Sensor and Simulation Notes

Note 335

December 12, 1991

A Prolate Spheroidal Uniform Isotropic Dielectric Lens Feeding A Circular Coax

Carl E. Baum

and

Joseph J. Sadler

Phillips Laboratory

and

Alexander P. Stone

Phillips Laboratory and

University of New Mexico

Abstract

In launching the TEM mode on a coaxial circular cylindrical transmission line at high frequences, one can use coaxial circular cones as a wave launcher. Matching the conical waveguide to the cylindrical waveguide, in this paper one makes the two characteristic impedances equal, and in the usual lens sense makes rays on the conical structure travel with equal time to an aperture plane perpendicular to the axis of the system. To accomplish this the conical region is filled with a uniform, isotropic, dielectric with frequency-independent dielectric constant (lossless and dispersionless). While the lens is not perfect in that there are small reflections at the lens surface, the high-frequency performance can be quite good for a large range of lens parameters.

I Introduction

EM lenses for transitioning TEM waves between certain types of transmission lines, without distortion or reflection, may be developed (as in [12]) through an analysis of exact solutions to EM boundary value problems. The important physical concepts are the matching of differential impedances along each ray, including the constraint of equal transit-times for all rays through the lens, (between appropriate positions and/or surfaces). In [12] a number of examples are given. These examples include inhomogeneous lenses, converging, diverging, and bending, which may be used to transition TEM waves between conical and/or cylindrical transmission lines. These examples represent exact solutions to the wave equation, though there could be approximations involved in practical applications.

In this paper we begin the study of a class of approximate lens designs in which the equal time requirement is maintained, but the impedance condition is relaxed from a differential one (along every ray) to a global one in which the transmission-line impedance is maintained constant on both sides of the lens as well as inside the lens. This is an average kind of impedance matching, allowing, some mismatch (preferably small) along ray paths (particularly at lens boundaries). Though a specific geometry is considered here, our solution technique will be applicable to a large class of problems.

We begin in Section II by considering a concentric circular conical uniform isotropic dielectric lens feeding a circular coaxial line. The lens (transition) region is to be specified so that a wave launched at an apex propagates through the lens region onto the coaxial line with minimum reflection at the boundary between the regions and with minimum distortion. We find that the lens shape for the equal-transit-time condition to be satisfied is a prolate spheriod. Moreover, the lens shape is only a function of the relative permittivity, ϵ_r, for high-frequency/early-time performance.

In Section III, a macroscopic impedance matching condition, allowing a uniform isotrpic lens, is imposed. On the two transmission lines perfect impedence matching along each ray (as in [12]) is a local requirement which has been relaxed here. The lens shape is thereby determined from the chosen relative permittivity of the lens region and the characteristic impedance of the coaxial line.

In section IV, the transmission of waves at the lens boundary in the high frequency limit is investigated. Snell's law is the governing relation and transmission and reflection coefficients are obtained. The special condition under which no reflection occurs at the lens boundary is the Brewster angle, which is determined by the relative permittivity, ϵ_r.

The fields which exist between the coaxial circular cones, and hence on the aperture plane, lead to an expression for a high frequency transfer function, T_V, which appears in Section VI. This function compares an initial voltage associated with the TEM mode on the coax, to a voltage on the conical section, and it is calculated as a function of ϵ_r and the coax impedance. Appendix A gives an asymptotic expansion of T_V for small characteristic impedances. Appendix B contains the tabular data for the various lens parameters as functions of relative permittivity ϵ_r and coax characteristic impedance Z_c (with a free-space medium).

II Lens Shape For Equal Transit Time to Aperture Plane: Prolate Spheroidal Lens

As in Fig. 2.1 consider a spherical TEM wave propagating radially outward from an apex at

$$(x, y, z) = (0, 0, -\ell) \tag{2.1}$$

We have coordinates (Ψ, ϕ, z) for the cylindrical transmission line

$$x = \Psi \cos(\phi), \quad y = \Psi \sin(\phi) \tag{2.2}$$

where $z = 0$ defines an aperture plane, the plane at which all the rays from the apex are to arrive simultaneously (equal time requirement). We have spherical coordinates (r, θ, ϕ) for the conical transmission line, centered on the apex, as

$$z + \ell = r \cos(\theta), \quad \Psi = r \sin(\theta) \tag{2.3}$$

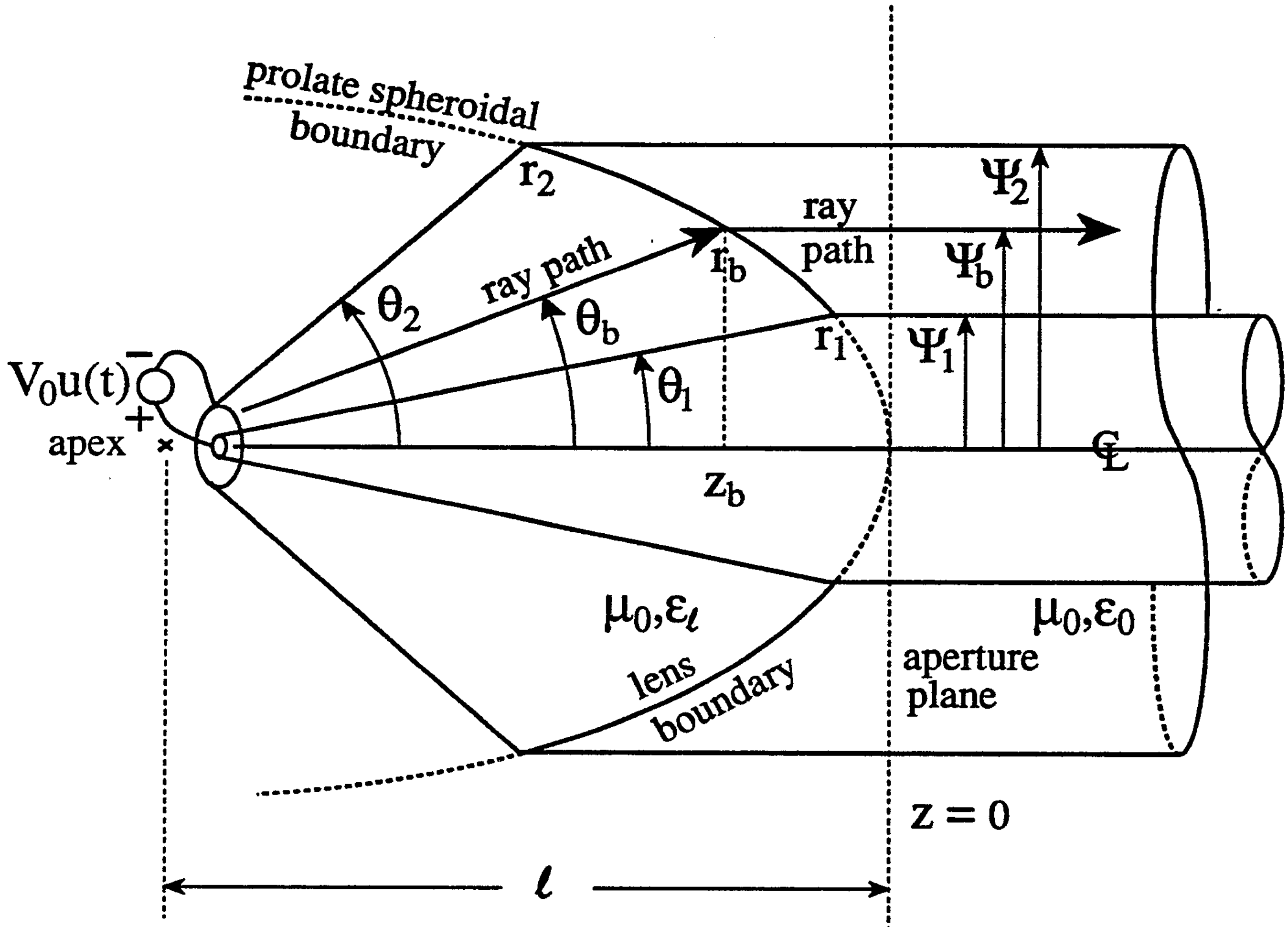
The lens boundary uses the above coordinates with a subscript b.

The ray path in Fig. 2.1 follows a radial line in the lens (spherical wavefront) with parameters

$$\begin{aligned} \mu_0 &\equiv \text{permeability} \\ \epsilon_\ell &\equiv \text{permittivity} \\ \epsilon_r &\equiv \frac{\epsilon_\ell}{\epsilon_0} \equiv \text{relative permittivity} > 1 \end{aligned} \tag{2.4}$$

In the coax the ray path is parallel to the z-axis (plane wavefront) so that all rays (transmitted) arrive at the aperture plane at the same time. The coax parameters are assumed to be those of free space. If desired the present results can allow for an $\epsilon > \epsilon_0$ by reinterpreting ϵ_r as relative to this permittivity for the coax dielectric. The propagation time on such a ray path is proportional to

$$\sqrt{\epsilon_r} r_b - z_b = \quad \text{constant} \quad = \sqrt{\epsilon_r} \ell \tag{2.5}$$



Example for $Z_c = 50\Omega$ (free space medium)
with $\varepsilon_r = 2.26$ (transformer oil or polyethylene)

Figure 2.1 Prolate Spheroidal Lens with Circular Conical Transmission Line Feeding Circular Coax

The constant is evaluated by taking the special limiting case of a ray on the z-axis ($\Psi_b = 0$) and letting the lens boundary there be tangent to the aperture plane. While (2.5) is in length units it is converted to time by dividing by the speed of light in vacuum

$$c = [\mu_0 \epsilon_0]^{-\frac{1}{2}} \tag{2.6}$$

Relating the spherical and cylindrical coordinates on the lens boundary we have

$$\frac{r_b}{\ell}\sin(\theta_b) = \frac{\Psi_b}{\ell}, \quad \frac{r_b}{\ell}\cos(\theta_b) = \frac{z_b}{\ell} + 1 \tag{2.7}$$

Incorporating the equal-time condition (2.5) we have

$$\frac{r_b}{\ell} = \frac{\sqrt{\epsilon_r} - 1}{\sqrt{\epsilon_r - \cos(\theta_b)}} = \frac{\Psi_b}{\ell}\sin^{-1}(\theta_b) = \left[\frac{z_b}{\ell} + 1\right]\cos^{-1}(\theta_b) \tag{2.8}$$

describing the lens boundary (r_b, Ψ_b, z_b in terms of θ_b). In cylindrical coordinates we have

$$\left(\frac{r_b}{\ell}\right)^2 = \left(\frac{\Psi_b}{\ell}\right)^2 + \left[\frac{z_b}{\ell} + 1\right]^2 \tag{2.9}$$

Then (2.5) gives

$$\left(\frac{\Psi_b}{\ell}\right)^2 + \left[\frac{z_b}{\ell} + 1\right]^2 = \left[\frac{1}{\sqrt{\epsilon_r}}\frac{z_b}{\ell} + 1\right]^2$$

$$\left(\frac{\Psi_b}{\ell}\right)^2 + 2\left[1 - \frac{1}{\sqrt{\epsilon_r}}\right]\frac{z_b}{\ell} + \left[1 - \frac{1}{\sqrt{\epsilon_r}}\right]\left(\frac{z_b}{\ell}\right)^2 = 0 \tag{2.10}$$

This is the equation of a prolate spheroid, a body of revolution with respect to the z-axis. It intersects the z-axis ($\Psi_b = 0$) at

$$\frac{z_b}{\ell} = 0, \ \frac{-2\sqrt{\epsilon_r}}{\sqrt{\epsilon_r + 1}} \tag{2.11}$$

It reaches maximum cylindrical radius at

$$\frac{z_b}{\ell} = -\frac{a}{\ell} = -\frac{\sqrt{\epsilon_r}}{\sqrt{\epsilon_r} + 1}, \quad \frac{a}{\ell} < 1$$

$a \equiv$ major radius of prolate spheroid

$$\frac{\Psi_b}{\ell} = \frac{b}{\ell} = \frac{\sqrt{\epsilon_r - 1}}{\sqrt{\epsilon_r} + 1} = \frac{\sqrt{\epsilon_r} - 1}{\sqrt{\epsilon_r - 1}} \tag{2.12}$$

$$b \equiv \text{minor radius of prolate spheroid.}$$

Note that for large ϵ_r we have on the $z$axis

$$\frac{z_b}{\ell} \longrightarrow 0, -2 \quad \text{as} \quad \epsilon_r \longrightarrow \infty \tag{2.13}$$

and at maximum cylindrical radius

$$\frac{z_b}{\ell} \longrightarrow -1, \quad \frac{\Psi_b}{\ell} \longrightarrow 1 \quad \text{as } \epsilon_r \longrightarrow \infty \tag{2.14}$$

which gives a spherical lens of radius ℓ. However for ϵ_r near 1 we have on the z-axis

$$\frac{z_b}{\ell} \longrightarrow 0, -1 \quad \text{as} \quad \epsilon_r \longrightarrow 1 \tag{2.15}$$

and at maximum cylindrical radius

$$\frac{z_b}{\ell} \longrightarrow -\frac{1}{2}, \quad \frac{\Psi_b}{\ell} \longrightarrow 0 \text{ as } \epsilon_r \longrightarrow 1 \tag{2.16}$$

so for ϵ_r near 1 and given maximum Ψ_b (from the cross section dimensions of the cylindrical transmission line) the lens becomes very long as one would expect.

