\frac{z_o^3}{a^2}\int_{S'}\{-2[x'^2+y'^2+z_o^2]^{-3/2}+3[x'^2+y'^2+z_o^2]^{-5/2}[x'^2+z_o^2]\}g_x(x',y')\ dS'$$

$$= \frac{1}{2\pi}\frac{z_o^3}{a^2}\int_o^a\int_o^{2\pi}\{-2[\Psi'^2+z_o^2]^{-3/2}+3[\Psi'^2+z_o^2]^{-5/2}[\Psi'^2\cos^2(\phi')+z_o^2]\}\ \Psi' d\phi' d\Psi'$$

$$= \frac{1}{2}\frac{z_o^3}{a^2}\int_o^a\{-4[\Psi'^2 + z_o^2]^{-3/2}+ 3[\Psi'^2 + z_o^2]^{-5/2}[\Psi'^2 + 2z_o^2]\}\ \Psi' d\Psi'$$

$$= \frac{1}{4}\frac{z_o^3}{a^2}\int_o^{a^2}\{-4[\nu + z_o^2]^{-3/2}+ 3[\nu + z_o^2]^{-5/2}[\nu + 2z_o^2]\}\ d\nu$$

$$= \frac{1}{4}\frac{z_o^3}{a^2}\{8\{[a^2+z_o^2]^{-1/2}-z_o^{-1}\}+6\{-[a^2+z_o^2]^{-1/2}+ z_o^{-1}+ \frac{z_o^2}{3}[a^2+z_o^2]^{-3/2}- \frac{1}{3}z_o^{-1}\}$$

$$- 4z_o^2\{[a^2+z_o^2]^{-3/2} - z_o^{-3}\}\}$$

$$= \frac{z_o^2}{a^2}\{\frac{1}{2}[1 + \frac{a^2}{z_o^2}]^{-1/2}- \frac{1}{2}[1 + \frac{a^2}{z_o^2}]^{-3/2}\}$$

Substituting these results in (3.9) we have

$$Z_0 H_y(z_0 \vec{1}_z, t) = E_0 \{ \frac{1}{2} \frac{z_0}{c} \{ [1 + \frac{a^2}{z_0^2}]^{1/2} - [1 + \frac{a^2}{z_0^2}]^{-1/2} \} \frac{\partial}{\partial t} f(t - \frac{z_0}{c})$$

$$+ \{ \frac{-1}{4} \ell n \, [1 + \frac{a^2}{z_0^2}] - \frac{3}{4} [[1 + \frac{a^2}{z_0^2}]^{-1} - 1] \} f(t - \frac{z_0}{c})$$

$$+ \frac{c}{z_0} \{ \frac{1}{2} [1 + \frac{a^2}{z_0^2}]^{-1/2} - \frac{1}{2} [1 + \frac{a^2}{z_0^2}]^{-3/2} \} \int_{-\infty}^{t} f(t' - \frac{z_0}{c}) \, dt' \} \qquad (5.6)$$

which again is valid for $z_0 > 0$. This can be expanded for small a^2/z_0^2 to give

$$Z_0 H_y(z_0 \vec{1}_z, t) = E_0 \{ \frac{a^2}{2 c z_0} [1 + O(\frac{a^2}{z_0^2})] \frac{\partial}{\partial t} f(t - \frac{z_0}{c})$$

$$+ \frac{1}{2} \frac{a^2}{z_0^2} [1 + O(\frac{a^2}{z_0^2})] f(t - \frac{z_0}{c})$$

$$+ \frac{1}{2} \frac{c a^2}{z_0^3} [1 + O(\frac{a^2}{z_0^2})] \int_{-\infty}^{t} f(t' - \frac{z_0}{c}) \, dt' \}$$

$$\text{as } \frac{a^2}{z_0^2} \to 0 \qquad (5.7)$$

Note in the far field the leading terms (going as z_0^{-1}) agree for electric and magnetic fields. As one approaches the aperture so that a/z_0 becomes not small compared to 1, even the derivative terms for electric and magnetic fields have different coefficients (i.e., not related as in a plane wave).

VI. Summary

The case of Section 5 is not the only specific aperture distribution one could consider. Considering a focused aperture other distributions of aperture fields can be investigated and the coefficients in Section 3 determined. Perhaps some other interesting distributions will give closed-form results as in Section 5.

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