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A Family of Canonical Examples for High Frequency Propagation on Unit Cell of Wave-Launcher Array

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Abstract

The available analytic solutions for a periodic array of symmetric in-line wave launchers have been used in characterizing the high-frequency propagation along the launchers. These analytic solutions are based on a high-frequency asymptotic formulation of the multiconductor transmission-line equations. In particular, a family of curves that permit a desired high-frequency voltage transfer ratio from the input at the apex to the aperture plane are computed and plotted.

symmetric transmission lines

Preface

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Contents

1. Introduction

Recent papers [1, 2 and 3] have considered the high-frequency (or equivalently early-time) propagation of a step pulse source at the apex, along the unit cell of a symmetric in-line, periodic array of wave launchers. Starting from multi-conductor transmission line equations specialized to a two-conductor (plus reference) model for the unit cell, [3] derives an expression for the voltage transfer ratio from the source to the aperture plane in the high frequency limit. It is noted that, the transmission line equations are strictly valid when the cross sectional dimensions of the unit cell are small and the launcher length is large compared with the shortest significant wavelength of interest. This results in the requirement that the launcher length be large compared to its cross sectional dimensions. Under the stated assumptions, the asymptotic transmission-line theory is useful in estimating the early-time propagation across the launchers. Of course, at late times (or equivalently low frequency propagation), the voltage transfer ratio from the apex to aperture plane tends to unity. The present interest however is in the high-frequency propagation which is a strong function of the characteristic impedance matrix (2x2) elements. It was observed in earlier papers [1 and 2], that three of the four elements of this matrix are explicitly known in terms of the cross-sectional dimensions for an incremental length of the launcher and the single unknown element can be numerically evaluated [2] for a given set of launcher cross sectional geometry. Conversely, a given impedance profile (within practical realizability constraints) can be realized by evaluating the geometry that produces it.

Denoting the normalized characteristic impedance matrix (2x2) by $(F_{n,m}(\zeta))$, we observe that $F_{1,2} = F_{2,1}$ could be a linear function of the normalized length coordinate ζ, which varies from 0 to 1 as one moves from the apex to aperture planes. $F_{2,2}$ has been shown to be a constant [1 and 2]. Under these conditions, [3] has analytically considered two special cases of constant and quadratically varying situations for $F_{1,1}$ in deriving the voltage transfer ratios in the limit of high frequencies. Under the same conditions for the off-diagonal and the(2,2) element, $F_{1,1}(\zeta)$ is allowed to vary linearly in ζ, and the voltage transfer ratio is derived analytically in this paper. In addition, a family of impedance profiles for $\nu(\zeta)=F_{1,1}$ are treated numerically. The usefulness of the results reported in this paper lies in the ability to pre-select a launcher geometry that produces a desired high-frequency propagation characteristics.

2. High-Frequency Voltage Transfer Characteristics Across the Unit Cell of a Periodic Array of Wave Launchers

Baum [3] has derived an expression for the high-frequency voltage transfer ratio $T(\zeta)$ in general, and $T(1)$ in particular gives the hi-frequency voltage transfer from the source at the apex to the aperture plane of the unit cell of a periodic array of wave launchers.

$$T(1) = \left[\frac{V_1(1)}{V_0}\right] = \cos\left(g(1) + \pi/4\right)/\sqrt{\alpha} \tag{1}$$

where

$$\begin{aligned}
\zeta &\equiv \text{normalized launcher length coordinate } (0 \le \zeta \le 1) \\
V_1(1) = V_2(1) &\equiv \text{voltage at } \zeta = 1 \text{ (aperture plane) in the limit of high frequencies} \\
V_0 u(t) &\equiv \text{step function input at the apex} \\
\alpha &\equiv \text{initial value of } F_{1,1} \text{ (normalized characteristic impedance matrix element)} \\
F_{1,1}(\zeta) &\equiv \nu(\zeta) \\
\nu(\zeta) &= Z_{1,1}(\zeta)/[Z_0(b/a)] \quad ; \quad \alpha \le \nu(\zeta) \le 1 \quad \text{for} \quad 0 \le \zeta \le 1 \\
Z_{1,1} &\equiv \text{(1,1) element of the characteristic impedance matrix} \\
2b &\equiv \text{plate separation in the aperture plane in the E-field direction} \\
2a &\equiv \text{plate width in the aperture plane in the H-field direction}
\end{aligned} \tag{2}$$

$$g(1) = \int_0^1 h(\zeta)\, d\zeta \tag{3}$$

$$h(\zeta) = \frac{1}{2}\left[(1-\nu)^2 + 4\zeta^2\right]^{-1}[\nu-\zeta^2]^{-1/2}\left(1 - \nu + \zeta\frac{d\nu}{d\zeta}\right)\left(1 + \nu - 2\sqrt{\nu-\zeta^2}\right) \tag{4}$$

$$\begin{aligned}
\nu &\equiv \nu(\zeta) \quad \text{with} \quad 0 \le \zeta \le 1 \\
\alpha &= \nu(0) \quad \text{and} \quad \nu(1) = 1
\end{aligned} \tag{5}$$

