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Theoretical Techniques and Computational Considerations for Determining the Electromagnetic Fields of a Biconical Antenna with Resistive Loading*

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ABSTRACT

This study develops a methodology for calculating the fields of a biconical antenna with resistive loading that appears to be computationally feasible. The resistively loaded case is found to require additional terms in the antenna region (compared to the lossless case) so that the interior boundary condition is satisfied. The solution of the interior boundary value equation requires the use of an orthogonal basis of functions in the radial direction, which reduces the problem to one that can be solved in matrix form. The computational implementation of this theory remains to be executed.

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I. INTRODUCTION

Wide-angle biconical antennas have been used for about two decades to produce both vertically and horizontally polarized electromagnetic fields for use in electromagnetic pulse (EMP) simulation. Approximate theoretical models have been used in conjunction with field mapping to provide a practical basis for testing and evaluating military systems against the EMP threat.

In recent years, however, there has been a greater interest in the development of biconical antenna EMP simulators that can deliver pulses with very fast risetimes. A major element of the waveform design for this type of simulator is the resistive loading of the antenna, which predominantly affects the intermediate- to late-time character of the radiated field. The issue of resistive loading was initially addressed by Baum [1] who rendered an approximate theory of a resistively loaded dipole. Wilton [2,3] has applied numerical techniques to address the problem of a loaded dipole over a ground plane. In this study we develop an analytical approach for predicting the interior and far fields of a resistively loaded wide-angle biconical antenna.

Schelkunoff [4] appears to have been the first to develop an exact theory of the biconical antenna in the absence of resistive loading. His method, although rigorously correct, presented formidable computational difficulties at the time of its inception (1941). The principal difficulties appeared to be the computation of the θ-dependent eigenfunctions that are characteristic of the biconical antenna with perfectly conducting interior surfaces. As shown in the appendix, this calculation requires the determination of the roots of polynomial equations involving hundreds of terms. While this computation is feasible today, it was virtually insurmountable in Schelkunoff's time.

Because of these numerical problems, emphasis was placed on obtaining approximate solutions in selected regimes. For example, Tai [5] developed a method for small-angle bicones, which is also briefly summarized by Kraus [6]. Techniques relevant to wide-angle bicones were developed by Smith [7] and Tai [8]. In all these cases, however, the issue of resistive loading was not addressed; perfectly interior conducting boundaries were assumed.

The purpose of this investigation is to theoretically evaluate the possibility of rigorously solving the biconical antenna problem with resistive loading. It appears that, at least in a formal sense, I have been successful inasmuch as the methodology leads to closure on a solution. My method is an extension of the Schelkunoff formalism in the absence of resistive loading. Since this case forms the basic building block for the method, I provide a comprehensive discussion of Schelkunoff's methodology [9] using Kong's formalism [10]. This review is rendered in section 2.

In section 3 I extend the theory for the case of a resistively loaded bicone. The resistively loaded case is found to require additional terms in the antenna region, compared to the lossless case, so that the interior boundary condition is satisfied. The resulting equations do not lend themselves to analytical solutions in closed form. Closure is achieved through the introduction of an orthogonal basis of radial functions, which ultimately reduces the problem to one that can be solved in matrix form. The computational implementation of this theory remains to be developed.

A key element in the computational algorithm is the evaluation of the θ-dependent eigenvalues, u_i, and eigenfunctions, $T_{u_i}(\cos\theta_0)$ of the biconical antenna in the absence of resistive loading. These entities are essential in determining the parameters of the matrix equation leading to the determination of the fields. The computational considerations for $T_{u_i}(\cos\theta_0)$ are presented in the appendix.

2. REVIEW OF SCHELKUNOFF'S METHOD USING KONG'S FORMALISM

This section establishes the basic formalism that will be used in the general treatment of the resistively loaded biconical antenna of section 3. This material is extracted from Kong's text [10]. No attempt is made to reproduce the step-by-step derivation of the results. The reader is referred to Kong's text for more details. Wherever possible I have used Kong's notation, the notable exception being the use of "j" for his "$-i$."

This summary is rendered in a manner which highlights those changes that are necessary to extend the theory from the lossless case to the resistively loaded bicone.