Since the rays launched onto the cylindrical transmission line are parallel to the z-axis, then z_b can be no smaller than the value in (2.12), corresponding to the maximum Ψ_b. Rays outside this region are not included. This gives a maximum allowable θ_b. From (2.7)

$$\tan(\theta_b) = \frac{\Psi_b}{\ell}\left[\frac{z_b}{\ell} + 1\right]^{-1} \tag{2.17}$$

which when evaluated at the coordinates of maximum cylindrical radius gives

$$\theta_{b_{\max}} = \arctan(\sqrt{\epsilon_r - 1})$$

(2.18)

$$0 \le \theta_b \le \theta_{b_{\max}} \quad \text{(allowable range)}$$

Limiting cases have

$$\theta_{b_{\max}} \longrightarrow \frac{\pi}{2} \quad \text{as} \quad \epsilon_r \longrightarrow \infty$$

(2.19)

$$\theta_{b_{\max}} \longrightarrow 0 \quad \text{as} \quad \epsilon_r \longrightarrow 1$$

This prolate shperiod, or ellipse revolved about its major axis, has an eccentricity (as it is usually termed [10]) as

$$\alpha = \left\{1 - \left(\frac{b}{a}\right)^2\right\}^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{\epsilon_r}} \tag{2.20}$$

It is also foci on the z-axis at

$$z_{f1} = -a[1+\alpha] = -\ell$$
$$z_{f2} = -a[1-\alpha] = -\frac{\sqrt{\epsilon_r}-1}{\sqrt{\epsilon_r}+1}\ell$$

Note that these two classical foci correspond to points with a constant sum of distances to any point on the prolate spheroidal surface. One also corresponds to the focusing of a plane wave propagating in the $-z$ direction outside the lens focusing at $z = -\ell$ inside the lens. In (2.21) we now have the result that the reflection (high frequency) from the lens boundary would focus at z_{f2} except for the presence of the inner conductor. Ideally this reflection is small.

It is interesting to note that the lens shape is only a function of ϵ_r, and not of the cross-section shape of the cylindrical transmission line. Tracing rays back from the cylindrical transmission line, one can extrapolate the conductors as well into appropriate conical-transmission-line conductors. The lens shape is a prolate sphere with major axis along the z-axis for all such cases. The transmission of waves through the lens boundary is, of course, a function of polarization, and thereby is a function of the conductor cross section chosen. In this paper we consider the simplest conductor cross section, coaxial circles, so that the transmission line is also a body of revolution, coaxial with the z-axis, and hence with the prolate spheroid.

III Matching Transmission-Line Characteristic Impedances

Having the lens shape based on the equal-time condition for high-frequency early-time performance, let us now consider impedances. Whereas the perfect matching (no reflection at lens boundary) requires a consideration of local or differential impedance matching along each ray [12], here the condition is relaxed to allow a uniform isotropic dielectric lens. The impedance condition is now taken in a more macroscopic form. Specifically let us try to match the transmission-line impedances between the circular-conical region (in the lens) and the circular-cylindrical coax.

The coaxial region has the characteristic impedance [1]

$$Z_c = \frac{Z_0}{2\pi}\ell n\left[\frac{\Psi_2}{\Psi_1}\right] \tag{3.1}$$

$$\begin{aligned} Z_0 &\equiv \sqrt{\frac{\mu_0}{\epsilon_0}} \equiv \quad \text{wave impedance of free space} \\ &\simeq 376.73\Omega \end{aligned}$$

Define a dimensionless parameter

$$\begin{aligned} \chi &\equiv \frac{\Psi_2}{\Psi_1} = e^{2\pi f_g}, \quad \zeta \equiv \ell n(\chi) = 2\pi f_g \\ f_g &\equiv \frac{Z_c}{Z_0} = \frac{1}{2\pi}\ell n(\chi) \equiv \quad \text{geometrical impedance factor} \end{aligned} \tag{3.2}$$

A given characteristic impedance implies a particular χ or ζ, in terms of which the various other parameters can be solved. Note again that $\epsilon > \epsilon_0$ can be used in the coax by scaling Z_c and ϵ_r of the lens appropriately.

The circular conical region has the characteristic impedance [2, 11]

$$Z_c = \frac{Z_0}{2\pi\sqrt{\epsilon_r}}\ell n\left[\frac{\tan\left(\frac{\theta_2}{2}\right)}{\tan\left(\frac{\theta_1}{2}\right)}\right] \tag{3.3}$$

Equate this to that of the coax and we have

$$\frac{\tan\left(\frac{\theta_2}{2}\right)}{\tan\left(\frac{\theta_2}{2}\right)} = \chi^{\sqrt{\epsilon_r}} = \frac{\sin(\theta_2)}{\sin(\theta_1)}\frac{1+\cos(\theta_1)}{1+\cos(\theta_2)} \tag{3.4}$$

For a second relation involving the two cone angles go back to the geometric constraints on the lens boundary. From (2.7) and (2.8) we have

$$\chi = \frac{r_2\sin(\theta_2)}{r_1\sin(\theta_1)} = \frac{\sqrt{\epsilon_r}-\cos(\theta_1)}{\sqrt{\epsilon_r}-\cos(\theta_2)}\frac{\sin(\theta_2)}{\sin(\theta_1)} \tag{3.5}$$

Dividing (3.4) by (3.5) gives

$$p \equiv \chi^{\sqrt{\epsilon_r}-1} = \frac{\sqrt{\epsilon_r}-\cos(\theta_2)}{\sqrt{\epsilon_r}-\cos(\theta_1)}\frac{1+\cos(\theta_1)}{1+\cos(\theta_2)} \tag{3.6}$$

This relates the two cosines as

$$\begin{aligned}\cos(\theta_2) &= \frac{\sqrt{\epsilon_r}[1-p]+[\sqrt{\epsilon_r}+p]\cos(\theta_1)}{p\sqrt{\epsilon_r}+1+[1-p]\cos(\theta_1)} \\ \cos(\theta_1) &= \frac{\sqrt{\epsilon_r}[1-p^{-1}]+[\sqrt{\epsilon_r}+p^{-1}]\cos(\theta_2)}{p^{-1}\sqrt{\epsilon_r}+1+[1-p^{-1}]\cos(\theta_2)}\end{aligned} \tag{3.7}$$

Note the symmetry in these results in that interchanging θ_1 and θ_2 replaces p by p^{-1}. The general form of these equations is refered to as a bilinear transformation [8].

Next square (3.4) and use trigonometric identities [9] to give

$$q \equiv \chi^{2\sqrt{\epsilon_r}} = \frac{1-\cos(\theta_2)}{1-\cos(\theta_2)}\frac{1+\cos(\theta_1)}{1+\cos(\theta_2)} \tag{3.8}$$

This is of the same form as (3.6) with $\sqrt{\epsilon_r}$ replaced by 1 and p by q giving

$$\begin{aligned}\cos(\theta_2) &= \frac{1-q+[1+q]\cos(\theta_1)}{1+q+[1-q]\cos(\theta_1)} \\ \cos(\theta_1) &= \frac{1-q^{-1}+[1+q^{-1}]\cos(\theta_2)}{1+q^{-1}+[1-q^{-1}]\cos(\theta_2)}\end{aligned} \tag{3.9}$$

Note again the symmetry on interchange of θ_1 and θ_2 corresponding to replacing q by q^{-1}.

Now solve for $\cos(\theta_1)$ by equating the two expressions for $\cos(\theta_2)$ in (3.7) and (3.9), giving a quadratic equation

$$[\sqrt{\epsilon_r}[1-q]+[1-q]-2[1-p]]\cos^2(\theta_1)+2[\sqrt{\epsilon_r}-1][1-p]\cos(\theta_1)$$
$$+[\sqrt{\epsilon_r}[2[1-p]-[1-q]]-[1-q]]=0\,. \tag{3.10}$$

In this form note that the coefficients are functions only of $\sqrt{\epsilon_r}$, $1-p$, and $1-q$. The quadradic equation for $\cos(\theta_2)$ is the same as this with (p,q) replaced by (p^{-1},q^{-1}). The quadradic can be solved by the usual formula, or by noting that (3.10) can be written in factored form

$$[\cos(\theta_1)+1][[\sqrt{\epsilon_r}[1-q]+[1-q]-2[1-p]]\cos(\theta_1)$$
$$+[\sqrt{\epsilon_r}[2[1-p]-[1-q]]-[1-q]]]=0 \tag{3.11}$$

Rejecting the -1 root for $\cos(\theta_1)$ as unphysical (as in (2.18)) we have

$$\cos(\theta_1)=\frac{\sqrt{\epsilon_r}[[1-q]-2[1-p]]+[1-q]}{[[1-q]-2[1-p]]+\sqrt{\epsilon_r}[1-q]} \tag{3.12}$$

Note the symmetry among the terms. Similarly we have

$$\cos(\theta_2)=\frac{\sqrt{\epsilon_r}[[1-q^{-1}]-2[1-p^{-1}]]+[1-q^{-1}]}{[[1-q^{-1}]-2[1-p^{-1}]]+\sqrt{\epsilon_r}[1-q^{-1}]} \tag{3.13}$$

giving now explicit solutions for both θ_1 and θ_2 in terms of ϵ_r and the characteristic impedance Z_c (via p and q as functions of ϵ_r and χ).

Recall (2.18) for the maximum allowable θ_b

$$\tan(\theta_{b_{\max}})=\sqrt{\epsilon_r-1}$$

$$\cos(\theta_{b_{\max}})=\frac{1}{\sqrt{\epsilon_r}},\quad \sin(\theta_{b_{\max}})=\left[1-\frac{1}{\epsilon_r}\right]^{\frac{1}{2}} \tag{3.14}$$

Evidently we have

$$0<\theta_1<\theta_2\leq\theta_{b_{\max}} \tag{3.15}$$

Equating θ_2 to $\theta_{b_{\max}}$ gives, for a given ϵ_r the maximum possible χ and hence the maximum coax impedance which can be matched by the lens.

This $\chi_{\max}$ is found from

$$\theta_2 \equiv \theta_{2_{\max}} = \theta_{b_{\max}} \tag{3.16}$$

For a given ϵ_r then from (3.13) we have

$$\cos(\theta_{b_{\max}}) = \frac{1}{\sqrt{\epsilon_r}} = \frac{\sqrt{\epsilon_r}[[1-q^{-1}] - 2[1-p^{-1}]] + [1-q^{-1}]}{[[1-q^{-1}] - 2[1-p^{-1}]] + \sqrt{\epsilon_r}[1-q^{-1}]} \tag{3.17}$$

This reduces to

$$[\epsilon_r - 1][[1-q^{-1}] - 2[1-p^{-1}]] = 0 \tag{3.18}$$

which for $\epsilon_r > 1$ gives

$$q^{-1} - 2p^{-1} + 1 = 0 \tag{3.19}$$

Substituting for p and q gives for $\chi_{\max}$

$$\begin{aligned} \chi_{\max}^{-2\sqrt{\epsilon_r}} - 2\chi_{\max}^{-\sqrt{\epsilon_r}+1} + 1 &= 0 \\ \chi_{\max}^{-\sqrt{\epsilon_r}} + \chi_{\max}^{\sqrt{\epsilon_r}} &= 2\chi_{\max} \\ \cosh(\sqrt{\epsilon_r}\ell n(\chi_{\max})) &= \chi_{\max} \end{aligned} \tag{3.20}$$

This has the implicit solution

$$\sqrt{\epsilon_r} = \frac{\text{arccosh }(\chi_{\max})}{\ell n(\chi_{\max})} = \frac{\ell n[\chi_{\max} + (\chi_{\max}^2 - 1)^{\frac{1}{2}}]}{\ell n(\chi_{\max})} \tag{3.21}$$

This is tabulated in Tables B.1 and B.2, (Appendix B). It establishes an acceptable range for Z_c or $\chi(< \chi_{\max})$ for a given ϵ_r. It also is the maximum acceptable value of ϵ_r for a given Z_c or $\chi(= \chi_{\max})$.

Now for small impedances we have

$$\begin{aligned} \zeta_{\max} &= \ell n(\chi_{\max}) \rightarrow 0 \\ \sqrt{\epsilon_r} &= \frac{\text{arccosh }(e^{\zeta_{\max}})}{\zeta_{\max}} = \frac{1}{\zeta_{\max}}\ell n[e^{\zeta_{\max}} + (e^{2\zeta_{\max}} - 1)^{\frac{1}{2}}] \end{aligned} \tag{3.22}$$

$$
\begin{aligned}
&= \frac{1}{\zeta_{\max}}\ell n[1+\zeta_{\max}+O(\zeta_{\max}^2)+[2\zeta_{\max}+O(\zeta_{\max}^2)]^{\frac{1}{2}}] \\
&= \frac{1}{\zeta_{\max}}\ell n[1+\zeta_{\max}+0(\zeta_{\max}^2)+\sqrt{2\zeta_{\max}}+[1+O(\zeta_{\max})]] \\
&= \frac{1}{\zeta_{\max}}\ell n[1+\sqrt{2\zeta_{\max}}+\zeta_{\max}+O(\zeta_{\max}^{\frac{3}{2}})] \\
&= \frac{1}{\zeta_{\max}}\left[\sqrt{2\zeta_{\max}}+\zeta_{\max}+O(\zeta_{\max}^{\frac{3}{2}})-\frac{1}{2}[\sqrt{2\zeta_{\max}}+O(\zeta_{\max})]^2\right] \\
&= \frac{1}{\zeta_{\max}}[\sqrt{2\zeta_{\max}}+O(\zeta_{\max}^{\frac{3}{2}})] \\
&= \sqrt{\frac{2}{\zeta_{\max}}}[1+O(\zeta_{\max})]
\end{aligned}
$$

Thus large ϵ_r are associated with small $\zeta_{\max}$, and hence small impedences. For large $\zeta_{\max}$ we have [9]

$$
\begin{aligned}
&\zeta_{\max} \longrightarrow \infty \\
&\sqrt{\epsilon_r} = \frac{\operatorname{arccosh}\ (\chi_{\max})}{\zeta_{\max}} = \frac{1}{\zeta_{\max}}\left[\ell n(2\chi_{\max})+O(\chi_{\max}^{-2})\right] \\
&= \frac{1}{\zeta_{\max}}[\zeta_{\max}+\ell n(2)+O(e^{-2\zeta_{\max}})] \\
&= 1+\frac{\ell n(2)}{\zeta_{\max}}+O(\zeta_{\max}^{-1}e^{-2\zeta_{\max}})
\end{aligned}
\tag{3.23}
$$

Thus small $\sqrt{\epsilon_r}-1$ allows large $\zeta_{\max}$ and hence large impedances.