From the above set of equations, it is evident that T(1) is in principle, readily computable for a given impedance profile $\nu(\zeta)$. Consistent with previous work [1 and 3], one can use an empirical form for $\nu(\zeta)$ such as

$$\nu(\zeta) = \alpha + (1-\alpha)\zeta^n \quad ; \quad (n \geq 0) \tag{6}$$

and vary the initial value α and the exponent n. Consequently, we can think of the voltage transfer ratio T to be a function of three variables and denote it by $T(\zeta, \alpha, n)$. Our interest is in the voltage or fields in the aperture plane ($\zeta = 1$) so that, we seek $T(1, \alpha, n)$. Observe that if $\alpha = 1$, ν becomes 1 for any n, corresponding to a constant impedance profile across the launcher. Another example is when n = 2, the impedance profile starts from α and quadratically increases to 1 in the aperture plane. These two examples of constant and quadratic variation were considered in [3] and the results can be summarized as follows

$$\begin{aligned} T(1, 1, n) &= (1/\sqrt{2}) \\ T(1, \alpha, 0) &= (1/\sqrt{2\alpha}) \qquad \text{(asymptotic hybrid case[3];limit of } n \to 0^+) \\ T(1, \alpha, 2) &= \frac{1}{\sqrt{\alpha}} \cos\left(\frac{\pi}{4\sqrt{\alpha}}\right) \end{aligned} \tag{7}$$

recalling that the three arguments of the voltage transfer ratio at high frequencies are indicated in $T(\zeta, \alpha, n)$. The result for n = 0 in (7) comes from observing that g(1) = 0 when n = 0 and $T(1, \alpha, 0)$ is simply $[\cos(\pi/4)/\sqrt{\alpha}]$.

In the next section, we consider yet another special case of linear variation in impedance profile given by $\nu(\zeta) = \alpha + (1-\alpha)\zeta$ or n=1.

3. Family of Canonical Examples of Wave Launchers

A. Special Case of n = 1

When the exponent n equals 1 in (6), we have

$$\nu(\zeta) = \alpha + (1-\alpha)\zeta = \alpha + \beta\zeta \quad \text{with} \quad \beta = (1-\alpha) \tag{8}$$

For such a linear impedance profile, we have the terms in (4) given by

$$\begin{aligned}\left[(1-\nu)^2 + 4\zeta^2\right] &= (1-\alpha-\beta\zeta)^2 + 4\zeta^2 = \beta^2 - 2\beta^2\zeta + (\beta^2+4)\zeta^2 \\ &= (\beta^2+4)(\zeta+p)(\zeta+q)\end{aligned} \tag{9}$$

with

$$p = -\beta/(\beta-i2) \quad \text{and} \quad q = -\beta/(\beta+i2) = p^* \tag{10}$$

$$(\nu-\zeta^2) = \alpha + \beta\zeta - \zeta^2 = (\zeta+\alpha)(1-\zeta) \tag{11}$$

$$\left[1 - \nu + \zeta\frac{d\nu}{d\zeta}\right] = (1-\alpha) = \beta \tag{12}$$

Combining we have from (4),

$$\begin{aligned}h(\zeta) &= \frac{\beta}{2(\beta^2+4)}\left[\frac{(1+\alpha) + \beta\zeta - 2\sqrt{\alpha + \beta\zeta - \zeta^2}}{(\zeta+p)(\zeta+q)\sqrt{\alpha + \beta\zeta - \zeta^2}}\right] \\ &= \frac{\beta}{2(\beta^2+4)}\left[h_1(\zeta) - h_2(\zeta)\right]\end{aligned} \tag{13}$$

where

$$h_1(\zeta) \equiv \left[\frac{(1+\alpha) + \beta\zeta}{(\zeta+p)(\zeta+q)\sqrt{\alpha + \beta\zeta - \zeta^2}}\right] \tag{14}$$

$$h_2(\zeta) \equiv \left[\frac{2}{(\zeta+p)(\zeta+q)}\right] \tag{15}$$

what we require is

$$g(1) = \int_0^1 h(\zeta)\, d\zeta = \left[\frac{\beta}{2(\beta^2+4)}\right](g_1 - g_2) \tag{16}$$

$$g_1 \equiv \int_0^1 h_1(\zeta)\, d\zeta \quad \text{and} \quad g_2 \equiv \int_0^1 h_2(\zeta)\, d\zeta \tag{17}$$