Figure 1 shows a model of the biconical antenna, and the corresponding coordinate system. For the cases of interest only radial currents are present, with the accompanying conditions

$$H_r = H_\theta = E_\phi = 0 \ , \tag{1}$$

$$\frac{\partial}{\partial \phi} = 0 \ . \tag{2}$$

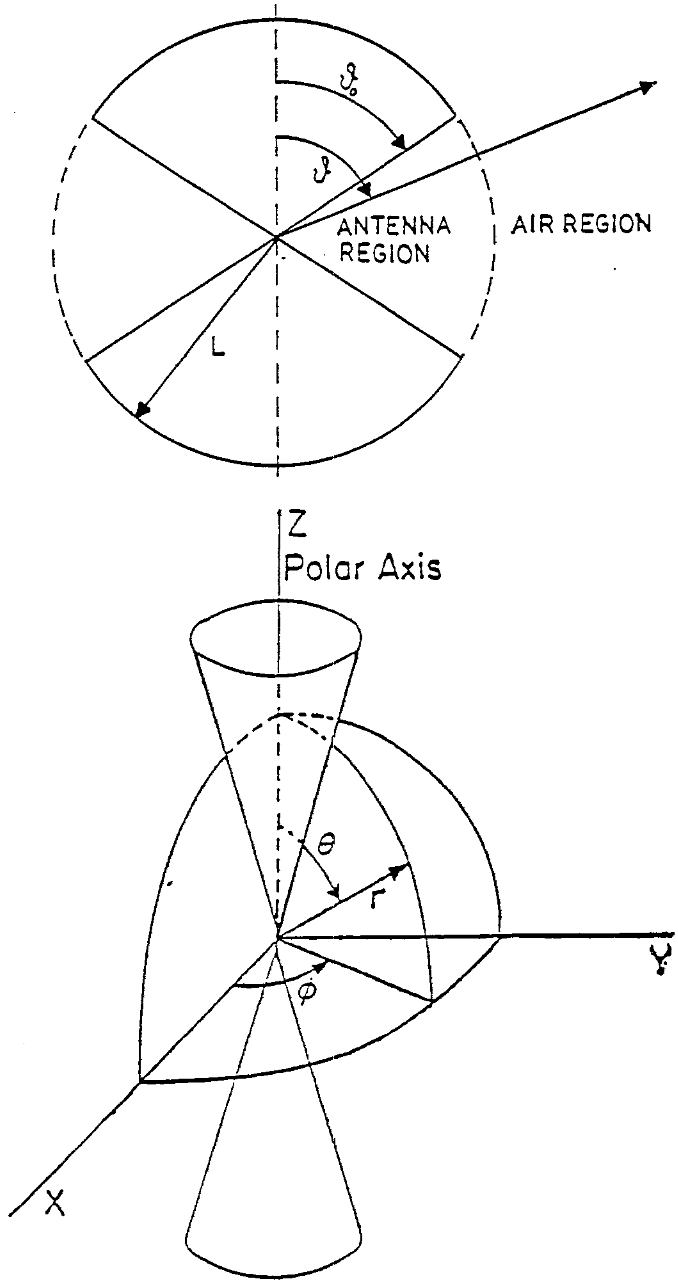

Figure 1. Geometric considerations for biconical antenna.

Under the foregoing conditions, Maxwell's equations reduce to

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial \theta}\right] = -j\omega\mu_0 H_\phi \ , \tag{3}$$

$$\frac{1}{r \sin\theta}\left[\frac{\partial}{\partial \theta}(H_\phi \sin\theta)\right] = j\omega\epsilon_0 E_r \ , \tag{4}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) = j\omega\epsilon_0 E_\theta \ , \tag{5}$$

where we have replaced $\partial/\partial t$ by $j\omega$.

In the air region, defined by the regime

$$r \geq L = \text{radius of bicone} \ , \tag{6}$$

the solution of equations (3) to (5) using separation of variables yields

$$H_\phi = \frac{1}{2\pi}\sum_{N=1}^{\infty} b_N h_N^{(2)}(kr)\dot{P}_N(\cos\theta) \ , \tag{7}$$

$$E_r = \frac{jZ_0}{2\pi kr}\sum_{N=1}^{\infty} N(N+1) b_N h_N^{(2)}(kr)\dot{P}_N(\cos\theta) \ , \tag{8}$$

$$E_\theta = \frac{jZ_0}{2\pi kr}\sum_{N=1}^{\infty} b_N \frac{d}{dr}\left[rh_N^{(2)}(kr)\right]\dot{P}_N(\cos\theta) \ , \tag{9}$$

where $P_N(\cos\theta)$ is the Legendre polynomial of order N, $h_N^{(2)}(kr)$ is the Hankel function of the second kind of degree N, the b_N's are constants to be determined from the solution of the problem, Z_0 is the impedance of free space $= \sqrt{\mu_0/\varepsilon_0}$, and