In the case of maximum impedance ($\zeta_{\max}$) for a given ϵ_r, the larger angle θ_2 is just given by (3.16). The smaller angle θ_1 can be taken as

$$\theta_1 \equiv \theta_{1_{\min}} \tag{3.24}$$

From (3.9) we readily obtain

$$\cos(\theta_{1_{\min}}) = \frac{1-\frac{\sqrt{\epsilon_r}-1}{\sqrt{\epsilon_r}+1}\chi_{\max}^{-2\sqrt{\epsilon_r}}}{1+\frac{\sqrt{\epsilon_r}-1}{\sqrt{\epsilon_r}+1}\chi_{\max}^{-2\sqrt{\epsilon_r}}} \tag{3.25}$$

Noting that

$$\tan\left(\frac{\theta_{2_{\max}}}{2}\right) = \left[\frac{1-\cos(\theta_{2_{\max}})}{1+\cos(\theta_{2_{\max}})}\right]^{\frac{1}{2}} = \left[\frac{\sqrt{\epsilon_r}-1}{\sqrt{\epsilon_r}+1}\right]^{\frac{1}{2}} \tag{3.26}$$

then another expression is obtained from (3.4) as

$$\tan\left(\frac{\theta_{1_{\max}}}{2}\right) = \left[\frac{\sqrt{\epsilon_r}-1}{\sqrt{\epsilon_r}+1}\right]^{\frac{1}{2}} \zeta_{\max}^{-\sqrt{\epsilon_r}} \tag{3.27}$$

which can also be obtained from the bilinear form in (3.25).

Now consider the case of small characteristic impedances for which $\zeta = \ell n(\chi)$ is an appropriate expansion parameter. Reorganize (3.12) as

$$\begin{aligned} \cos(\theta_1) &= \frac{\sqrt{\epsilon_r}+1-2\sqrt{\epsilon_r}\dfrac{1-p}{1-q}}{\sqrt{\epsilon_r}+1-2\dfrac{1-p}{1-q}} \\ &= \frac{\sqrt{\epsilon_r}+1-2\sqrt{\epsilon_r}\sqrt{\dfrac{p}{q}}X}{\sqrt{\epsilon_r}+1-2\sqrt{\dfrac{p}{q}}X} \end{aligned} \tag{3.28}$$

$$\begin{aligned} X &\equiv \frac{\sqrt{p}-\dfrac{1}{\sqrt{p}}}{\sqrt{q}-\dfrac{1}{\sqrt{q}}} = \frac{e^{\frac{\sqrt{\epsilon_r}-1}{2}\zeta} - e^{\frac{\sqrt{\epsilon_r}-1}{2}\zeta}}{e^{\sqrt{\epsilon_r}\zeta} - e^{\sqrt{\epsilon_r}\zeta}} \\ &= \frac{\sinh\left(\dfrac{\sqrt{\epsilon_r}-1}{2}\zeta\right)}{\sinh(\sqrt{\epsilon_r}\zeta)} \\ \sqrt{\frac{p}{q}} &= e^{-\frac{\sqrt{\epsilon_r}+1}{2}\zeta} \end{aligned}$$

Expanding terms

$$X = \frac{\sqrt{\epsilon_r}-1}{2\sqrt{\epsilon_r}}[1+0(\zeta^2)]$$

$$\sqrt{\frac{p}{q}} = 1 - \frac{\sqrt{\epsilon_r}+1}{2}\zeta + O(\zeta^2)$$

we have

$$\cos(\theta_1) = \frac{\sqrt{\epsilon_r}+1-[\sqrt{\epsilon_r}-1][1-\dfrac{\sqrt{\epsilon_r}+1}{2}\zeta+O(\zeta^2)]}{\sqrt{\epsilon_r}+1-\dfrac{\sqrt{\epsilon_r}-1}{\sqrt{\epsilon_r}}[1-\dfrac{\sqrt{\epsilon_r}+1}{2}\zeta+O(\zeta^2)]}$$

$$
\begin{aligned}
&= \sqrt{\epsilon_r}\frac{2+\frac{\epsilon_r-1}{2}\zeta+O(\zeta^2)}{\epsilon_r+1+\frac{\epsilon_r-1}{2}\zeta+O(\zeta^2)} \\
&= \frac{2\sqrt{\epsilon_r}}{\epsilon_r+1}\frac{1+\frac{\epsilon_r-1}{4}\zeta+O(\zeta^2)}{1+\frac{1}{2}\frac{\epsilon_r-1}{\epsilon_r+1}\zeta+O(\zeta^2)} \\
&= \frac{2\sqrt{\epsilon_r}}{\epsilon_r+1}[1+\frac{1}{4}\frac{[\epsilon_r-1]^2}{\epsilon_r+1}\zeta+O(\zeta^2)] \qquad (3.29)
\end{aligned}
$$

Identifying terms we have

$$\cos(\theta_B)=\frac{2\sqrt{\epsilon_r}}{\epsilon_r+1} \qquad (3.30)$$

which we will later encounter as the Brewster angle with no reflection at the lens boundary. Write

$$
\begin{aligned}
\theta_1 &= \theta_B-\Delta\theta \\
\cos(\theta_1) &= \cos(\theta_B)+\sin(\theta_B)\Delta\theta+O((\Delta\theta)^2) \text{ as } \Delta\theta\to O
\end{aligned}
\qquad (3.31)
$$

Then we have

$$
\begin{aligned}
\sin(\theta_B) &= \frac{\epsilon_r-1}{\epsilon_r+1} \\
\Delta\theta &= \frac{\sqrt{\epsilon_r}}{2}\frac{\epsilon_r-1}{\epsilon_r+1}\zeta+O(\zeta^2)
\end{aligned}
\qquad (3.32)
$$

Applying this to θ_2 then (3.28) takes the form

$$
\begin{aligned}
\cos(\theta_2) &= \frac{\sqrt{\epsilon_r}+1-2\sqrt{\epsilon_r}\frac{1-p^{-1}}{1-q^{-1}}}{\sqrt{\epsilon_r}+1-2\frac{1-p^{-1}}{1-q^{-1}}} \\
&= \frac{\sqrt{\epsilon_r}+1-2\sqrt{\epsilon_r}\sqrt{\frac{q}{p}}X}{\sqrt{\epsilon_r}+1-2\sqrt{\frac{q}{p}}X} \qquad (3.33)
\end{aligned}
$$

The change from $\sqrt{p/q}$ to $\sqrt{q/p}$ changes the sign of the coefficient of ζ in (3.29) and (3.30) giving

$$\cos(\theta_2) = \frac{2\sqrt{\epsilon_r}}{\epsilon_r + 1}[1 - \frac{1}{4}\frac{[\epsilon_r - 1]^2}{\epsilon_r + 1}\zeta + O(\zeta^2)] \tag{3.34}$$

This allows us to write (to first order)

$$\begin{gathered} \theta_2 = \theta_B + \Delta\theta \\ \cos(\theta_2) = \cos(\theta_B) - \sin(\theta_B)\Delta\theta + O((\Delta\theta)^2) \text{ as } \Delta\theta \to O \end{gathered} \tag{3.35}$$

with $\Delta\theta$ the same as in (3.33).

So we see now that for small characteristic impedances θ_1 and θ_2 are equally spaced on both sides of θ_B. This is shown in Appendix B (Tables B.4) where θ_1 is reasonably approximated by $\theta_B - \Delta\theta$ and θ_2 reasonably approximated by $\theta_B + \Delta\theta$ for small Z_c.

IV Transmission of Waves at Lens Boundary in High-Frequency Limit

As indicated in Fig. 4.1 a ray incident on the lens boundary has angles as indicated with respect to the normal $\vec{1}_b$ to the lens surface. As in the usual derivations [4, 5] the angles are determined by matching the phase velocities along the local boundary giving

$$\begin{aligned} \psi_r &= \psi_i \\ \sqrt{\epsilon_r}\sin(\psi_i) &= \sin(\psi_t) \end{aligned} \tag{4.1}$$

The relation between transmitted angle ψ_t and incident angle ψ_i is known as Snell's law. It is a condition of equal time, local to the ray path, also known as Fermat's principle. Note that in this paper the equal time principle is made a global condition, the same transit time to the aperture plane being constrained for all incident plus transmitted rays.

Note that with respect to $\vec{1}_b$ the wave is polarized in the plane of incidence and is referred to as an E (or TM) wave. Summarizing the results for the transmission and reflection of the various fields

$$\begin{aligned} R_e &\equiv \frac{E_{refl}}{E_{inc}} = \frac{H_{refl}}{H_{inc}} = R_h \\ T_h &= 1 + R_h = \frac{H_{trans}}{H_{inc}} \\ T_e &= \frac{Z_0}{Z_\ell} T_h = \frac{E_{trans}}{E_{inc}} = \sqrt{\epsilon_r} T_h \end{aligned} \tag{4.2}$$

Here the subscripts apply to the fields of concern. Here the fields are given as vector components with orientations as in Fig. 4.1, All the above parameters can be expressed in terms of [4,5]

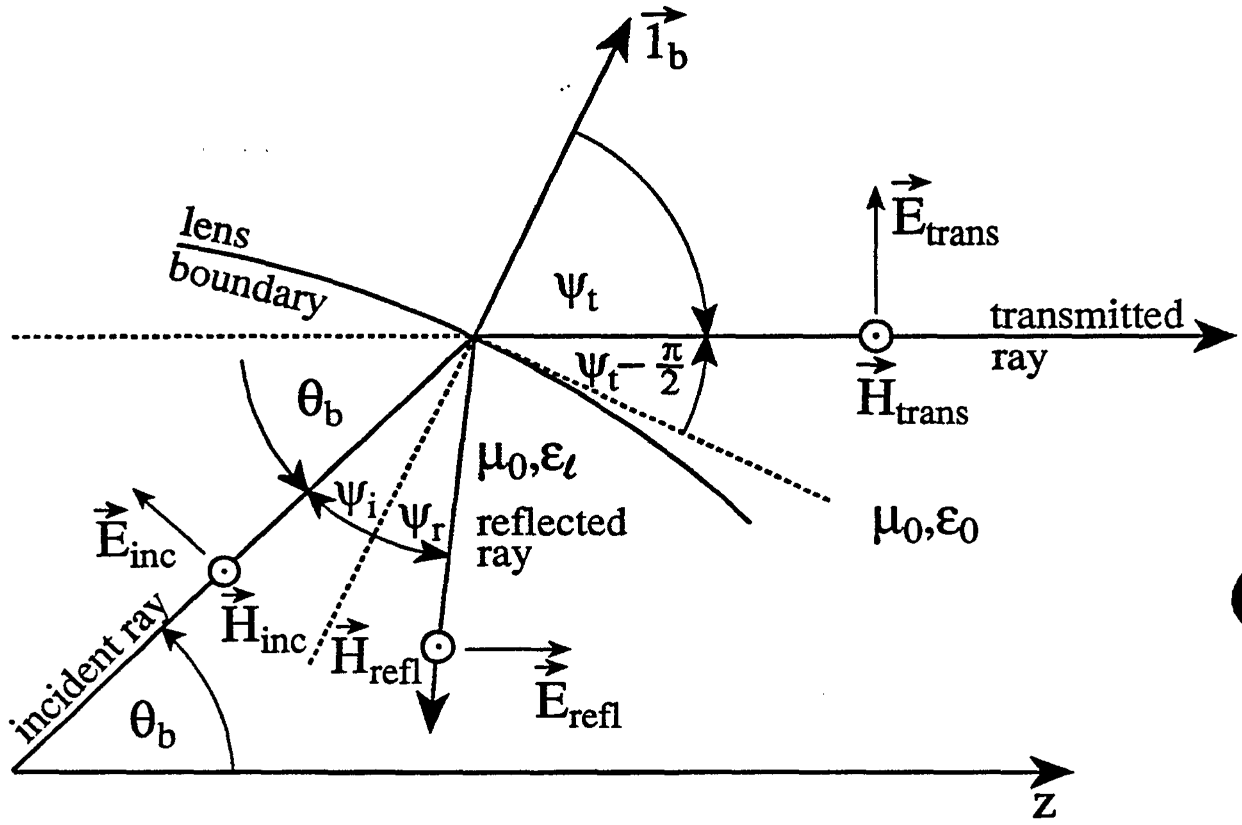


Figure 4.1 Transmission and Reflection at Lens Boundary

$$
\begin{aligned}
R_e = R_h &= \frac{\frac{1}{\sqrt{\epsilon_r}}\cos(\psi_i) - [1-\epsilon_r \sin^2(\psi_i)]^{\frac{1}{2}}}{\frac{1}{\sqrt{\epsilon_r}}\cos(\psi_i) + [1-\epsilon_r \sin^2(\psi_i)]^{\frac{1}{2}}} \\
&= \frac{[1-\frac{1}{\epsilon_r}\sin^2(\psi_t)]^{\frac{1}{2}} - \sqrt{\epsilon_r}\cos(\psi_t)}{[1-\frac{1}{\epsilon_r}\sin^2(\psi_t)]^{\frac{1}{2}} + \sqrt{\epsilon_r}\cos(\psi_t)}
\end{aligned}
\tag{4.3}
$$

There is a special condition under which there is no reflection at the lens boundary, referred to as the Brewster angle which we designate with a subscript B. From (4.3) this is given by

$$
\begin{aligned}
\cos(\psi_{iB}) &= \left[\frac{\epsilon_r}{\epsilon_r + 1}\right]^{\frac{1}{2}} = \sin(\psi_{tB}) \\
\sin(\psi_{iB}) &= [\epsilon_r + 1]^{-\frac{1}{2}} = \cos(\psi_{tB}) \\
\cot(\psi_{iB}) &= \sqrt{\epsilon_r} = \tan(\psi_{tB}) \\
\psi_{iB} + \psi_{tB} &= \frac{\pi}{2}
\end{aligned}
\tag{4.4}
$$

Recall the lens-boundary parameters from Section II. From (2.10) we have

$$
\begin{aligned}
\Psi_b &= \left\{-\frac{\epsilon_r - 1}{\epsilon_r} z_b^2 - 2\frac{\sqrt{\epsilon_r} - 1}{\sqrt{\epsilon_r}} \ell z_b\right\}^{\frac{1}{2}} \\
\frac{d\Psi_b}{dz_b} &= \Psi_b^{-1}\left\{-\frac{\epsilon_r - 1}{\epsilon_r} z_b - \frac{\sqrt{\epsilon_r} - 1}{\sqrt{\epsilon_r}} \ell\right\} \\
&= -\frac{\epsilon_r - 1}{\epsilon_r}\frac{z}{\Psi_b} - \frac{\sqrt{\epsilon_r} - 1}{\sqrt{\epsilon_r}}\frac{\ell}{\Psi_b}
\end{aligned}
\tag{4.5}
$$