Consider g_2 for the present,

$$g_2 = \int_0^1 \frac{2}{(\zeta+p)(\zeta+q)}\, d\zeta$$

$$= \frac{2}{(q-p)}\, \ell n\left(\frac{\zeta+p}{\zeta+q}\right)\Bigg|_0^1 = \frac{(\beta^2+4)}{2i\beta}\left[\ell n\left(\frac{p+1}{q+1}\right) - \ell n\left(\frac{p}{q}\right)\right]$$

$$= \frac{(\beta^2+4)}{2i\beta}\left[\ell n\left(\frac{q(p+1)}{p(q+1)}\right)\right] = \frac{1}{2i\beta}(\beta^2+4)[\ell n(-1)]$$

$$= \frac{(\beta^2+4)}{2i\beta}\left[\ell n(e^{i\pi})\right] = \left[\frac{(\beta^2+4)\pi}{2\beta}\right] \tag{18}$$

Substituting for g_2 in (16), we get

$$g(1) = \left[\frac{\beta}{2(\beta^2+4)}\, g_1\right] - \frac{\pi}{4}$$

$$\left(g(1) + \frac{\pi}{4}\right) = \left[\frac{\beta}{2(\beta^2+4)}\quad g_1\right] \tag{19}$$

$$g_1 = \int_0^1 \left[\frac{(1+\alpha) + \beta\zeta}{(\zeta+p)(\zeta+q)\sqrt{\alpha + \beta\zeta - \zeta^2}}\right] d\zeta$$

Using a tabulated integral (2.282-5 in [4]), we have

$$g_1 = \left[\frac{(1+\alpha)-p\beta}{q-p}\right]\int_0^1 \frac{d\zeta}{(\zeta+p)\sqrt{(\zeta+\alpha)(1-\alpha)}} + \left[\frac{(1+\alpha)-q\beta}{p-q}\right]\int_0^1 \frac{d\zeta}{(\zeta+q)\sqrt{(\zeta+\alpha)(1-\zeta)}} \tag{20}$$

Denoting the two integrals in the above by I_p and I_q respectively, we have

$$g_1 = \left[\frac{(\beta^2+4)}{i4\beta}\,[(1+\alpha) - p\beta]\; I_p\right] - \left[\frac{(\beta^2+4)}{i4\beta}\,[(1+\alpha) - q\beta]\; I_q\right] \tag{21}$$

Note that since p is the complex conjugate of q (see (10)), the two terms in (21) are also conjugates of each other resulting in

$$g_1 = \frac{(\beta^2+4)}{2\beta}\, \mathrm{Im}\left[\left[(1+\alpha) - p\beta\right] I_p\right] \tag{22}$$

Using the above in (19), we have

$$\left(g(1)+\frac{\pi}{4}\right) = \frac{1}{4}\,\mathrm{Im}\left[\left[(1+\alpha) - p(1-\alpha)\right] I_p\right] \tag{23}$$

It now remains to evaluate I_p, which is given by

$$I_p = \int_0^1 \frac{d\zeta}{(\zeta+p)\sqrt{(\zeta+\alpha)(1-\zeta)}} \tag{24}$$

Once again, using a tabulated integral [5] as noted below,

$$\int \frac{dx}{(x-a)\sqrt{(x-b)(c-x)}} = \frac{2}{\sqrt{(b-a)(c-a)}}\arctan\left(\sqrt{\frac{(x-b)(c-a)}{(c-x)(b-a)}}\right) \tag{25}$$

we have

$$I_p = \frac{2}{\sqrt{(p-\alpha)(p+1)}}\left[\arctan\left(\sqrt{\frac{(\zeta+\alpha)(p+1)}{(1-\zeta)(p-\alpha)}}\right)\right]\Bigg|_0^1$$

$$= \left[\frac{2}{\sqrt{(p-\alpha)(p+1)}}\arctan\left(\sqrt{\frac{p-\alpha}{\alpha(p+1)}}\right)\right] \tag{26}$$

Using (26) in (23), we have

$$\left[g(1)+\frac{\pi}{4}\right] = \frac{1}{2}\,\mathrm{Im}\left[\frac{(1+\alpha)-p(1-\alpha)}{\sqrt{(p-\alpha)(p+1)}}\arctan\left(\sqrt{\frac{p-\alpha}{\alpha(p+1)}}\right)\right] \tag{27}$$