$$\dot{P}_N(\cos\theta) = dP_N(\cos\theta)/d\theta \ . \tag{10}$$

It is important to note that the functional form of equations (7) to (9) will be the same with or without resistive loading. The mathematical structure of the external fields stems from the requirement that only outgoing waves be present outside the antenna, as well as the condition of symmetry,

$$H_\phi(\pi-\theta) = H_\phi(\theta) \ , \qquad \text{(a)}$$

$$E_\theta(\pi-\theta) = E_\theta(\theta) \ . \qquad \text{(b)} \qquad (11)$$

When equation (11) applies, only odd values of N are allowed in the summations of equations (7) to (9). The first Hankel function, $h_1^{(2)}$, is given by

$$h_1^{(2)}(kr) = -\left[1-\frac{j}{kr}\right]\frac{e^{-jkr}}{kr} \ , \qquad (12)$$

while the first Legendre polynomial and its derivative are given by

$$P_1 = \cos\theta \ , \qquad (13)$$

$$\dot{P}_1 = -\sin\theta \ . \qquad (14)$$

Using equations (12) to (14), the leading terms of H_ϕ, E_θ, and E_r in the far field ($kr >> 1$) become

$$H_\phi = (-b_1\dot{P}_1)\frac{e^{-jkr}}{kr} \ , \qquad (15)$$

$$E_\theta = Z_o(-b_1\dot{P}_1)\frac{e^{-jkr}}{kr} = Z_o H_\theta \ , \qquad (16)$$

$$E_r = jZ_o(-2b_1P_1)\frac{e^{-jkr}}{(kr)^2} \ . \qquad (17)$$

Examination of equations (15) to (17) shows that for practical purposes a determination of b_1 will be sufficient to determine the fields in the test volume.

After some mathematical manipulation it can be shown that the interior fields, valid in the region $\theta_0 \le \theta \le \pi - \theta_0$, can be written in the form

$$H_\phi = \frac{I_0(r)}{2\pi r \sin\vartheta} + \frac{1}{2\pi}\sum_u a_u j_u(kr)\dot{T}_u(\cos\theta) \ , \qquad (18)$$

$$E_r = \frac{jZ_0}{2\pi kr} \sum_u u(u+1)\, a_u j_u(kr) T_u(\cos\theta) \ , \tag{19}$$

$$E_\theta = \frac{Z_0 V_0(r)}{Z_c 2\pi r \sin\theta} + \frac{jZ_0}{2\pi kr} \sum_u a_u \frac{d}{dr}[r j_u(kr)] \dot{T}_u(\cos\theta) \ , \tag{20}$$

where

$$T_u = \frac{1}{2}\left[P_u(\cos\theta) - P_u(-\cos\theta)\right] \ , \tag{21}$$

$$\dot{T}_u = dT_u/d\theta \ , \tag{22}$$

$$I_0(r) = \frac{V_0(L)}{Z_c}\left[j \sin k(L-r) + Y_t Z_c \cos k(L-r)\right] \ , \tag{23}$$

$$V_0(r) = V_0(L)\left[\cos k(L-r) + jY_t Z_c \, \text{sinc}\, k(L-r)\right] \ , \tag{24}$$

$$Z_c = \frac{Z_0 \ln \cot \frac{\theta_0}{2}}{\pi} = \text{characteristic impedance of bicone} \ , \tag{25}$$

$j_u(kr)$ is the spherical Bessel function, and the a_u's are constants, which, like the b_N's of the exterior region, are to be determined from the solution to the problem. The terminating admittance Y_t is likewise determined from the solution.

In contrast to equations (7) to (9), in which the summation index is defined, the values of u in equations (18) to (20) are determined from the boundary condition in the antenna region. In the absence of resistive loading we require that

$$E_r(\theta_0) = E_r(\pi - \theta_0) = 0 \ . \tag{26}$$

Based on equation (19), the specific values of u which will satisfy equation (26) are solutions of the equation

$$T_u(\cos\theta_0) = T_u(\cos(\pi - \theta_0)) = 0 \ , \qquad \text{(a)}$$

which reduce to (27)

$$P_u(\cos\theta_0) - P_u(-\cos\theta_0) = 0 \ . \qquad \text{(b)}$$

The P_u's of equation (27) are not Legendre polynomials, but are Legendre functions since u is not generally an integer. Appropriate analytical expressions for P_u that are necessary to solve equation (27), and hence determine the values of u, can be found in Schelkunoff [9], Abramowitz and Stegor [11], and Erdélyi et al [12]. For the time being we simply assume that equation (27) has been solved and the values of u determined. Thus, the summation index of equations (18) to (20) is defined.