From (2.7) and (2.8) we have

$$
\begin{aligned}
\frac{z_b}{\Psi_b} &= \cot(\theta_b) - \frac{\ell}{\Psi_b} \\
\frac{\ell}{\Psi_b} &= \frac{1}{\sin(\theta_b)}\frac{\sqrt{\epsilon_r} - \cos(\theta_0)}{\sqrt{\epsilon_r - 1}} \\
\frac{z_b}{\Psi_b} &= \cot(\theta_b) - \frac{1}{\sin(\theta_b)}\frac{\sqrt{\epsilon_r} - \cos(\theta_b)}{\sqrt{\epsilon_r - 1}}
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
&= \frac{\sqrt{\epsilon_r}}{\sqrt{\epsilon_r}-1}\frac{\cos(\theta_b)-1}{\sin(\theta_b)} \\
&= -\frac{\sqrt{\epsilon_r}}{\sqrt{\epsilon_r}-1}\tan\left(\frac{\theta_b}{2}\right)
\end{aligned}
$$

Substituting back we have

$$
\begin{aligned}
\cot(\psi_t) &= -\frac{d\Psi_b}{dz_b} \\
&= \frac{\sqrt{\epsilon_r}+1}{\sqrt{\epsilon_r}}\frac{\cos(\theta_b)-1}{\sin(\theta_b)} + \frac{\sqrt{\epsilon_r}-\cos(\theta_b)}{\sqrt{\epsilon_r}\sin(\theta_b)} \\
&= \frac{\sqrt{\epsilon_r}\cos(\theta_b)-1}{\sqrt{\epsilon_r}\sin(\theta_b)}
\end{aligned}
\tag{4.7}
$$

For any given θ_b, ψ_t can now be directly computed, and hence R_e and T_e. Alternately from (2.7) and (2.8) we have

$$
\begin{aligned}
\sin(\theta_b) &= \frac{\Psi_b}{r_b} = \frac{\Psi_b}{\ell}\frac{\sqrt{\epsilon_r}-\cos(\theta_b)}{\sqrt{\epsilon_r}-1} \\
\frac{\Psi_b}{\ell} &= [\sqrt{\epsilon_r}-1]\frac{\sin(\theta_b)}{\sqrt{\epsilon_r}-\cos(\theta_b)}
\end{aligned}
\tag{4.8}
$$

relating Ψ_b to θ_b. So in principle we can compute T_e as a function of Ψ_b or θ_b for each transmitted ray.

V Fields Between Coaxial Circular Cones

In general the electric potential in the lens region (for the incident spherical TEM wave) takes the form (at constant r) [2, 11]

$$U = \ell n[\tan\left(\frac{\theta}{2}\right)] \tag{5.1}$$

This has already been used for the characteristic impedance in Section II. If V is the potential between the two cones (0 on θ_2, V on θ_1) we can write a normalized potential as

$$f_V(\theta) = \frac{\ell n\left[\frac{\tan\left(\frac{\theta}{2}\right)}{\tan\left(\frac{\theta_2}{2}\right)}\right]}{\ell n\left[\frac{\tan\left(\frac{\theta_1}{2}\right)}{\tan\left(\frac{\theta_2}{2}\right)}\right]} = \begin{cases} 0 & \text{for} \quad \theta = \theta_2 \\ 1 & \text{for} \quad \theta = \theta_1 \end{cases} \tag{5.2}$$

The electric field then takes the form

$$\begin{aligned} E_\theta &= -\frac{V}{r}\frac{df_V(\theta)}{d\theta} \\ f_E(\theta) &\equiv -\frac{df_V(\theta)}{d\theta} \\ &= \left\{\sin(\theta)\ell n\left[\frac{\tan\left(\frac{\theta_2}{2}\right)}{\tan\left(\frac{\theta_1}{2}\right)}\right]\right\}^{-1} \end{aligned} \tag{5.3}$$

Applying a step voltage at the apex we have (before any reflections)

$$\begin{aligned} V &= V_0 u(t - \frac{r}{c}\sqrt{\epsilon_r}) \\ E_\theta &= \frac{V_0}{r} f_E(\theta) u(t - \frac{r}{c}\sqrt{\epsilon_r}) \end{aligned} \tag{5.4}$$

On going through the lens boundary these rays are bent to become parallel to the z-axis and the electric field at $r = r_b$ is multiplied by T_e. These rays arrive at the $z = 0$ plane (the aperture plane) at a time

$$t_a = \frac{\ell}{c}\sqrt{\epsilon_r} \tag{5.5}$$

VI Launching TEM Mode on Coax

Having the fields on the aperture plane, let us consider a representation in terms of the waveguide modes appropriate to the cylindrical transmission line. As discussed in [3, 7] the modes are mutually orthogonal as well as orthogonal to a continuous spectrum known as the radiation field. The orthogonality is over a plane of constant z (e.g. the aperture plane). Furthermore an enclosed cylindrical system (e.g. a coax) has no radiation field, the field being zero outside the outer cylindrical boundary.

The general form of the modal orthogonality involves an integral over the aperture plane S_a of the cross product of electric and magnetic fields of the pair of modes. However, for the TEM mode as one of these, this reduces to a dot product of the electric fields of the two modes. Fundamental to this is the representation of the TEM mode as

$$\begin{aligned} \tilde{\vec{E}}_{TEM}(\vec{r},s) &= \tilde{V}(s)e^{-\gamma z}\vec{e}_0(x,y) \\ \tilde{\vec{H}}_{TEM}(\vec{r},s) &= \tilde{V}(s)e^{-\gamma z}\frac{1}{Z_0}\vec{h}_0(x,y) \\ \vec{h}_0(x,y) &= \vec{1}_z \times \vec{e}_0(x,y) \end{aligned} \tag{6.1}$$

The normalization uses

$$\vec{e}_0(x,y) = -\nabla u(x,y) \tag{6.2}$$

where u is zero on the outer conductor and one on the inner conductor. V then represents the voltage in the TEM mode.

So now let us assume that there is some electric field $\vec{E}$ on the aperture plane. Then from [3] we have

$$\tilde{V}(s) = \frac{\int_{S_a} \tilde{\vec{E}}(x,y,s)\cdot\vec{e}_0(x,y)dS}{\int_{S_a} \vec{e}_0(x,y)\cdot\vec{e}_0(x,y)dS} \tag{6.3}$$

where only the components of $\vec{E}$ transverse to z are important. Note that this formula is equally simple in time domain as

$$V(t) = \frac{\int_{S_a} \vec{E}(x,y,t) \cdot \vec{e}_0(x,y) dS}{\int_{S_a} \vec{e}_0(x,y) \cdot \vec{e}_0(x,y) dS} \tag{6.4}$$

since the TEM transverse distribution is frequency independent. Note that per (6.1) the TEM mode takes the time-domain form

$$\vec{E}_{TEM}(\vec{r}, s) = V(t - \frac{z}{c})\vec{e}_0(x,y) \tag{6.5}$$

Our case of rotation symmetry (about the z-axis) further reduces the complexity. Analogens to (5.2) a circular coax has a normalized potential distribution

$$\begin{aligned} u(\Psi) &= \frac{\ell n\left(\frac{\Psi}{\Psi_2}\right)}{\ell n\left(\frac{\Psi_1}{\Psi_2}\right)} = \begin{cases} 0 & \text{for} \quad \Psi = \Psi_2 \\ 1 & \text{for} \quad \Psi = \Psi_1 \end{cases} \\ \vec{e}_0(\Psi) &= -\vec{1}_\Psi \frac{du}{d\Psi} = \left\{ \Psi \ell n\left(\frac{\Psi_2}{\Psi_1}\right) \right\}^{-1} \end{aligned} \tag{6.6}$$

Constraining the aperture field by the rotation symmetry we have

$$\vec{E}(x,y,t) = E_\Psi(\Psi,t)\vec{1}_\Psi + E_z(\Psi,t)\vec{1}_z \tag{6.7}$$

Then (6.4) becomes

$$\begin{aligned} V(t) &= \ell n\left(\frac{\Psi_2}{\Psi_1}\right) \frac{\int_{\Psi_1}^{\Psi_2} E_\Psi(\Psi,t) d\Psi}{\int_{\Psi_1}^{\Psi_2} \frac{d\Psi}{\Psi}} \\ &= \int_{\Psi_1}^{\Psi_2} E_\Psi(\Psi,t) d\Psi \end{aligned} \tag{6.8}$$

Noting that the initial signal arrives at t_a on the aperture plane, let us define an initial voltage on the coax as (6.8) evaluated at $t = t_{a+}$. This gives the early-time/high-frequency form of the coax voltage. Comparing this to the step-rising voltage on the conical section as in (5.4) we can define a high-frequency transfer function as

$$T_V = \frac{V(t_{a+})}{V_0} = \frac{1}{V_0} \int_{\Psi_1}^{\Psi_2} E_\Psi(\Psi,t) d\Psi \tag{6.9}$$

Based on power considerations, since the conical and cylindrical sections have the same characteristic impedance $|T_V|$ is bounded by 1.

Now, to find the initial electric field on the aperture plane, recall from (5.4), evaluating the coordinates on the lens surface, the initial field there

$$E_\theta = \frac{V_0}{r_b} f_E(\theta_b) \tag{6.10}$$

Then from (4.2) the initial field on the aperture plane is

$$E_\Psi(\Psi, t_{a+}) = T_e(\theta_b) E_\theta = \frac{V_0}{r_b} T_e(\theta_b) f_E(\theta_b) \tag{6.11}$$

This gives, using lens-boundary variables,

$$T_V = \int_{\Psi_1}^{\Psi_2} \frac{1}{r_b} T_e(\theta_b) f_E(\theta_b) d\Psi_b \tag{6.12}$$

Recall the factors in the integrand as

$$\begin{aligned} \frac{r_b}{\ell} &= \frac{\sqrt{\epsilon_r} - 1}{\sqrt{\epsilon_r} - \cos(\theta_b)} \\ \frac{\Psi_b}{\ell} &= \frac{\Psi_b}{r_v} \frac{r_b}{\ell} = \sin(\theta_b) \frac{\sqrt{\epsilon_r} - 1}{\sqrt{\epsilon_r} - \cos(\theta_b)} \end{aligned} \tag{6.13}$$

which puts all variables in terms of θ_b. Then using

$$\begin{aligned} \frac{d\Psi_b}{\ell} &= \frac{[\sqrt{\epsilon_r} - 1][\sqrt{\epsilon_r}\cos(\theta_b) - 1]}{[\sqrt{\epsilon_r} - \cos(\theta_b)]^2} d\theta_b \\ \frac{d\Psi_b}{r_b} &= \frac{\sqrt{\epsilon_r}\cos(\theta_b) - 1}{\sqrt{\epsilon_r} - \cos(\theta_b)} d\theta_b \end{aligned} \tag{6.14}$$

we have

$$T_V = \int_{\theta_1}^{\theta_2} T_e(\theta_b) f_E(\theta_b) \frac{\sqrt{\epsilon_r}\cos(\theta_b) - 1}{\sqrt{\epsilon_r} - \cos(\theta_b)} d\theta_b \tag{6.15}$$

and recalling f_E from (5.3) we have

$$T_V = \left\{ \ell n \left[\frac{\tan\left(\frac{\theta_2}{2}\right)}{\tan\left(\frac{\theta_1}{2}\right)} \right] \right\}^{-1} \int_{\theta_1}^{\theta_2} T_e(\theta_b) \frac{\sqrt{\epsilon_r}\cos(\theta_b) - 1}{\sqrt{\epsilon_r} - \cos(\theta_b)} \frac{d\theta_b}{\sin(\theta_b)} \tag{6.16}$$

From Section IV we have

$$
\begin{aligned}
T_e = \sqrt{\epsilon_r}[2 + R_e] &= \frac{2\sqrt{\epsilon_r}[1 - \frac{1}{\epsilon_r}\sin^2(\psi_t)]^{\frac{1}{2}}}{[1 - \frac{1}{\epsilon_r}\sin^2(\psi_t)]^{\frac{1}{2}} + \sqrt{\epsilon_r}\cos(\psi_t)} \\
\cot(\psi_t) &= \frac{\sqrt{\epsilon_r}\cos(\theta_b) - 1}{\sqrt{\epsilon_r}\sin(\theta_b)} \\
\sin(\psi_t) &= \frac{\sqrt{\epsilon_r}\sin(\theta_b)}{[\epsilon_r + 1 - 2\sqrt{\epsilon_r}\cos(\theta_b)]^{\frac{1}{2}}} \\
\cos(\psi_t) &= \frac{\sqrt{\epsilon_r}\cos(\theta_b) - 1}{[\epsilon_r + 1 - 2\sqrt{\epsilon_r}\cos(\theta_b)]^{\frac{1}{2}}} \qquad (6.17)
\end{aligned}
$$

Substituting, we find

$$
T_E(\theta_b) = 2\sqrt{\epsilon_r}\frac{\sqrt{\epsilon_r} - \cos(\theta_b)}{[\epsilon_r - 1]\cos(\theta_b)} \qquad (6.18)
$$

With (6.16) this gives

$$
T_V = \frac{2\sqrt{\epsilon_r}}{\epsilon_r - 1}\left\{\ell n\left[\frac{\tan\left(\frac{\theta_2}{2}\right)}{\tan\left(\frac{\theta_1}{2}\right)}\right]\right\}^{-1}\int_{\theta_1}^{\theta_2}\frac{\sqrt{\epsilon_r}\cos(\theta_b) - 1}{\sin(\theta_b)\cos(\theta_b)}d\theta_b \qquad (6.19)
$$