By substituting for p from (10) in the factor in front of the arctan( ) in the above and after some algebraic manipulations, we find

$$\left[g(1)+\frac{\pi}{4}\right] = \frac{1}{2}\,\mathrm{Im}\left[2i\arctan\left(\sqrt{\frac{p-\alpha}{\alpha(p+1)}}\right)\right]$$

$$= \mathrm{Re}\left[\arctan\left(\sqrt{\frac{p-\alpha}{\alpha(p+1)}}\right)\right] \tag{28}$$

Substituting for p from (10) into the above, we find

$$\left[g(1) + \frac{\pi}{4}\right] = \mathrm{Re}\left[\arctan\left(-\frac{(\alpha+i)(1+i)}{2\sqrt{\alpha}}\right)\right]$$

$$= \mathrm{Re}\left[\arctan\left(\frac{(1-\alpha) - i\,(1+\alpha)}{2\sqrt{\alpha}}\right)\right] = \frac{1}{2}\arctan\left(\frac{2\sqrt{\alpha}}{(\alpha-1)}\right) \qquad (29)$$

In writing (29) use is made of the identity

$$\mathrm{Re}\left[\arctan(x+iy)\right] = (1/2)\arctan\left(\frac{2x}{1-x^2-y^2}\right) \qquad (30)$$

Continuing the simplification,

$$\left[g(1) + \frac{\pi}{4}\right] = (1/2)\arccos\left(\frac{1}{1+\left[\frac{2\sqrt{\alpha}}{\alpha-1}\right]^2}\right)^{1/2}$$

$$= (1/2)\arccos\left(\frac{(\alpha-1)^2}{(\alpha+1)^2}\right)^{1/2} = (1/2)\arccos\left(\frac{\alpha-1}{\alpha+1}\right) \qquad (31)$$

Use has been made of the identity

$$\arctan(x) = \arccos\left(\frac{1}{1+x^2}\right)^{1/2} \qquad (32)$$

Taking cosine of both sides of (31) we have

$$\cos\left(g(1) + (\pi/4)\right) = \cos\left[\frac{1}{2}\arccos\left(\frac{\alpha-1}{1+\alpha}\right)\right] = \left[\frac{1}{2} + \frac{1}{2}\frac{(\alpha-1)}{(1+\alpha)}\right]^{1/2}$$

$$= [\alpha/(\alpha+1)]^{1/2} \qquad (33)$$

$$T(1, \alpha, 1) = \cos\left(g(1) + (\pi/4)\right)/\sqrt{\alpha} = (\alpha + 1)^{-1/2} \qquad (34)$$

which indeed is a compact expression. Summarizing the analytical results for $T(\zeta, \alpha, n)$, we have

$$\begin{aligned} T(1, 1, n) &= 1/\sqrt{2} \\ T(1, \alpha, 0) &= 1/(\sqrt{2\alpha}) \\ T(1, \alpha, 1) &= 1/(\sqrt{\alpha+1}) \\ T(1, \alpha, 2) &= \frac{1}{\sqrt{\alpha}}\cos\left(\frac{\pi}{4\sqrt{\alpha}}\right) \end{aligned} \qquad (35)$$

For the cases of n = 1 and n = 2, g(1) was evaluated numerically by employing a Simpson's rule for integration and the last two results in (34) were evaluated numerically as well. The analytical and numerical results for the four cases of (34) are reported in Table 1. The analytical results of (34) are also plotted in figure 1.

It is noted that n = 0 and $\alpha = 0.5$ leads to a high-frequency voltage transfer ratio of 1. Baum [3] had earlier considered a "hybrid" launcher which is made up of two sections. The first section has a transformer action resulting in a voltage of $V_0 \sqrt{2}$ and the second section with a constant impedance has a transfer ratio of $1/\sqrt{2}$, resulting in a net ratio of 1 from source to aperture plane. The relative lengths of the two sections is an open issue. In the context of such a "hybrid" launcher, it is observed that the n = 0 case considered here is the asymptotic limit of the hybrid case. For example, starting with an $\alpha = 0.5$, and n = 0.1 say, the impedance ν (i.e., normalized $z_{1,1}$) will start from 0.5 and rapidly rise and asymptotically go to 1 as ζ tends to 1. Such a launcher array has the desired high-frequency propagation characteristics.