The complete solution to the problem is now found by matching the boundary conditions at $r = L$. Using equation (23) we have

$$I_0(L) = Y_t V_0(L) \quad , \tag{28}$$

which when inserted in equation (18) gives

$$H_\phi(r = L) = \frac{Y_t V_0(L)}{2\pi L \sin\theta} + \frac{1}{2\pi} \sum_{u'} a_{u'} j_{u'}(kL)\, \dot{T}_{u'}(\cos\theta) \ . \tag{29}$$

Multiplying both sides of equation (29) by $\sin\theta\, \dot{T}_{u'}(\cos\theta)$ and integrating from θ_0 to $\pi - \theta_0$ gives

$$a_u = \frac{2\pi}{N_u j_u(kL)} \int_{\theta_0}^{\pi-\theta_0} \sin\theta H_\phi(r = L) \dot{T}_u(\cos\theta)\, d\theta \ , \tag{30}$$

where we have used the normalization

$$\int_{\theta_0}^{\pi-\theta_0} \sin\theta \dot{T}_u(\cos\theta)\, \dot{T}_{u'}(\cos\theta)\, d\theta = \begin{cases} 0 \text{ if } u \neq u' \\ N_u \text{ if } u = u' \end{cases} \ . \tag{31}$$

Equation (31) is a general property of the Legendre functions which satisfies the boundary condition of equation (27), with N_u being the associated normalization constant.

Equation (30) provides a connection between the a_u's and the b_N's through substitution of equation (7). We have

$$a_u = \sum_{N=1}^{\infty} \alpha_{uN} b_N \quad , \tag{32}$$

where

$$\alpha_{uN} = \frac{h_N^{(2)}(kL)}{N_u j_u(kL)} \int_{\theta_0}^{\pi-\theta_0} \sin\theta\, \dot{T}_u(\cos\theta)\, \dot{P}_N(\cos\theta)\, d\theta \ . \tag{33}$$

We should also recall in passing that the functional form of H_ϕ and E_θ, as given by equations (18) and (20), respectively, ensures that for both exterior and interior regions the following conditions will be satisfied:

$$H_\phi(\theta) = H_\phi(\pi - \theta) \quad , \qquad \text{(a)}$$

$$E_\theta(\theta) = E_\theta(\pi - \theta) \quad . \qquad \text{(b)} \qquad (34)$$

If we now define a_u and b_N to be the components of column vectors $\hat{a}$ and $\hat{b}$, respectively, and α_{uN} to be the elements of a matrix $\tilde{A}$, we can write equation (32) in the compact matrix form

$$\hat{a} = \tilde{A}\hat{b} \quad . \qquad (35)$$

The second relationship between the a_u's and b_N's is obtained by requiring continuity of E_θ at the boundary. From equation (9) we have

$$LE_\theta(r = L) = \frac{jZ_0}{2\pi k} \sum_{N=1}^{\infty} b_N G_N(kL) \dot{P}_N(\cos\theta) \quad , \qquad (36)$$

where

$$G_N(kL) = \left\{ \frac{d}{dr} \left[r h_N^{(2)}(kr) \right] \right\}_{r=L} \quad . \qquad (37)$$

Multiplying both sides of equation (36) by $\sin\theta \, \dot{P}_N(\cos\theta)$ and integrating from 0 to π gives

$$b_N = \frac{2N(N+1)}{2N+1} \, \frac{2\pi k}{jZ_0 G_N(kL)} \int_0^\pi \sin\theta \, LE_\theta(r = L) \, \dot{P}_N(\cos\theta) \, d\theta \quad , \qquad (38)$$

where we have used the relationship

$$\int_0^\pi \sin\theta \, d\theta \, \dot{P}_N(\cos\theta) \, \dot{P}_{N'}(\cos\theta) = \begin{cases} \dfrac{2N+1}{2N(N+1)} & \text{if } N = N' \\ 0 & \text{if } N \neq N' \end{cases} \quad . \qquad (39)$$