Substituting from (3.4) rewrite the integral as

$$
\begin{aligned}
T_V &= \frac{2\sqrt{\epsilon_r}}{\epsilon_r - 1}\left[\ell n(\sqrt{q})\right]^{-1}\int_{\theta_1}^{\theta_2}\left[\frac{\sqrt{\epsilon_r}}{\sin(\theta_b)} - \frac{1}{\sin(\theta_b)\cos(\theta_b)}\right]d\theta_b \\
&= \frac{2\sqrt{\epsilon_r}}{\epsilon_r - 1}\left[\ell n(\sqrt{q})\right]^{-1}\left\{\sqrt{\epsilon_r}\ell n\left[\frac{\tan\left(\frac{\theta_2}{2}\right)}{\tan\left(\frac{\theta_1}{2}\right)}\right] - \ell n\left[\frac{\tan(\theta_2)}{\tan(\theta_1)}\right]\right\} \\
&= \frac{2\sqrt{\epsilon_r}}{\epsilon_r - 1}\left\{\sqrt{\epsilon_r} - \frac{1}{\sqrt{\epsilon_r}}\frac{1}{\ell n(\chi)}\ell n\left[\frac{\tan(\theta_2)}{\tan(\theta_1)}\right]\right\} \\
&= \frac{2\sqrt{\epsilon_r}}{[\epsilon_r - 1]\ell n(\sqrt{q})}\left\{\sqrt{\epsilon_r}\ell n(\sqrt{q}) - \ell n\left[\frac{\tan(\theta_2)}{\tan(\theta_1)}\right]\right\} \qquad (6.20)
\end{aligned}
$$

From (3.12) we have

$$
\begin{aligned}
\tan(\theta_1) &= \left[\frac{1}{\cos^2(\theta_1)} - 1\right]^{\frac{1}{2}} \\
&= 2[\epsilon_r - 1]^{\frac{1}{2}}\frac{[[1 - q][1 - p] - [1 - p]^2]^{\frac{1}{2}}}{\sqrt{\epsilon_r}[[1 - q] - 2[1 - p]] + [1 - q]} \qquad (6.21)
\end{aligned}
$$

Similarly from (3.13) or just substituting $p \rightarrow p^{-1}, q \rightarrow q^{-1}$, gives (noting now negative denominator when taking square root of a square)

$$\begin{aligned}\tan(\theta_2) &= -2[\epsilon_r - 1]^{\frac{1}{2}} \frac{[[1-q^{-1}][1-p^{-1}] - [1-p^{-1}]^2]^{\frac{1}{2}}}{\sqrt{\epsilon_r}[[1-q^{-1}] - 2[1-p^{-1}]] + [1-q^{-1}]} \\ &= -2[\epsilon_r - 1]^{\frac{1}{2}} \frac{1}{p\sqrt{q}} \frac{[[1-q][1-p] - [1-p]^2]^{\frac{1}{2}}}{\sqrt{\epsilon_r}[[1-q^{-1}] - 2[1-p^{-1}] + [1-q^{-1}]]} \end{aligned} \quad (6.22)$$

Noting the common factors

$$\frac{\tan(\theta_2)}{\tan(\theta_1)} = \frac{-1}{p\sqrt{q}} - \frac{\sqrt{\epsilon_r}[[1-q] - 2[1-p]] + [1-q]}{\sqrt{\epsilon_r}[[1-q^{-1}] - 2[1-p^{-1}]] + [1-q^{-1}]} \quad (6.23)$$

Substituting in (6.20) gives

$$\begin{aligned}T_V &= \frac{2\sqrt{\epsilon_r}}{[\epsilon_r - 1]\ell n(\sqrt{q})} \left\{ [\sqrt{\epsilon_r} + 1]\ell n(\sqrt{q}) + \ell n(p) \right. \\ &- \left. \ell n \left[- \frac{[\sqrt{\epsilon_r} + 1][1-q] - 2\sqrt{\epsilon_r}[1-p]}{[\sqrt{\epsilon_r} + 1][1-q^{-1}] - 2\sqrt{\epsilon_r}[1-p^{-1}]} \right] \right\} \end{aligned} \quad (6.24)$$

and recalling from Section III

$$\begin{aligned} p &= \chi^{\sqrt{\epsilon_r} - 1} = e^{[\sqrt{\epsilon_r} - 1]\zeta} \\ q &= \chi^{2\sqrt{\epsilon_r}} = e^{2\sqrt{\epsilon_r}\zeta} \\ \zeta &\equiv \ell n(\chi) \end{aligned} \quad (6.25)$$

gives the expression

$$\begin{aligned}T_V &= \frac{2}{[\epsilon_r - 1]\zeta} \left\{ [\sqrt{\epsilon_r} + 1]\ell n(\sqrt{q}) + \ell n(p) \right. \\ &- \left. \ell n \left[q \frac{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\frac{1-p}{1-q}}{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\frac{q}{p}\frac{1-p}{1-q}} \right] \right\} \\ &= \frac{2}{[\epsilon_r - 1]\zeta} \left\{ [\epsilon_r - 1]\zeta - \ell n \left[\frac{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\frac{1-p}{1-q}}{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\frac{q}{p}\frac{1-p}{1-q}} \right] \right\} \\ &= 2 - \frac{2}{[\epsilon_r - 1]\zeta} \ell n \left[\frac{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\frac{1-p}{1-p}}{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\frac{q}{p}\frac{1-p}{1-q}} \right] \end{aligned} \quad (6.26)$$

In this form we can readily see that T_V is an even function of ζ by substituting

$$\zeta \rightarrow -\zeta$$

$$p \rightarrow \frac{1}{p}, \; q \rightarrow \frac{1}{q} \tag{6.27}$$

thereby interchanging the numerator and denominator of the last term and thereby changing its sign. Dividing by ζ then keeps the result unchanged and (6.26) is even in $\ell n(\chi)$.

For convenience in evaluation write

$$\begin{aligned} T_V &= 2 - \frac{2}{[\epsilon_r - 1]\zeta} \ell n \left[\frac{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\sqrt{\frac{p}{q}} X}{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\sqrt{\frac{q}{p}} X} \right] \\ X &\equiv \frac{\sqrt{p} - \frac{1}{\sqrt{p}}}{\sqrt{q} - \frac{1}{\sqrt{q}}} \end{aligned} \tag{6.28}$$

where we have

$$\begin{aligned} \sqrt{p} - \frac{1}{\sqrt{p}} &= \chi^{\frac{\sqrt{\epsilon_r}-1}{2}} - \chi^{\frac{-\sqrt{\epsilon_r}+1}{2}} = 2\sinh\left(\frac{\sqrt{\epsilon_r}-1}{2}\zeta\right) \\ \sqrt{q} - \frac{1}{\sqrt{q}} &= \chi^{\sqrt{\epsilon_r}} - \chi^{-\sqrt{\epsilon_r}} = 2\sinh\left(\sqrt{\epsilon_r}\zeta\right) \\ X &= \frac{\sinh\left(\frac{\sqrt{\epsilon_r}-1}{2}\zeta\right)}{\sinh(\sqrt{\epsilon_r}\zeta)} \\ \sqrt{\frac{q}{p}} &= \chi^{\frac{\sqrt{\epsilon_r}+1}{2}} = e^{\frac{\sqrt{\epsilon_r}+1}{2}\zeta} \end{aligned} \tag{6.29}$$

so that T_V is now expressible in terms of ϵ_r and $\ell n(\chi)$ alone.

Since we expect T_V to be near (slightly less than) 1, we can also write

$$T_V = 1 - \frac{2}{[\epsilon_r - 1]\zeta} \ell n \left[e^{-\frac{\epsilon_r - 1}{2}\zeta} \frac{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\sqrt{\frac{p}{q}} X}{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\sqrt{\frac{q}{p}} X} \right] \tag{6.30}$$

where the second term is the deviation (negative) from 1. As shown in Appendix A, for small ζ (small impedance) we have

$$T_V = 1 - \frac{1}{48}[\epsilon_r - 1]^2\zeta^2 + 0(\zeta^4) \text{ as } \zeta \to 0 \tag{6.31}$$

As seen by consulting Appendix B, Tables B.4 (a through j), this is an extremely good approximation for small $\epsilon_r - 1$ and small impedance. Here it is labelled as T_{V_a} (including terms through ζ^2, noting the missing ζ^3 term).

VII Concluding Remarks

It is found that over a useful range of coax impedance and lens dielectric constant, the lens works quite well. The equal transmission-line-characteristic-impedance constraint assures perfect low-frequency performance (no reflection or unity transmission). The high-frequency/early-time performance, as manifest in the TEM-mode coefficient T_V is quite good in such cases, approaching to within a few percent of unity. Note that T_V^2 represents the power in the TEM mode on the coax. The remaining power $1 - T_V^2$ represents the power in the reflections at the lens boundary plus all the higher order modes on the coax.

The coaxial geometry considered here has introduced some important simplifications. The rotation (C_∞) symmetry has only a ϕ component for $\vec{H}$, no E_ϕ, and all radial and axial components independent of ϕ. The equal-time requirement (Section II), however, did not use this assumption. Hence, any uniform isotropic dielectric lens taking a spherical wave from a conical apex (not necessarily for a circular cone) to a plane wave on a cylindrical transmission line (not necessarily for a circular cylinder as in a coax) has a prolate spheroidal shape. For such problems the relation of a, b, ℓ and ϵ_r is found in the equations of Section II.

Taking the more general cylindrical transmission line with propagation parallel to the z-axis one first fixes the location of a cross section with respect to the z-axis. Then extend the conductors back toward the lens (negative z) until they reach the lens boundary forming paths of intersection (one for each conductor on the lens surface). Assuming no portions of these conductors bypass the lens (or readjusting the parameters to make this true) then from each path of intersection extend conducting conical sheets toward the conical apex. Next one can consider the transmission-line characteristic impedance in the two regions and attempt to match them (as nearly as possible) by adjustment of the various lens and cylindrical transmission-line parameters.

In calculating T_V the problem will be considerably complicated by the details of the two electric-field components on each ray first passing through the lens boundary and second reaching the aperture plane with each component having different values (or different magnitude and orientation) from those of the TEM mode of the cylindrical transmission line. Furthermore some rays may have $\theta > \theta_{b_{max}}$ (as in (2.18) so that they will not be deflected to paralleling the $+z$-axis outside the lens. Such rays then do not contribute to the integral for T_V.

Such a more general conical to cylindrical transmission-line lens will then need to be optimized for best matching. Location of the cylindrical-transmission-line cross section with respect to the prolate-spheroidal lens axis (z-axis) and scaling of ℓ (compared to cross-section dimensions) give some flexibility in this process. Just how well such lenses can perform remains to be seen.

Appendix A: Asymptotic Expansion of T_V for Small Characteristic Impedance

In evaluating the early-time voltage transfer through the lens boundary (to the coaxial TEM mode) we have (6.29) and (6.30) giving this in terms of ϵ_r and $\zeta = \ell n(\chi)$. For small impedances then ζ is the appropriate expansion parameter. Furthermore we already know that T_V is an even function of ζ which we will later see in even powers of ζ.

Write the basic formula as

$$\begin{aligned} T_V &= 1 - \frac{2}{[\epsilon_r - 1]\zeta} \ell n(1 + A) \\ 1 + A &= e^{-\frac{\epsilon_r - 1}{2}\zeta} \, \frac{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\sqrt{\frac{p}{q}}X}{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\sqrt{\frac{q}{p}}X} \\ A &= \frac{[\sqrt{\epsilon_r} + 1][e^{-\frac{\epsilon_r - 1}{2}\zeta} - 1] + 2\sqrt{\epsilon_r}[\sqrt{\frac{q}{p}} - \sqrt{\frac{p}{q}} e^{-\frac{\epsilon_r - 1}{2}\zeta}]X}{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\sqrt{\frac{q}{p}}X} \end{aligned} \qquad \text{(A.1)}$$

where we have

$$\begin{aligned} X &= \frac{\sinh\left(\frac{\sqrt{\epsilon_r} - 1}{2}\zeta\right)}{\sinh(\sqrt{\epsilon_r}\zeta)} \\ \sqrt{\frac{q}{p}} &= e^{\frac{\sqrt{\epsilon_r} + 1}{2}\zeta} \end{aligned} \qquad \text{(A.2)}$$

Rewrite A in the more convenient form

$$A = 2e^{-\frac{\epsilon_r - 1}{4}\zeta} \, \frac{-[\sqrt{\epsilon_r} + 1]\sinh\left(\frac{\epsilon_r - 1}{4}\zeta\right) + 2\sqrt{\epsilon_r}\sinh\left(\frac{[\sqrt{\epsilon_r} + 1]^2}{4}\zeta\right) X}{\sqrt{\epsilon_r} + 1 - 2\sqrt{\epsilon_r}\sqrt{\frac{q}{p}}X} \qquad \text{(A.3)}$$