B. General Case of Varying n

We are now in a position to compute $T(1, \alpha, n)$ for a parametric set of values. The high-frequency voltage transfer ratio T is computed numerically, tabulated (see Table 2) and plotted as a function of α ranging from 0 to 1, for different values of n in the range of 0 to 2, in figure 2. In this figure T = 1 line leads one to pick out pairs of values of α and n for which the high-frequency voltage transfer ratio is nominally 1.

Such values of α and n for which $T \simeq 1$, are shown plotted in figure 3 and it is observed that such pairs of α and n form a smooth curve, while resulting in a T of nominally 1. It is also evident that $\alpha = 0.5$ and n = 0 is a desirable set of values from purely high-frequency propagation characteristics.

C. Realizing a Particular Impedance Profile $\nu(\zeta)$

In [2], normalized impedance $\nu(\zeta)$, therein called $F_{1,1}$ was evaluated for a given launcher geometry. Conversely one can determine the geometry of the launchers to attain a prescribed impedance profile using the nomographs in [2], within certain practical constraints. We have thus established that certain optimal launcher geometries exist and they are computable to accomplish desired results at high frequencies, propagating along the launchers.

In concluding this section, it is observed that propagation of intermediate times or frequencies are not addressed yet. Consequently, actual computations of launcher geometries and their eventual optimization should be performed after

$\alpha = 1$; $\nu = 1$

n	$T(1,1,n)$ (numerical)
0.100	0.707107
0.200	0.707107
0.333	0.707107
0.500	0.707107
0.600	0.707107
0.750	0.707107
0.800	0.707107
1.000	0.707107
1.250	0.707107
1.500	0.707107
2.000	0.707107

Asymptotic hybrid
$n \to 0^+$; $\nu = \alpha + (1-\alpha)\zeta^n$; see [3]

α	$T(1,\alpha,0) = 1/(\sqrt{2\alpha})$
0.0	infinity
0.1	2.236
0.2	1.581
0.3	1.291
0.4	1.118
0.5	1.000
0.6	0.913
0.7	0.845
0.8	0.791
0.9	0.745
1.0	0.707

$n = 1$; $\nu = \alpha + (1-\alpha)\zeta$

α	$T(1,\alpha,1)$	
	numerical	analytical
0.0	-	1.000
0.1	0.965	0.953
0.2	0.919	0.912
0.3	0.881	0.877
0.4	0.848	0.845
0.5	0.818	0.816
0.6	0.792	0.790
0.7	0.767	0.766
0.8	0.745	0.745
0.9	0.725	0.725
1.0	0.707	0.707

$n = 2$; $\nu = \alpha + (1-\alpha)\zeta^2$

α	$T(1,\alpha,2)$	
	numerical	analytical
0.0	-	- infinity
0.1	-2.464	-2.502
0.2	-0.385	-0.412
0.3	0.264	0.250
0.4	0.520	0.510
0.5	0.633	0.628
0.6	0.686	0.683
0.7	0.708	0.706
0.8	0.715	0.714
0.9	0.713	0.713
1.0	0.707	0.707