Using equation (20) applied for $r = L$ in equation (38) yields

$$b_N = \sum_u \beta_{Nu} \, a_u + K_N \frac{V_0(L)}{Z_c} \quad , \qquad (40)$$

where the relevant constants are defined as follows:

$$\beta_{Nu} = \frac{2N(N+1)}{2N+1} \frac{F_u(kL)}{G_N(kL)} \int_{\theta_0}^{\pi-\theta_0} \sin\theta \, \dot{P}_N(\cos\theta) \, \dot{T}_u(\cos\theta) \, d\theta \quad , \tag{41}$$

$$K_N = -j \, \frac{4N(N+1)}{2N+1} \frac{k}{G_N(kL)} P_N(\cos\theta_0) \quad , \tag{42}$$

$$F_u(kL) = \left\{ \frac{d}{dr} \left[r j_u(kr) \right] \right\}_{r=L} \quad . \tag{43}$$

The derivation of equation (40) was based on the requirement that $E_\theta\,(r = L) = 0$ when θ lies outside the limits $\theta_0 \le \theta \le \pi - \theta_0$.

By identifying K_N as the N^{th} component of the column vector $\hat{K}$ and β_{Nu} as the element of the matrix $\tilde{B}$, we can recast equation (40) in the form

$$\hat{b} = \tilde{B}\,\hat{a} + \left(V_0(L)/Z_c\right) \hat{K} \quad . \tag{44}$$

The termination impedance, Y_t, can now be determined from equations (35), (44), (29), and (7). This is accomplished by first integrating equation (29) from θ_0 to $\pi - \theta_0$. We have

$$Y_t = \frac{Z_0 L}{Z_c V_0(L)} \int_{\theta_0}^{\pi-\theta_0} H_\phi\,(r = L)\, \mathrm{d}\theta \quad . \tag{45}$$

We now substitute equation (7) into equation (45) to obtain

$$Y_t = \frac{-Z_0 L}{Z_c V_0(L)} \frac{1}{\pi} \sum_{N=1}^{\infty} b_N h_N^{(2)}(kL) P_N(\cos\theta_0) \quad . \tag{46}$$

The foregoing expression can also be cast in matrix form by introducing the transpose of the column vector $\hat{\xi}$ whose components are $h_N^{(2)}(kL)P_N(\cos\theta_0)$. Thus, we write equation (46) in the form

$$Y_t = \frac{-Z_0 L}{Z_c V_0(L)} \frac{1}{\pi} \hat{\xi}^T \hat{b} \quad , \tag{47}$$

where $\hat{\xi}^T$ is the transpose of $\hat{\xi}$ and is a row vector.

In an actual computation we will truncate the series for u and the series for N after a finite number of terms. However, the maximum number of terms used in the respective series may differ since the number of terms required to describe the fields in the interior and exterior regions may not be the same. This does not place any restriction on the theory or the method of solution. It is convenient in this report to conceptually regard the u and N series as being truncated after the same number of terms, say $\overline{n}$. Let us assume that this is done so that $\tilde{A}$ and $\tilde{B}$ are now $\tilde{n} \times \tilde{n}$ matrices and $\hat{a}$, $\hat{b}$, $\hat{K}$, and $\hat{\xi}$ are $\tilde{n}$-dimensional vectors. We now substitute equation (35) into equation (44) to obtain

$$\tilde{U}\hat{b} = \frac{V_0(L)}{Z_c}\hat{K} \quad , \tag{48}$$

where $\tilde{U}$ is the matrix

$$\tilde{U} = \tilde{I} - \tilde{B}\tilde{A} \quad , \tag{49}$$

and $\tilde{I}$ is the identity matrix. The solution of equation (48) is

$$\hat{b} = \frac{V_0(L)}{Z_c}\tilde{U}^{-1}\hat{K} \quad , \tag{50}$$

where $\tilde{U}^{-1}$ is the inverse of $\tilde{U}$.

Substituting equation (50) into equation (47) gives the following expression for the admittance:

$$Y_t = \frac{-Z_0 L}{Z_c^2 \pi}\hat{\xi}^T\tilde{U}^{-1}\hat{K} \quad . \tag{51}$$

Examination of the terms of equation (51) shows that Y_t is a function of the bicone angle θ_0, radial dimension L, and wavenumber (frequency) k. For specified values of L and θ_0, the admittance can be expressed as a function of frequency ω.

Once $Y_t(\omega)$ is determined from equation (51), the remaining calculation proceeds as follows: We initially determine $V_0(L)$ from a knowledge of the source voltage $V_0(0)$. Using equation (24) we have

$$V_0(L) = (\cos kL + jY_t Z_c \sin kL)^{-1} V_0(0) \quad . \tag{52}$$

Substituting equation (52) into equation (50) gives $\hat{b}$, and the external fields are then determined from equation (7).