Consider the terms in (A.3) for small ζ as $\zeta \to 0$

$$\begin{aligned}
e^{-\frac{\epsilon_r - 1}{4}\zeta} &= 1 + 0(\zeta) \\
X &= \frac{\frac{\sqrt{\epsilon_r} - 1}{2}\zeta + \frac{[\sqrt{\epsilon_r} - 1]^3}{48}\zeta^3 + 0(\zeta^5)}{\sqrt{\epsilon_r}\zeta + \frac{\epsilon_r^{3/2}}{6}\zeta^3 + 0(\zeta^5)} \\
&= \frac{\sqrt{\epsilon_r} - 1}{2\sqrt{\epsilon_r}} \frac{1 + \frac{[\sqrt{\epsilon_r} - 1]^2}{24}\zeta^2 + 0(\zeta^4)}{1 + \frac{\epsilon_r}{6}\zeta^2 + 0(\zeta^4)} \\
&= \frac{\sqrt{\epsilon_r} - 1}{2\sqrt{\epsilon_r}} \left[1 + \frac{1}{24}[[\sqrt{\epsilon_r} - 1]^2 - 4\epsilon_r]\zeta^2 + 0(\zeta^4)\right] \\
\sqrt{\frac{q}{p}} &= 1 + 0(\zeta)
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\sqrt{\epsilon_r} + 1 &- 2\sqrt{\epsilon_r}\sqrt{\frac{q}{p}}X = 2 + 0(\zeta) \\
\sinh\left(\frac{\epsilon_r - 1}{4}\zeta\right) &= \frac{\epsilon_r - 1}{4}\zeta + \frac{1}{6}\frac{[\epsilon_r - 1]^3}{64}\zeta^3 + 0(\zeta^5) \\
\sinh\left(\frac{[\sqrt{\epsilon_r} + 1]^2}{4}\zeta\right) &= \frac{[\sqrt{\epsilon_r} + 1]^2}{4}\zeta + \frac{1}{6}\frac{[\sqrt{\epsilon_r} + 1]^6}{64}\zeta^3 + 0(\zeta^5) \\
&- [\sqrt{\epsilon_r} + 1]\sinh\left(\frac{\epsilon_r - 1}{4}\zeta\right) + 2\sinh\left(\frac{[\sqrt{\epsilon_r} + 1]^2}{4}\zeta\right) X \\
&= -[\sqrt{\epsilon_r} + 1]\left\{\frac{\epsilon_r - 1}{4}\zeta + \frac{1}{6}\frac{[\epsilon_r - 1]^3}{64}\zeta^3\right\} \\
&+ [\sqrt{\epsilon_r} - 1]\left\{\frac{[\sqrt{\epsilon_r} + 1]^2}{4}\zeta\right. \\
&+ \left.\frac{1}{6}\frac{1}{64}\left[4[[\sqrt{\epsilon_r} - 1]^2 - 4\epsilon_r][\sqrt{\epsilon_r} + 1]^2 + [\sqrt{\epsilon_r} + 1]^6\right]\zeta^3\right\} \\
&= \frac{1}{96}[\epsilon_r - 1]^3 + 0(\zeta^5)
\end{aligned}$$

Collecting these we have

$$A = \frac{1}{96}[\epsilon_r - 1]^3\zeta^3 + 0(\zeta^5) \tag{A.5}$$

and hence

$$T_V = 1 - \frac{1}{48}[\epsilon_r - 1]^2 \zeta^2 + 0(\zeta^4) \tag{A.6}$$

Appendix B: Tabular Data

Here numerical tables are presented for various parameters of interest. Tables B.1 and B.2 are concerned with the lens under maximum impedance conditions as derived in Section III. In B.1 ϵ_r is varied through practical small values. In B.2 this is specialized to some important materials [6] where ϵ_r is taken for microwave (not optical) frequencies. Along with $Z_{c_{\max}}$ the associated lens angles $\theta_{1_{\min}}$ and $\theta_{2_{\max}}$ (with θ_B in between) are included (in degrees for convenience) as well as the sizing parameters $\chi_{\max}(= \Psi_2/\Psi_1)$ and ℓ/Ψ_2 (from Sections II and III). These tables are finished off with the high-frequency TEM-mode transfer function T_V (Section VI).

The remaining tables go into more detail. Tables B.3 (a through f) consider the selected special values of ϵ_r and vary $Z_c(< Z_{c_{max}})$ through an appropriate range to give the remaining lens parameters $\chi, \theta_1, \theta_2, \ell/\Psi_2$, and T_V. Since ϵ_r is fixed for each table a single value of the Brewster angle θ_B applies to the entire table. Tables B.4 (a through j) consider selected values of Z_c and vary ϵ_r up to 6.0 unless this makes Z_c exceed $Z_{c_{\max}}$ in which case ϵ_r is truncated before this condition is reached. Here, for small Z_c, we include $\theta_B - \Delta\theta$ and $\theta_B + \Delta\theta$ as good approximations to θ_1 and θ_2 respectively, where $\Delta\theta$ is taken as the first-order approximation (in $\zeta = \ell n(\chi)$) in (3.33). This approximation is seen from the tables to be good for small ϵ_r and small Z_c. Here T_V is also compared to its approximation T_{V_a} in (6.31) (through third order in ξ, noting only even powers appear). As can be seen from the tables this approximation is quite good for small Z_c. Since T_V is near 1.0 it is more convenient to look at $1 - T_V$ and $1 - T_{V_a}$ with a scale magnification ($\times 100$) to better appreciate the accuracy.

ε_r	χ_{max}	$Z_{c_{max}}$ (ohms)	$\theta_{1_{min}}$ (deg.)	θ_B (deg.)	$\theta_{2_{max}}$ (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100
1.200	1425.468	435.43	.00	5.22	24.09	4.69	1.864
1.300	140.438	296.48	.10	7.49	28.71	3.91	2.583
1.400	43.927	226.79	.38	9.59	32.31	3.45	3.193
1.500	21.798	184.78	.84	11.54	35.26	3.15	3.713
1.600	13.619	156.58	1.44	13.34	37.76	2.92	4.161
1.700	9.704	136.26	2.15	15.03	39.92	2.75	4.549
1.800	7.507	120.86	2.93	16.60	41.81	2.62	4.888
1.900	6.135	108.77	3.75	18.08	43.49	2.51	5.186
2.000	5.212	98.99	4.59	19.47	45.00	2.41	5.451
2.100	4.554	90.89	5.45	20.78	46.36	2.34	5.687
2.200	4.065	84.08	6.31	22.02	47.61	2.27	5.899
2.300	3.689	78.26	7.16	23.20	48.75	2.21	6.091
2.400	3.391	73.22	8.01	24.32	49.80	2.15	6.265
2.500	3.151	68.81	8.84	25.38	50.77	2.11	6.424
2.600	2.953	64.92	9.66	26.39	51.67	2.07	6.569
2.700	2.787	61.46	10.46	27.35	52.51	2.03	6.703
2.800	2.647	58.36	11.25	28.27	53.30	1.99	6.825
2.900	2.526	55.57	12.01	29.16	54.04	1.96	6.939
3.000	2.422	53.03	12.77	30.00	54.74	1.93	7.044
3.200	2.250	48.62	14.22	31.59	56.01	1.88	7.234
3.400	2.114	44.89	15.60	33.06	57.16	1.84	7.400
3.600	2.005	41.70	16.92	34.42	58.19	1.80	7.545
3.800	1.915	38.95	18.17	35.69	59.14	1.76	7.676
4.000	1.839	36.54	19.37	36.87	60.00	1.73	7.790
4.200	1.775	34.41	20.51	37.98	60.79	1.70	7.895
4.400	1.720	32.52	21.60	39.02	61.53	1.68	7.990
4.600	1.672	30.83	22.65	40.01	62.21	1.66	8.075
4.800	1.630	29.31	23.65	40.93	62.84	1.64	8.152
5.000	1.593	27.93	24.61	41.81	63.43	1.62	8.225
5.200	1.560	26.68	25.53	42.64	63.99	1.60	8.290
5.400	1.531	25.53	26.41	43.43	64.51	1.58	8.350
5.600	1.504	24.48	27.25	44.18	65.00	1.57	8.407
5.800	1.480	23.51	28.07	44.90	65.47	1.56	8.459
6.000	1.458	22.62	28.85	45.58	65.91	1.54	8.508

Table B.1. Lens parameters for maximum impedance condition as a function of ε_r.

ϵ_r	χ_{max}	$Z_{c_{max}}$ (ohms)	$\theta_{1_{min}}$ (deg.)	θ_B (deg.)	$\theta_{2_{max}}$ (deg.)	ℓ/Ψ_2	$(1-T_V) \times 100$
1.020	1.789E+30	4176.68	90.00	.57	8.05	14.21	.000
1.050	1.548E+12	1682.93	.00	1.40	12.60	9.05	.529
2.260	3.831E+00	80.54	6.82	22.74	48.30	2.23	6.025
2.550	3.048E+00	66.82	9.25	25.89	51.23	2.09	6.500
4.000	1.844E+00	36.68	19.32	36.87	60.00	1.73	7.861
6.000	1.461E+00	22.75	28.77	45.58	65.91	1.54	8.620
78.000	1.026E+00	1.54	70.79	77.08	83.50	1.12	9.669

ϵ_r	Material
1.02	foam polyethylene
1.05	foam polyethylene
2.26	polyethylene and transformer oil
2.55	polystyrene
4.00	typical of certain plastics
6.00	typical of glass
78.00	distilled water

Table B.2. Lens parameters for maximum impedance condition for special (practical) values of ϵ_r.

Z_c (ohms)	χ	θ_1 (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100
1.00	1.02	1.39	1.410	41.149	.0000
2.50	1.04	1.37	1.428	40.645	.0000
5.00	1.09	1.34	1.458	39.823	.0000
10.00	1.18	1.28	1.519	38.269	.0001
25.00	1.52	1.11	1.705	34.212	.0009
50.00	2.30	.86	2.022	29.061	.0035
100.00	5.30	.48	2.640	22.643	.0124
150.00	12.20	.25	3.204	19.024	.0241
200.00	28.10	.12	3.709	16.769	.0369

Table B.3a: Lens parameters versus Z_c for $\varepsilon_r = 1.05$ ($\theta_B = 1.398^\circ$) (foam polyethylene)

Z_c (ohms)	χ	θ_1 (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100
1.00	1.02	22.46	23.016	2.962	.0009
2.50	1.04	22.05	23.436	2.926	.0057
5.00	1.09	21.37	24.146	2.870	.0230
10.00	1.18	20.05	25.593	2.766	.0920
15.00	1.28	18.77	27.075	2.675	.2069
20.00	1.40	17.55	28.590	2.596	.3678
25.00	1.52	16.37	30.134	2.527	.5744
30.00	1.65	15.25	31.703	2.467	.8269
35.00	1.79	14.17	33.294	2.416	1.1253
40.00	1.95	13.16	34.905	2.372	1.4696
45.00	2.12	12.19	36.531	2.336	1.8600
50.00	2.30	11.28	38.170	2.306	2.2970
55.00	2.50	10.42	39.820	2.281	2.7809
60.00	2.72	9.62	41.477	2.262	3.3127
65.00	2.96	8.86	43.140	2.248	3.8932
70.00	3.21	8.15	44.806	2.238	4.5238
75.00	3.49	7.49	46.474	2.232	5.2063
80.00	3.80	6.88	48.142	2.230	5.9427

Table B.3b. Lens parameters versus Z_c for $\varepsilon_r = 2.26$ ($\theta_B = 22.74^\circ$) (polyethylene and transformer oil)

Z_c (ohms)	χ	θ_1 (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100
1.00	1.02	25.56	26.223	2.653	.0014
2.50	1.04	25.06	26.728	2.621	.0087
5.00	1.09	24.25	27.580	2.571	.0348
10.00	1.18	22.67	29.321	2.480	.1393
15.00	1.28	21.15	31.108	2.402	.3137
20.00	1.40	19.69	32.936	2.335	.5584
25.00	1.52	18.30	34.800	2.277	.8740
30.00	1.65	16.97	36.695	2.229	1.2612
35.00	1.79	15.71	38.619	2.189	1.7213
40.00	1.95	14.52	40.565	2.157	2.2556
45.00	2.12	13.39	42.530	2.131	2.8662
50.00	2.30	12.33	44.510	2.112	3.5555
55.00	2.50	11.34	46.501	2.098	4.3266
60.00	2.72	10.41	48.499	2.090	5.1837
65.00	2.96	9.55	50.502	2.086	6.1315

Table B.3c. Lens parameters versus Z_c for $\varepsilon_r = 2.55$ ($\theta_B = 25.89^\circ$) (polystyrene)

Z_c (ohms)	χ	θ_1 (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100
1.00	1.02	36.30	37.445	1.984	.0052
2.50	1.04	35.45	38.317	1.960	.0326
5.00	1.09	34.06	39.791	1.924	.1306
10.00	1.18	31.37	42.812	1.863	.5255
15.00	1.28	28.81	45.921	1.816	1.1939
20.00	1.40	26.39	49.106	1.780	2.1522
25.00	1.52	24.10	52.351	1.755	3.4252
30.00	1.65	21.96	55.645	1.739	5.0494
35.00	1.79	19.96	58.972	1.732	7.0764

Table B.3d. Lens parameters versus Z_c for $\varepsilon_r = 4.00$ ($\theta_B = 36.87^\circ$) (typical of certain plastics)

Z_c (ohms)	χ	θ_1 (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100
1.00	1.02	44.75	46.424	1.676	.0145
2.50	1.04	43.52	47.697	1.657	.0907
3.00	1.05	43.11	48.125	1.651	.1307
4.00	1.07	42.30	48.985	1.640	.2327
5.00	1.09	41.50	49.851	1.629	.3644
6.00	1.11	40.71	50.724	1.619	.5261
7.00	1.12	39.92	51.603	1.609	.7184
8.00	1.14	39.15	52.487	1.601	.9419
9.00	1.16	38.38	53.377	1.593	1.1971
10.00	1.18	37.62	54.272	1.585	1.4850
15.00	1.28	33.97	58.817	1.558	3.4507
20.00	1.40	30.55	63.451	1.544	6.4363
21.00	1.42	29.89	64.385	1.543	7.1801
22.00	1.44	29.25	65.322	1.543	7.9801

Table B.3e. Lens parameters versus Z_c for $\varepsilon_r = 6.00$ ($\theta_B = 45.585^o$) (typical of glass)

Z_c (ohms)	χ	θ_1 (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100
.10	1.00	76.67	77.492	1.127	.0360
.20	1.00	76.26	77.904	1.126	.1378
.30	1.01	75.85	78.316	1.125	.3115
.40	1.01	75.44	78.729	1.124	.5551
.50	1.01	75.03	79.142	1.124	.8722
.60	1.01	74.62	79.555	1.123	1.2650
.70	1.01	74.21	79.969	1.123	1.7366
.80	1.01	73.81	80.383	1.122	2.2902
.90	1.02	73.40	80.797	1.122	2.9314
1.00	1.02	72.99	81.212	1.121	3.6654
1.05	1.02	72.79	81.419	1.121	4.0693
1.10	1.02	72.59	81.627	1.121	4.4992
1.15	1.02	72.39	81.834	1.121	4.9559
1.20	1.02	72.18	82.042	1.121	5.4409
1.25	1.02	71.98	82.249	1.121	5.9554
1.30	1.02	71.78	82.457	1.121	6.5008
1.35	1.02	71.58	82.665	1.121	7.0786
1.40	1.02	71.38	82.872	1.121	7.6904
1.45	1.02	71.17	83.080	1.120	8.3382
1.50	1.03	70.97	83.288	1.120	9.0240