Table 1. Some Special Cases of $T(\zeta,\alpha,n)$

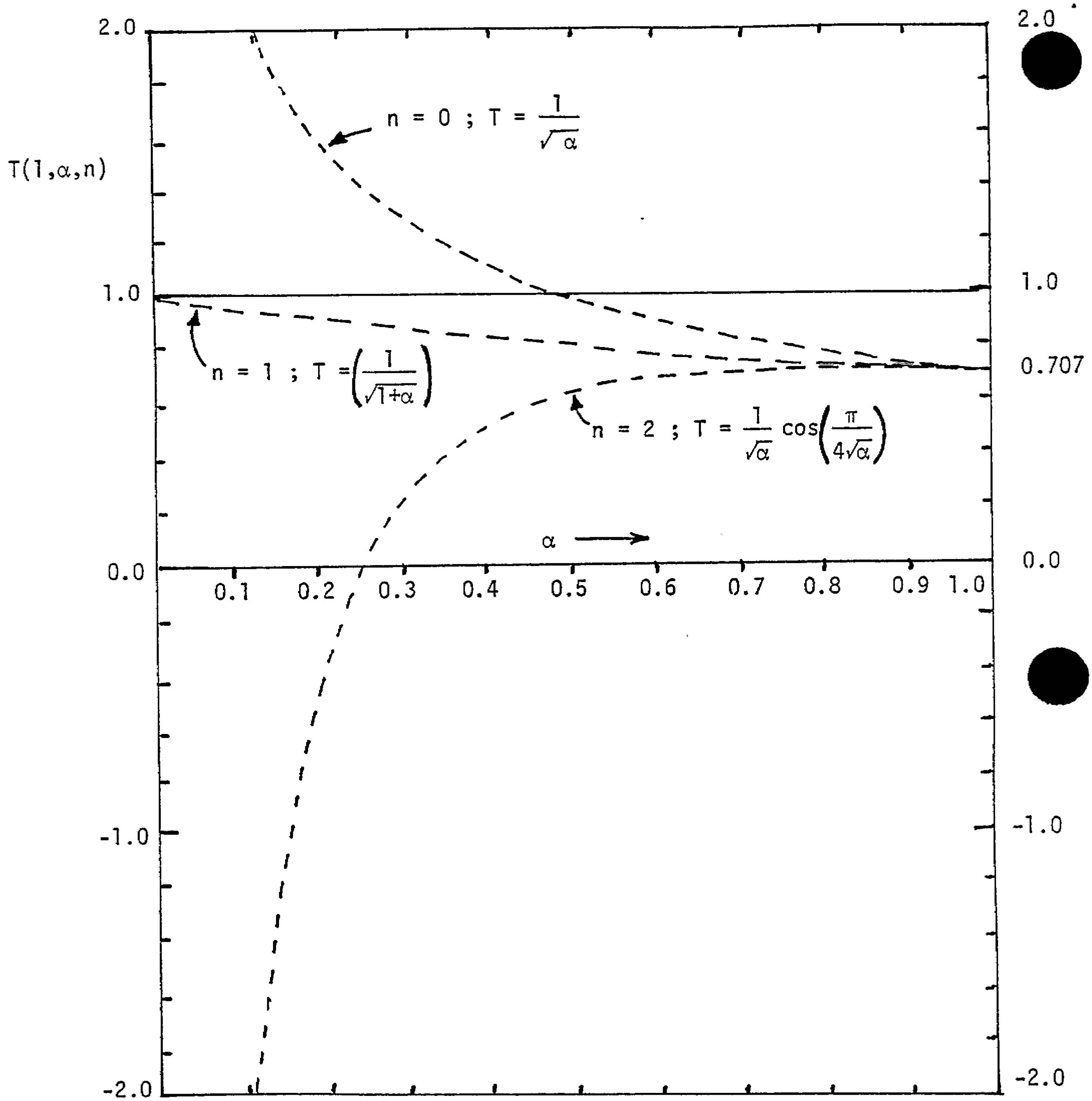


Figure 1. Three special cases of impedance profiles $\nu(\zeta)$ for which the high-frequency voltage transfer ratios are analytically derived.

TABLE 2. High-Frequency Voltage Transfer Ratio T(1,α,n)

α \ n	0*	0.1	0.2	0.333	0.5	0.6	0.75	0.8	1.0*	1.25	1.5	1.75	2.0*
0	+ ∞	-	-	-	-	-	-	-	1.000	-	-	-	-
0.1	2.236	2.153	2.061	1.922	1.725	1.594	1.379	1.302	0.953	0.460	-0.187	-1.087	-2.464
0.2	1.581	1.529	1.474	1.395	1.289	1.221	1.114	1.072	0.912	0.697	0.430	0.090	-0.385
0.3	1.291	1.254	1.217	1.165	1.097	1.056	0.992	0.971	0.877	0.760	0.624	0.464	0.264
0.4	1.118	1.091	1.064	1.028	0.983	0.957	0.916	0.903	0.845	0.777	0.701	0.616	0.520
0.5	1.000	0.980	0.961	0.936	0.905	0.888	0.861	0.853	0.816	0.775	0.731	0.684	0.633
0.6	0.913	0.898	0.885	0.868	0.847	0.836	0.819	0.813	0.790	0.765	0.739	0.713	0.686
0.7	0.845	0.835	0.826	0.815	0.802	0.794	0.784	0.781	0.766	0.752	0.737	0.722	0.708
0.8	0.791	0.784	0.779	0.772	0.765	0.760	0.754	0.753	0.745	0.737	0.729	0.722	0.715
0.9	0.745	0.742	0.740	0.737	0.733	0.732	0.729	0.728	0.725	0.722	0.719	0.716	0.713
1.0	0.707	0.707	0.707	0.707	0.707	0.707	0.707	0.707	0.707	0.707	0.707	0.707	0.707