3. SOLUTION WITH RESISTIVE LOADING

The deduction of the solution with resistive loading is not an obvious extension of the lossless case. A fundamental aspect of the lossless case was that the interior fields were made up of modes with index u determined from the solution of the interior boundary condition

$$E_r(\theta_0) = E_r(\pi - \theta_0) = 0 \quad . \tag{53}$$

This led to equation (27) and the determination of the values of u.

When resistive loading is considered, the new boundary condition at the walls becomes

$$E_r(\theta_0) = E_r(\pi - \theta_0) = Z_\omega(r)\, I_\omega(r) \tag{54}$$

where $I_\omega(r)$ is the wall current and $Z_\omega(r)$ is the surface impedance. The wall current is given by

$$I_\omega(r) = 2\pi r \sin\theta_0 H_\phi(\theta_0) \quad . \tag{55}$$

The existence of resistive losses requires modification of the functional form of the fields in the interior region. In the absence of resistive loading, the interior fields are given by equations (18) to (20) with the assumed noninteger values of u being determined from the solution of equation (27).

When the boundary condition of equation (54) replaces that of equation (53) because of resistive loading, the structure of the fields given by equations (18) to (20) does not appear to be sufficient in itself to solve the problem. I have not been able to find a means to uniquely determine the values of u that satisfy equation (54) and the other boundary conditions using the field representation of equations (18) to (20).

It appears, however, that a unique solution to the problem can be obtained if one adds to the field components of equations (18) to (20) additional terms involving integer values of the summation index, which are also solutions of the basic equations in the antenna region. These terms are necessary to satisfy the boundary condition of equation (54) and do not exist in the lossless case.

The deduction of the θ dependence of equations (18) to (20) was based on the fact that the two linearly independent solutions for H_ϕ in the θ dimension are derivatives of

Legendre functions of the first and second kind, $P_\upsilon(\cos\theta)$ and $Q_\upsilon(\cos\theta)$, respectively [9-12]. In the condition where

$$\upsilon = u \neq \text{integer} \ , \tag{56}$$

the function Q_u satisfies the equation

$$Q_u(\cos\theta) = P_u(-\cos\theta) \ . \tag{57}$$

Using the result

$$\cos(\pi - \theta) = -\cos\theta \ , \tag{58}$$

combined with equation (57) and the symmetry requirement

$$H_\phi(\pi - \theta) = H_\phi(\theta) \ , \tag{59}$$

led to the choice of

$$T_u = \frac{1}{2}(P_u - Q_u) = \frac{1}{2}\left[P_u(\cos\theta) - P_u(-\cos\theta)\right] \tag{60}$$

as the only candidate θ-dependent solution (compare equation (21)).

When we allow υ to be an integer, the θ-dependent part of the solution changes as follows. For the case where

$$\upsilon = N = \text{integer} \ ,$$

we have

$$P_N(\cos\theta) = \sum_{q=0}^{N} \frac{(-1)^q (N+q)!}{(N-q)!(q!)^2} \sin^{2q}\left(\frac{\theta}{2}\right) , \tag{61}$$

$$Q_N(\cos\theta) = P_N(\cos\theta)\ln\left(\cot\frac{\theta}{2}\right) - \sum_{m=1}^{N} \frac{P_{N-m}P_{m-1}}{m} \ . \tag{62}$$

Using the foregoing equations we easily see that

$$P_N(-\cos\theta) = (-1)^N P_N(\cos\theta) \ , \tag{63}$$

$$Q_N(-\cos\theta) = (-1)^{N+1} Q_N(\cos\theta) \ . \tag{64}$$

Employing the symmetry condition of equation (59) requires that only odd powers of N be used for the P_N terms and even powers of N for the Q_N terms.

When integer terms are added to equations (18) to (20), the expressions for the fields can be written in the form

$$H_\phi = \frac{I_0(r)}{2\pi r \sin\theta} + \frac{1}{2\pi}\sum_u a_u j_u(kr)\dot{T}_u(\cos\theta) + \frac{1}{2\pi}\sum_m \bar{a}_m j_m(kr)\dot{\Gamma}_m(\cos\theta) \ , \qquad (65)$$

$$E_r = \frac{jZ_0}{2\pi kr}\sum_u u(u+1)\, a_u j_u(kr)\, T_u(\cos\theta) + \frac{jZ_0}{2\pi kr}\sum_m m(m+1)\,\bar{a}_m j_m(kr)\,\Gamma_m(\cos\theta) \ , \qquad (66)$$