Table B.3f. Lens parameters versus Z_c for $\varepsilon_r = 78.0$ ($\theta_B = 77.08^o$) (distilled water)

ε_r	θ_1 (deg.)	$\theta_B-\Delta\theta$ (deg.)	θ_B (deg.)	$\theta_B+\Delta\theta$ (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100	$(1-T_{Va})$ ×100
1.01	.28	.28	.29	.29	.29	199.857	.0000	.0000
1.02	.56	.56	.57	.57	.57	100.655	.0000	.0000
1.05	1.39	1.39	1.40	1.41	1.41	41.151	.0000	.0000
1.10	2.71	2.71	2.73	2.75	2.75	21.309	.0000	.0000
1.20	5.17	5.17	5.22	5.26	5.26	11.381	.0000	.0000
1.30	7.42	7.42	7.49	7.57	7.57	8.066	.0000	.0001
1.40	9.50	9.50	9.59	9.69	9.69	6.404	.0001	.0001
1.50	11.42	11.42	11.54	11.66	11.66	5.404	.0001	.0001
1.60	13.20	13.20	13.34	13.48	13.48	4.735	.0002	.0002
1.70	14.86	14.86	15.03	15.19	15.19	4.255	.0003	.0003
1.80	16.42	16.42	16.60	16.79	16.79	3.894	.0003	.0004
1.90	17.87	17.87	18.08	18.29	18.29	3.612	.0005	.0005
2.00	19.24	19.24	19.47	19.70	19.70	3.386	.0006	.0006
2.10	20.54	20.54	20.78	21.03	21.03	3.200	.0007	.0007
2.20	21.76	21.76	22.02	22.29	22.29	3.044	.0008	.0009
2.30	22.91	22.91	23.20	23.49	23.49	2.911	.0010	.0010
2.40	24.01	24.01	24.32	24.62	24.62	2.797	.0011	.0012
2.50	25.05	25.05	25.38	25.70	25.71	2.698	.0013	.0013
2.60	26.04	26.04	26.39	26.73	26.73	2.611	.0015	.0015
2.70	26.99	26.99	27.35	27.72	27.72	2.534	.0017	.0017
2.80	27.89	27.89	28.27	28.66	28.66	2.465	.0019	.0019
2.90	28.76	28.75	29.16	29.56	29.56	2.402	.0021	.0021
3.00	29.58	29.58	30.00	30.42	30.42	2.346	.0023	.0024
3.20	31.14	31.14	31.59	32.04	32.04	2.249	.0028	.0029
3.40	32.57	32.57	33.06	33.54	33.54	2.167	.0033	.0034
3.60	33.90	33.90	34.42	34.94	34.94	2.097	.0039	.0040
3.80	35.14	35.14	35.69	36.23	36.24	2.036	.0046	.0046
4.00	36.29	36.29	36.87	37.45	37.45	1.983	.0052	.0053
4.20	37.37	37.37	37.98	38.59	38.59	1.937	.0060	.0061
4.40	38.39	38.38	39.02	39.66	39.66	1.895	.0069	.0068
4.60	39.34	39.34	40.01	40.67	40.67	1.858	.0076	.0077
4.80	40.24	40.24	40.93	41.63	41.63	1.825	.0085	.0085
5.00	41.09	41.09	41.81	42.53	42.53	1.794	.0094	.0095
5.20	41.90	41.90	42.64	43.39	43.39	1.766	.0103	.0104
5.40	42.66	42.66	43.43	44.20	44.21	1.741	.0114	.0115
5.60	43.39	43.39	44.18	44.98	44.98	1.718	.0126	.0125
5.80	44.08	44.08	44.90	45.72	45.73	1.696	.0136	.0136
6.00	44.74	44.74	45.58	46.43	46.43	1.676	.0148	.0148

Table B.4a. Lens parameters versus ε_r for $Z_c = 1\ \Omega$ ($\chi = 1.017$)

ϵ_r	θ_1 (deg.)	$\theta_B-\Delta\theta$ (deg.)	θ_B (deg.)	$\theta_B+\Delta\theta$ (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100	$(1-T_{Va})$ ×100
1.01	.28	.28	.29	.29	.29	197.368	.0000	.0000
1.02	.56	.56	.57	.58	.58	99.412	.0000	.0000
1.05	1.37	1.37	1.40	1.43	1.43	40.636	.0000	.0000
1.10	2.67	2.67	2.73	2.79	2.79	21.044	.0000	.0000
1.20	5.10	5.10	5.22	5.34	5.34	11.240	.0001	.0001
1.30	7.32	7.32	7.49	7.67	7.67	7.966	.0003	.0003
1.40	9.36	9.36	9.59	9.83	9.83	6.325	.0006	.0006
1.50	11.24	11.24	11.54	11.83	11.83	5.337	.0009	.0009
1.60	12.99	12.99	13.34	13.69	13.70	4.677	.0013	.0013
1.70	14.62	14.62	15.03	15.43	15.44	4.203	.0018	.0018
1.80	16.14	16.14	16.60	17.06	17.07	3.847	.0024	.0024
1.90	17.57	17.56	18.08	18.60	18.60	3.568	.0030	.0030
2.00	18.91	18.90	19.47	20.04	20.04	3.344	.0037	.0037
2.10	20.17	20.16	20.78	21.40	21.41	3.161	.0045	.0045
2.20	21.36	21.35	22.02	22.70	22.70	3.007	.0053	.0053
2.30	22.48	22.48	23.20	23.92	23.93	2.876	.0062	.0062
2.40	23.55	23.55	24.32	25.09	25.09	2.763	.0072	.0072
2.50	24.57	24.56	25.38	26.19	26.20	2.665	.0083	.0083
2.60	25.53	25.52	26.39	27.25	27.26	2.579	.0095	.0095
2.70	26.45	26.44	27.35	28.26	28.27	2.503	.0107	.0107
2.80	27.33	27.32	28.27	29.23	29.24	2.435	.0119	.0120
2.90	28.16	28.15	29.16	30.16	30.16	2.373	.0133	.0133
3.00	28.96	28.96	30.00	31.04	31.05	2.318	.0148	.0148
3.20	30.47	30.46	31.59	32.72	32.73	2.222	.0179	.0179
3.40	31.85	31.84	33.06	34.27	34.28	2.141	.0213	.0213
3.60	33.14	33.12	34.42	35.71	35.72	2.072	.0249	.0250
3.80	34.33	34.31	35.69	37.06	37.07	2.012	.0289	.0290
4.00	35.44	35.42	36.87	38.32	38.33	1.960	.0332	.0332
4.20	36.47	36.46	37.98	39.50	39.52	1.914	.0378	.0378
4.40	37.45	37.43	39.02	40.62	40.63	1.873	.0427	.0427
4.60	38.36	38.34	40.01	41.67	41.68	1.836	.0479	.0479
4.80	39.22	39.20	40.93	42.66	42.68	1.803	.0533	.0533
5.00	40.03	40.01	41.81	43.61	43.63	1.773	.0591	.0591
5.20	40.80	40.78	42.64	44.51	44.52	1.746	.0652	.0651
5.40	41.53	41.51	43.43	45.36	45.38	1.721	.0716	.0715
5.60	42.22	42.20	44.18	46.17	46.19	1.698	.0782	.0781
5.80	42.87	42.85	44.90	46.95	46.97	1.676	.0852	.0851
6.00	43.50	43.47	45.58	47.69	47.72	1.657	.0924	.0923

Table B.4b. Lens parameters versus ϵ_r for $Z_c = 2.5\ \Omega$ ($\chi = 1.043$)

ε_r	θ_1 (deg.)	$\theta_B - \Delta\theta$ (deg.)	θ_B (deg.)	$\theta_B + \Delta\theta$ (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100	$(1-T_{Va})$ ×100
1.01	.27	.27	.29	.30	.30	193.409	.0000	.0000
1.02	.54	.54	.57	..59	.59	97.411	.0000	.0000
1.05	1.34	1.34	1.40	1.46	1.46	39.824	.0000	.0000
1.10	2.61	2.61	2.73	2.85	2.85	20.624	.0001	.0001
1.20	4.98	4.98	5.22	5.45	5.46	11.017	.0006	.0006
1.30	7.14	7.14	7.49	7.85	7.85	7.808	.0013	.0013
1.40	9.13	9.12	9.59	10.07	10.07	6.200	.0023	.0023
1.50	10.96	10.95	11.54	12.12	12.13	5.232	.0036	.0036
1.60	12.65	12.64	13.34	14.04	14.05	4.585	.0052	.0052
1.70	14.23	14.22	15.03	15.83	15.84	4.121	.0071	.0071
1.80	15.70	15.69	16.60	17.52	17.53	3.771	.0093	.0093
1.90	17.07	17.06	18.08	19.10	19.12	3.499	.0117	.0117
2.00	18.36	18.34	19.47	20.60	20.61	3.279	.0145	.0145
2.10	19.57	19.55	20.78	22.01	22.03	3.099	.0175	.0175
2.20	20.72	20.70	22.02	23.35	23.37	2.949	.0209	.0209
2.30	21.79	21.77	23.20	24.63	24.65	2.821	.0245	.0245
2.40	22.82	22.79	24.32	25.84	25.86	2.710	.0284	.0284
2.50	23.78	23.76	25.38	27.00	27.02	2.615	.0326	.0326
2.60	24.70	24.68	26.39	28.10	28.13	2.530	.0371	.0371
2.70	25.58	25.55	27.35	29.16	29.19	2.455	.0419	.0419
2.80	26.41	26.38	28.27	30.17	30.20	2.389	.0470	.0470
2.90	27.21	27.17	29.16	31.14	31.17	2.329	.0524	.0523
3.00	27.97	27.93	30.00	32.07	32.10	2.275	.0580	.0580
3.20	29.39	29.35	31.59	33.83	33.87	2.180	.0702	.0702
3.40	30.70	30.65	33.06	35.46	35.50	2.101	.0836	.0835
3.60	31.90	31.85	34.42	36.98	37.03	2.034	.0981	.0980
3.80	33.02	32.97	35.69	38.40	38.45	1.975	.1138	.1137
4.00	34.06	34.00	36.87	39.74	39.79	1.924	.1307	.1305
4.20	35.03	34.97	37.98	40.99	41.05	1.879	.1488	.1485
4.40	35.93	35.87	39.02	42.18	42.24	1.839	.1680	.1676
4.60	36.78	36.71	40.01	43.30	43.36	1.804	.1884	.1879
4.80	37.58	37.50	40.93	44.36	44.43	1.771	.2101	.2094
5.00	38.32	38.25	41.81	45.37	45.44	1.742	.2328	.2320
5.20	39.03	38.95	42.64	46.33	46.41	1.716	.2568	.2557
5.40	39.70	39.61	43.43	47.25	47.33	1.691	.2820	.2807
5.60	40.33	40.24	44.18	48.13	48.21	1.669	.3083	.3068
5.80	40.93	40.84	44.90	48.96	49.05	1.648	.3359	.3340
6.00	41.50	41.40	45.58	49.77	49.85	1.629	.3647	.3625

Table B.4c. Lens parameters versus ε_r for $Z_c = 5\ \Omega$ (χ = 1.087)

ε_r	θ_1 (deg.)	$\theta_B-\Delta\theta$ (deg.)	θ_B (deg.)	$\theta_B+\Delta\theta$ (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100	$(1-T_{Va})$ ×100
1.01	.26	.26	.29	.31	.31	185.855	.0000	.0000
1.02	.52	.52	.57	.61	.62	93.622	.0000	.0000
1.05	1.28	1.28	1.40	1.52	1.52	38.277	.0001	.0001
1.10	2.50	2.49	2.73	2.97	2.97	19.825	.0006	.0006
1.20	4.75	4.74	5.22	5.69	5.70	10.593	.0023	.0023
1.30	6.80	6.79	7.49	8.20	8.22	7.510	.0052	.0052
1.40	8.67	8.65	9.59	10.53	10.55	5.965	.0092	.0092
1.50	10.40	10.37	11.54	12.70	12.73	5.035	.0144	.0144
1.60	11.99	11.95	13.34	14.73	14.76	4.413	.0207	.0208
1.70	13.46	13.42	15.03	16.64	16.67	3.967	.0282	.0283
1.80	14.82	14.77	16.60	18.43	18.47	3.632	.0369	.0369
1.90	16.10	16.04	18.08	20.12	20.17	3.370	.0467	.0467
2.00	17.29	17.22	19.47	21.72	21.78	3.160	.0576	.0577
2.10	18.41	18.33	20.78	23.23	23.30	2.987	.0697	.0698
2.20	19.45	19.37	22.02	24.68	24.75	2.843	.0830	.0830
2.30	20.44	20.35	23.20	26.05	26.13	2.720	.0974	.0974
2.40	21.37	21.28	24.32	27.36	27.45	2.614	.1130	.1130
2.50	22.25	22.15	25.38	28.61	28.70	2.522	.1298	.1297
2.60	23.09	22.97	26.39	29.80	29.91	2.442	.1477	.1476
2.70	23.88	23.75	27.35	30.95	31.06	2.370	.1668	.1666
2.80	24.63	24.50	28.27	32.05	32.17	2.306	.1871	.1868
2.90	25.34	25.20	29.16	33.11	33.24	2.249	.2085	.2081
3.00	26.02	25.87	30.00	34.13	34.26	2.197	.2312	.2306
3.20	27.29	27.12	31.59	36.05	36.20	2.107	.2800	.2791
3.40	28.45	28.26	33.06	37.85	38.01	2.032	.3335	.3321
3.60	29.51	29.31	34.42	39.53	39.71	1.968	.3918	.3898
3.80	30.48	30.27	35.69	41.10	41.30	1.912	.4549	.4520
4.00	31.39	31.15	36.87	42.59	42.80	1.864	.5229	.5189
4.20	32.22	31.97	37.98	43.99	44.21	1.821	.5957	.5904
4.40	33.00	32.73	39.02	45.32	45.55	1.783	.6734	.6665
4.60	33.72	33.43	40.01	46.58	46.82	1.750	.7561	.7473
4.80	34.39	34.09	40.93	47.77	48.03	1.719	.8437	.8326
5.00	35.02	34.71	41.81	48.91	49.19	1.692	.9364	.9225
5.20	35.61	35.28	42.64	50.00	50.29	1.667	1.0342	1.0171
5.40	36.17	35.82	43.43	51.05	51.34	1.644	1.1371	1.1163
5.60	36.69	36.32	44.18	52.04	52.35	1.623	1.2452	1.2201
5.80	37.18	36.80	44.90	53.00	53.32	1.603	1.3586	1.3285
6.00	37.64	37.25	45.58	53.92	54.25	1.586	1.4773	1.4415