*analytical expressions in closed form are available for these three cases only

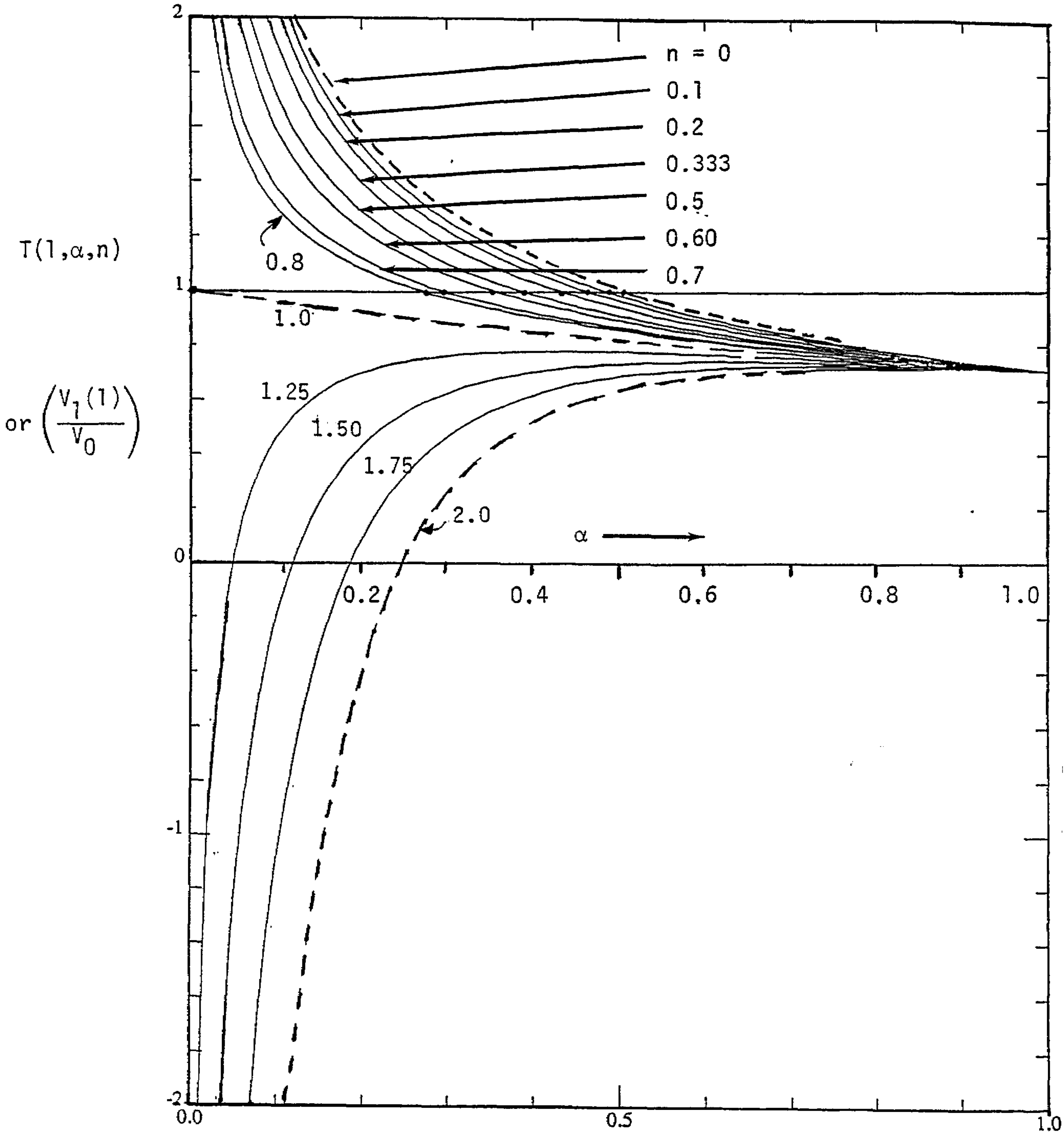
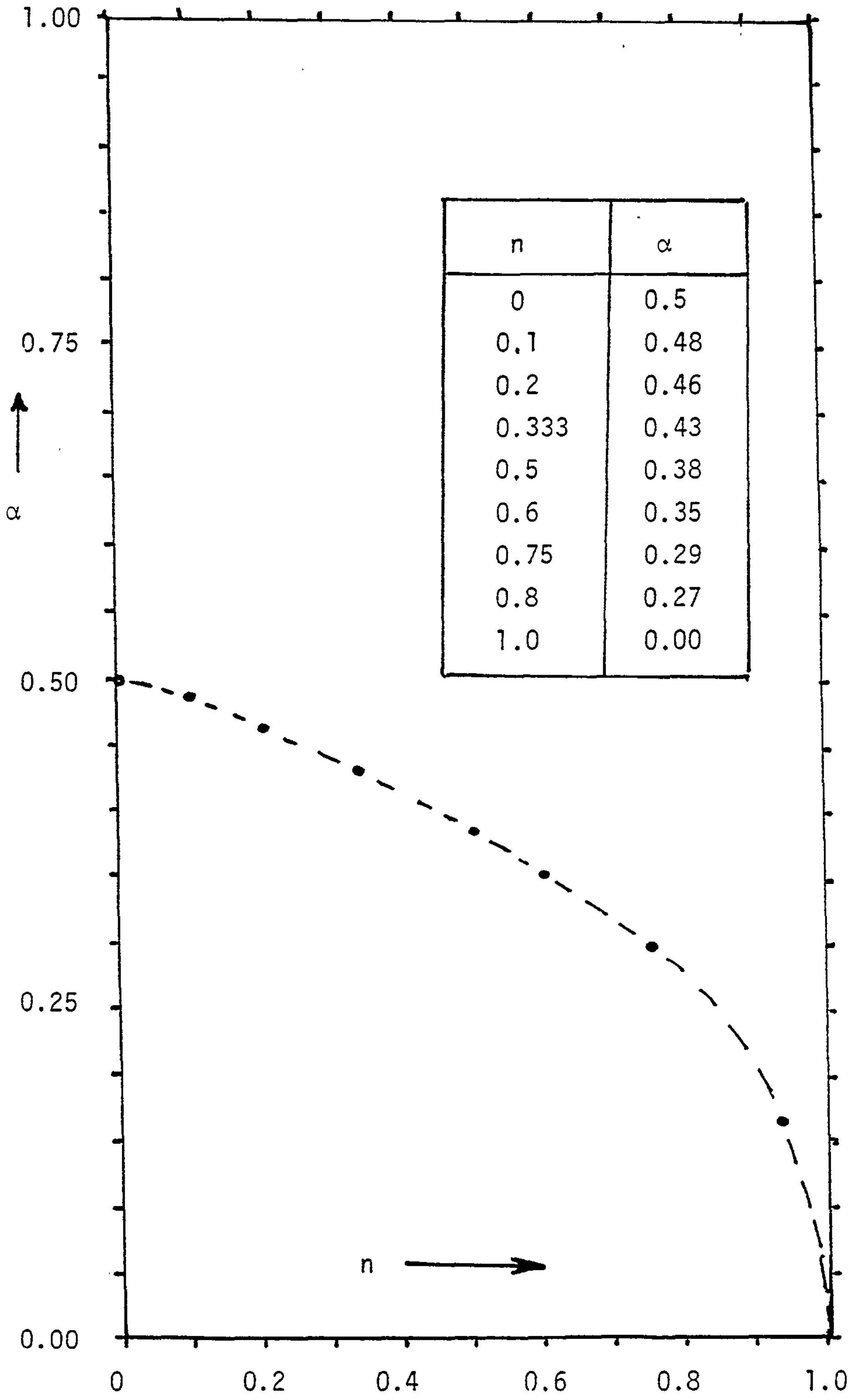