$$E_\theta = \frac{Z_0 V_0(r)}{Z_c 2\pi r \sin\theta} + \frac{jZ_0}{2\pi kr}\sum_u a_u F_u(kr)\,\dot{T}_u(\cos\theta) + \frac{jZ_0}{2\pi kr}\sum_m \bar{a}_m F_m(kr)\,\dot{\Gamma}_m(\cos\theta) \ , \qquad (67)$$

where

$$F_u(kr) = \frac{d}{dr}\left[r j_u(kr)\right] \ , \qquad (68)$$

$$F_m(kr) = \frac{d}{dr}\left[r j_m(kr)\right] \ , \qquad (69)$$

$$\Gamma_m(\cos\theta) = P_m(\cos\theta) \quad \text{if} \quad m = \text{odd} \ , \qquad \text{(a)}$$

$$\Gamma_m(\cos\theta) = Q_m(\cos\theta) \quad \text{if} \quad m = \text{even} \ , \qquad \text{(b)} \ (70)$$

and the $\bar{a}_m$'s are additional constants to be determined from the problem in the resistively loaded case. The index m is an integer.

We assume that at the boundary $\theta = \theta_0$. In addition, we continue to retain the equation

$$T_u(\cos\theta_0) = 0 \ , \qquad (71)$$

so that the noninteger values of u remain unchanged, and hence are known.

The relationship between the $\bar{a}_m$'s and $\bar{a}_u$'s is determined from the boundary condition of equation (54). Using equations (23) and (55) we have

$$Z_\omega\left[jV_o(L)Z_c^{-1}\sin k(L-r) + V_o(L)Y_t\cos k(L-r)\right]$$

$$+\left[Z_\omega r\, sin\, \theta_0 \times \sum_{v} a_u j_u(kr)\, \dot{T}_u(\cos\theta_0)\right] + Z_\omega r \sin\theta_0 \sum_{m} \bar{a}_m j_m \dot{\Gamma}_m(\cos\theta_0) =$$

$$\frac{jZ_0}{2\pi kr} \sum_{m} m(m+1)\, \bar{a}_m j_m(kr) \Gamma_m(\cos\theta_0) \quad . \tag{72}$$

There does not appear to be a simple way to relate the $\bar{a}_m$'s to the a_u's, but it can be accomplished in matrix form by assuming that in the region

$$0 \leq r \leq L \ , \tag{73}$$

we can expand all the r-dependent functions in an orthonormal basis using a complete (and as yet unspecified) set of functions $\phi_n(r)$ which satisfy the standard conditions

$$\int_0^L \phi_n(r)\, \phi_m(r)\, dr = \delta_{mn} \quad , \tag{74}$$

where n and m are integers and δ_{mn} is the Kronecker delta.

The aforementioned procedure is accomplished by (1) multiplying both sides of equation (72) by r, (2) setting

$$\gamma(r) = Z_\omega / Z_0 \quad ,$$

and (3) using the following relationships:

$$r\gamma \sin k(L-r) = \sum_{n=0}^{\infty} d_n \phi_n(r) \quad , \tag{75}$$

$$r\gamma \cos k(L-r) = \sum_{n=0}^{\infty} f_n \phi_n(r) \quad , \tag{76}$$

$$\gamma r^2 j_u(kr) = \sum_{n=0}^{\infty} g_{nu} \phi_n(r) \quad , \tag{77}$$

$$\gamma r^2 j_m(kr) = \sum_{n=0}^{\infty} g_{nm}\, \phi_n(r) \quad , \tag{78}$$

$$j_m(kr) = \sum_{n=0}^{\infty} h_{nm}\phi_n(r) \quad , \tag{79}$$

where the d_n's, f_n's, g_{nu}'s, g_{nm}'s, and h_{nm}'s are constants determined from the orthogonality condition of equation (74). For example,

$$g_{\text{nm}} = \int_0^L r^2 j_m(kr)\phi_n(r)\, dr \quad . \tag{80}$$

Since the ϕ_n's form a complete set, the substitution of equations (75) to (79) into equation (72) yields

$$jV_0(L)Z_c^{-1}\, d_n + V_0(L)Y_t f_n + \sin\theta_0 \sum_u g_{nu}\dot{T}_u(\cos\theta_0)\, a_u$$

$$+ \sin\theta_0 \sum_m g_{nm}\dot{\Gamma}_m(\cos\theta_0)\, \bar{a}_m = \frac{j}{2\pi k} \sum_m m(m+1)\, h_{nm}\bar{a}_m \quad . \tag{81}$$