Table B.4d. Lens parameters versus ε_r for Z_c = 10 Ω (χ = 1.181)

ε_r	θ_1 (deg.)	$\theta_B-\Delta\theta$ (deg.)	θ_B (deg.)	$\theta_B+\Delta\theta$ (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100	$(1-T_{Va})$ ×100
1.01	.23	.23	.29	.34	.35	166.010	.0000	.0000
1.02	.45	.45	.57	.69	.69	83.638	.0001	.0001
1.05	1.11	1.10	1.40	1.70	1.70	34.214	.0009	.0009
1.10	2.16	2.13	2.73	3.33	3.34	17.737	.0036	.0036
1.20	4.09	4.03	5.22	6.40	6.45	9.494	.0143	.0145
1.30	5.82	5.72	7.49	9.27	9.34	6.743	.0322	.0326
1.40	7.38	7.24	9.59	11.95	12.05	5.365	.0573	.0579
1.50	8.79	8.61	11.54	14.46	14.60	4.537	.0896	.0905
1.60	10.09	9.86	13.34	16.83	17.00	3.984	.1291	.1303
1.70	11.27	10.99	15.03	19.06	19.28	3.588	.1758	.1773
1.80	12.35	12.03	16.60	21.18	21.44	3.290	.2299	.2316
1.90	13.35	12.97	18.08	23.19	23.49	3.059	.2913	.2931
2.00	14.27	13.84	19.47	25.10	25.44	2.873	.3600	.3618
2.10	15.13	14.64	20.78	26.92	27.31	2.721	.4363	.4378
2.20	15.92	15.38	22.02	28.66	29.09	2.594	.5200	.5210
2.30	16.66	16.07	23.20	30.33	30.81	2.486	.6113	.6115
2.40	17.35	16.70	24.32	31.93	32.45	2.394	.7102	.7091
2.50	17.99	17.29	25.38	33.47	34.03	2.314	.8168	.8141
2.60	18.59	17.83	26.39	34.94	35.55	2.243	.9313	.9262
2.70	19.16	18.34	27.35	36.37	37.01	2.182	1.0536	1.0456
2.80	19.69	18.81	28.27	37.74	38.43	2.127	1.1840	1.1723
2.90	20.18	19.25	29.16	39.06	39.79	2.077	1.3225	1.3061
3.00	20.65	19.66	30.00	40.34	41.11	2.033	1.4692	1.4472
3.20	21.51	20.40	31.59	42.77	43.63	1.957	1.7878	1.7512
3.40	22.27	21.05	33.06	45.06	45.99	1.893	2.1410	2.0840
3.60	22.95	21.61	34.42	47.22	48.23	1.840	2.5301	2.4458
3.80	23.56	22.11	35.69	49.26	50.34	1.794	2.9563	2.8366
4.00	24.11	22.54	36.87	51.20	52.34	1.755	3.4214	3.2563
4.20	24.60	22.92	37.98	53.04	54.25	1.720	3.9272	3.7049
4.40	25.05	23.26	39.02	54.79	56.06	1.690	4.4757	4.1825
4.60	25.45	23.54	40.01	56.47	57.80	1.664	5.0691	4.6890
4.80	25.81	23.80	40.93	58.07	59.46	1.641	5.7101	5.2245
5.00	26.14	24.01	41.81	59.61	61.05	1.620	6.4015	5.7889
5.20	26.44	24.20	42.64	61.08	62.57	1.601	7.1466	6.3823
5.40	26.71	24.36	43.43	62.51	64.04	1.585	7.9491	7.0046
5.60	26.96	24.49	44.18	63.88	65.45	1.570	8.8131	7.6558
5.50	26.84	24.43	43.82	63.21	64.76	1.577	8.3803	7.3320

Table B.4e. Lens parameters versus ε_r for $Z_c = 25\ \Omega$ ($\chi = 1.517$)

ε_r	θ_1 (deg.)	$\theta_B-\Delta\theta$ (deg.)	θ_B (deg.)	$\theta_B+\Delta\theta$ (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100	$(1-T_{Va})$ ×100
1.01	.18	.17	.29	.40	.41	140.666	.0001	.0001
1.02	.35	.33	.57	.81	.82	70.914	.0006	.0006
1.05	.86	.80	1.40	1.99	2.02	29.063	.0035	.0036
1.10	1.66	1.54	2.73	3.92	3.98	15.113	.0139	.0145
1.20	3.11	2.84	5.22	7.59	7.73	8.139	.0555	.0579
1.30	4.37	3.94	7.49	11.05	11.28	5.816	.1250	.1303
1.40	5.49	4.88	9.59	14.30	14.65	4.655	.2227	.2317
1.50	6.48	5.69	11.54	17.39	17.86	3.960	.3488	.3621
1.60	7.36	6.37	13.34	20.31	20.91	3.498	.5037	.5214
1.70	8.14	6.95	15.03	23.10	23.84	3.169	.6880	.7097
1.80	8.85	7.45	16.60	25.76	26.64	2.923	.9023	.9269
1.90	9.48	7.86	18.08	28.30	29.32	2.733	1.1471	1.1731
2.00	10.05	8.21	19.47	30.73	31.90	2.582	1.4235	1.4483
2.10	10.56	8.50	20.78	33.07	34.38	2.460	1.7322	1.7524
2.20	11.03	8.74	22.02	35.31	36.77	2.358	2.0744	2.0856
2.30	11.45	8.93	23.20	37.47	39.08	2.273	2.4512	2.4476
2.40	11.83	9.08	24.32	39.55	41.31	2.201	2.8641	2.8387
2.50	12.17	9.19	25.38	41.56	43.46	2.140	3.3145	3.2587
2.60	12.49	9.27	26.39	43.51	45.54	2.086	3.8042	3.7077
2.70	12.77	9.32	27.35	45.39	47.55	2.040	4.3351	4.1856
2.80	13.03	9.34	28.27	47.21	49.50	2.000	4.9093	4.6925
2.90	13.27	9.34	29.16	48.97	51.40	1.964	5.5294	5.2284
3.00	13.49	9.31	30.00	50.69	53.23	1.933	6.1981	5.7932
3.20	13.86	9.21	31.59	53.97	56.75	1.880	7.6937	7.0098
3.13	13.74	9.25	31.08	52.90	55.61	1.896	7.1744	6.5946

Table B.4f. Lens parameters versus ε_r for Z_c = 50 Ω (χ = 2.302)

ε_r	θ_1 (deg.)	$\theta_B-\Delta\theta$ (deg.)	θ_B (deg.)	$\theta_B+\Delta\theta$ (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100	$(1-T_{Va})$ ×100
1.01	.13	.11	.29	.46	.47	122.288	.0003	.0003
1.02	.27	.21	.57	.93	.94	61.707	.0012	.0013
1.05	.65	.50	1.40	2.29	2.34	25.361	.0074	.0081
1.10	1.24	.94	2.73	4.52	4.62	13.250	.0297	.0326
1.20	2.30	1.65	5.22	8.78	9.02	7.202	.1189	.1304
1.30	3.20	2.17	7.49	12.82	13.24	5.193	.2685	.2933
1.40	3.97	2.53	9.59	16.66	17.29	4.194	.4795	.5215
1.50	4.63	2.76	11.54	20.31	21.18	3.600	.7535	.8148
1.60	5.20	2.88	13.34	23.80	24.92	3.207	1.0925	1.1733
1.70	5.70	2.91	15.03	27.14	28.52	2.931	1.4990	1.5970
1.80	6.13	2.87	16.60	30.34	31.99	2.727	1.9760	2.0859
1.90	6.50	2.75	18.08	33.41	35.34	2.571	2.5274	2.6399
2.00	6.83	2.58	19.47	36.36	38.57	2.449	3.1576	3.2592
2.10	7.11	2.36	20.78	39.21	41.69	2.351	3.8718	3.9436
2.20	7.36	2.09	22.02	41.95	44.71	2.272	4.6763	4.6932
2.30	7.58	1.79	23.20	44.61	47.63	2.208	5.5786	5.5080
2.36	7.70	1.59	23.91	46.24	49.42	2.173	6.2031	6.0566

Table B.4g. Lens parameters versus ε_r for Z_c = 75 Ω (χ = 3.493)

ε_r	θ_1 (deg.)	$\theta_B-\Delta\theta$ (deg.)	θ_B (deg.)	$\theta_B+\Delta\theta$ (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100	$(1-T_{Va})$ ×100
1.01	.10	.05	.29	.52	.53	108.668	.0005	.0006
1.02	.20	.09	.57	1.05	1.06	54.899	.0020	.0023
1.05	.48	.20	1.40	2.59	2.64	22.643	.0124	.0145
1.10	.91	.34	2.73	5.12	5.23	11.899	.0496	.0580
1.20	1.66	.46	5.22	9.97	10.29	6.542	.1992	.2318
1.30	2.28	.39	7.49	14.60	15.18	4.770	.4511	.5216
1.40	2.79	.17	9.59	19.02	19.91	3.895	.8086	.9273
1.50	3.22	-.17	11.54	23.24	24.49	3.378	1.2767	1.4489
1.60	3.58	-.61	13.34	27.29	28.92	3.041	1.8615	2.0864
1.70	3.88	-1.13	15.03	31.18	33.21	2.807	2.5711	2.8398
1.80	4.13	-1.71	16.60	34.92	37.37	2.637	3.4155	3.7092
1.90	4.34	-2.36	18.08	38.52	41.38	2.511	4.4075	4.6944
1.99	4.50	-2.97	19.32	41.61	44.84	2.424	5.4222	5.6641

Table B.4h. Lens parameters versus ε_r for Z_c = 100 Ω (χ = 5.301)

ε_r	θ_1 (deg.)	$\theta_B-\Delta\theta$ (deg.)	θ_B (deg.)	$\theta_B+\Delta\theta$ (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100	$(1-T_{Va})$ ×100
1.01	.05	-.07	.29	.64	.64	90.298	.0010	.0013
1.02	.10	-.15	.57	1.28	1.28	45.745	.0038	.0052
1.05	.25	-.39	1.40	3.19	3.20	19.024	.0241	.0326
1.10	.46	-.85	2.73	6.31	6.39	10.133	.0968	.1304
1.20	.82	-1.92	5.22	12.35	12.68	5.719	.3918	.5216
1.30	1.10	-3.16	7.49	18.15	18.88	4.277	.8955	1.1735
1.40	1.31	-4.54	9.59	23.73	24.96	3.578	1.6241	2.0863
1.50	1.48	-6.02	11.54	29.09	30.93	3.177	2.6003	3.2598
1.60	1.61	-7.58	13.34	34.26	36.75	2.925	3.8553	4.6941
1.63	1.64	-8.05	13.85	35.74	38.42	2.870	4.2797	5.1617

Table B.4i. Lens parameters versus ε_r for Z_c = 150 Ω (χ = 12.204)

ε_r	θ_1 (deg.)	$\theta_B - \Delta\theta$ (deg.)	θ_B (deg.)	$\theta_B + \Delta\theta$ (deg.)	θ_2 (deg.)	ℓ/Ψ_2	$(1-T_V)$ ×100	$(1-T_{Va})$ ×100
1.01	.03	-.19	.29	.76	.74	78.652	.0015	.0023
1.02	.05	-.39	.57	1.52	1.48	39.966	.0059	.0093
1.05	.12	-.99	1.40	3.79	3.71	16.769	.0369	.0579
1.10	.23	-2.04	2.73	7.50	7.43	9.063	.1490	.2318
1.20	.39	-4.30	5.22	14.73	14.90	5.260	.6098	.9272
1.30	.51	-6.72	7.49	21.71	22.36	4.038	1.4127	2.0862
1.40	.59	-9.25	9.59	28.44	29.79	3.465	2.6037	3.7088
1.46	.63	-10.78	10.75	32.27	34.10	3.260	3.5075	4.8741

Table B.4j. Lens parameters versus ε_r for $Z_c = 200\ \Omega$ ($\chi = 28.096$)

References

[1] C. E. Baum, "Impedances and Field Distributions for Parallel Plate Transmission Line Simulators," Sensor and Simulation Note 21, June 1966.

[2] C. E. Baum, "A Circular Conical Antenna Simulator," Sensor and Simulation Note 36, March 1967.

[3] C. E. Baum, "General Principles for the Design of ATLAS I and II," Part V: Some Approximate Figures of Merit for Comparing the Waveforms Launched by Imperfect Pulser Arrays onto TEM Tarnsmission Lines, Sensor and Simulation Note 148, May 1972.

[4] C. E. Baum, "The Reflection of Pulsed Waves From the Surface of a Conducting Dielectric," Theoretical Note 25, February 1967.

[5] J. A. Stratton, *Electromagnetic Theory*, McGraw Hill, 1941.

[6] H. P. Westman (ed.) *Reference Data for Radio Engineers, 4th ed.*, IT&T, 1956.

[7] R. E. Collin, *Field Theory of Guided Waves*, McGraw Hill, 1960.

[8] R. V. Churchill, *Complex Variables and Applications*, McGraw Hill, 1960.

[9] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, AMS 55, U. S. Gov't Printing Office, 1964.

[10] G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers, 2nd ed.*, McGraw Hill, 1968.

[11] W. R. Smythe, *Static and Dynamic Electricity, 3rd ed.*, Hemisphere Publishing Corp. 1989.

[12] C. E. Baum and A. P. Stone, *Transient Lens Synthesis: Differential Geometry in Electromagnetic Theory*, Hemisphere Publishing Corp., 1991.