Figure 2. Family of high-frequency voltage transfer ratios as a function of α for varying n.
Note: n = 0, 1 and 2 plotted in broken lines have been analytically derived (here and [3]).



n	α
0	0.5
0.1	0.48
0.2	0.46
0.333	0.43
0.5	0.38
0.6	0.35
0.75	0.29
0.8	0.27
1.0	0.00

Figure 3. Pairs of values of (n and α) that result in $(V_1(1)/V_0) \equiv T \simeq 1$.

the intermediate times propagation characteristics are analyzed and understood. Such an analysis valid for all time regimes can be performed by numerically solvi the Telegrapher's equations for voltages and currents in the time domain. The two conductor (plus reference) model is available and one can consider a variety of impedance profiles and compute the aperture voltage ($V_1(1)$ and $V_2(1)$) for an idealized unit step input in the time domain. This analysis will be the subject of future studies.

4. Summary

This paper has extended the earlier work of Baum [3] which considered the high-frequency propagation characteristics across the unit cell of a periodic array of wave launchers. A special case of linear variation of the launcher impedance as a function of the length coordinate along the launcher is analyzed for its high-frequency propagation characteristics. In addition, a family of curves, indicative of the high-frequency voltage transfer characteristics from the source to the aperture plane are computed numerically, tabulated and plotted. Such results are useful in designing launcher geometries that are optimal from purely high-frequency considerating. Additional analysis is required to evaluate the propagation characteristics in the intermediate times (or equivalently mid-frequency regime).

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