By defining d_n, f_n, a_u, and $\bar{a}_m$ to be components in the same dimensional vector space, we can cast equation (81) in the matrix form

$$jV_0(L)\, Z_c^{-1}\hat{d} + V_0(L)Y_t\hat{f} + \tilde{G}\hat{a} + \tilde{\tilde{G}}\,\hat{\hat{a}} = \tilde{\tilde{H}}\,\hat{\hat{a}} \quad , \tag{82}$$

where

$$\tilde{G}_{nu} = \sin\theta_0\, g_{nu}\dot{T}_u(\cos\theta_0) \quad , \qquad \text{(a)}$$

$$\tilde{\tilde{G}}_{nm} = \sin\theta_0\, g_{nm}\dot{\Gamma}_m(\cos\theta_0) \quad , \qquad \text{(b)} \tag{83}$$

$$\tilde{\tilde{H}}_{nm} = \frac{j}{2\pi k}(m)\,(m+1)h_{nm} \quad , \qquad \text{(c)}$$

The solution of equation (82) which renders $\hat{\hat{a}}$ in terms of $\hat{a}$ is given by

$$\hat{\hat{a}} = \tilde{V}\,\hat{a} + j\, V_0(L)\, Z_c^{-1}\hat{\lambda} + V_0(L)\, Y_t\,\hat{\eta} \quad , \tag{84}$$

where

$$\tilde{V} = (\tilde{\tilde{H}} - \tilde{\tilde{G}})^{-1}\tilde{\tilde{G}} \quad , \tag{85}$$

$$\hat{\lambda} = (\tilde{\tilde{H}} - \tilde{\tilde{G}})^{-1}\hat{d} \quad , \tag{86}$$

$$\hat{\eta} = (\tilde{\tilde{H}} - \tilde{\tilde{G}})^{-1}\hat{f} \quad . \tag{87}$$

Using equation (84), the determination of the admittance Y_t and the constants $\hat{a}, \hat{\bar{a}}$, and $\hat{b}$ follows in a manner analogous to the lossless case considered in the previous section (compare the analysis beginning with eq (28)). For brevity, some of the obvious intermediate steps will be omitted. From equation (65) we have

$$H_\phi(r=L) = \frac{Y_t V_0(L)}{2\pi L \sin\theta} + \frac{1}{2\pi}\sum_{u'} a_u j_{u'}(kL)\dot{T}_{u'}(\cos\theta) + \frac{1}{2\pi}\sum_m \bar{a}_m j_m(kL)\,\dot{\Gamma}_m(\cos\theta) \tag{88}$$

If we now multiply equation (88) by $\sin\theta\ \dot{T}_u(\cos\theta)$and integrate from θ_0 to $\pi - \theta_0$, and then use equation (7) for $H_\phi(r = L)$, we obtain the following result:

$$\hat{a} = \tilde{A}\hat{b} - \tilde{\Omega}\,\hat{\bar{a}} \tag{89}$$

where $\tilde{A}$ is the previously defined matrix, and $\tilde{\Omega}$ is a matrix whose components are

$$\Omega_{\mu m} = \frac{j_m(kL)}{N_u j_u(kL)} \int_{\theta_0}^{\pi-\theta_0} \sin\theta\,\dot{T}_u(\cos\theta)\,\dot{\Gamma}_m(\cos\theta)\,d\theta \quad . \tag{90}$$

From equation (38) we have the relationship which relates b_N to the interior fields. However, we must now use equation (67) for E_θ. Inserting equation (67) into (38) gives

$$b_N = \sum_u \beta_{N\mu} a_u + K_N V_o(L) Z_c^{-1} + \sum_m \bar{\beta}_{Nm}\bar{a}_m \quad , \tag{91}$$

where

$$\bar{\beta}_{N\mu} = \frac{2N(2N+1)}{2N+1}\,\frac{F_m(kL)}{G_N(kL)} \int_{\theta_0}^{\pi-\theta_0} \sin\theta\,\dot{P}_N(\cos\theta)\dot{\Gamma}_m(\cos\theta)\,d\theta \ . \tag{92}$$

It is observed that equation (91) is similar in mathematical structure to equation (40); the matrix equivalent becomes

$$\hat{b} = \tilde{B}\,\hat{a} + K_N V_0(L)\,Z_c^{-1} + \tilde{\tilde{B}}\,\hat{\hat{a}} \quad , \tag{93}$$

which is likewise analogous to equation (44). $\tilde{\tilde{B}}$ is the matrix whose elements are given by equation (92).

The final required equation for determining the unknowns in the system is found by integrating equation (88) between θ_0 and $\pi - \theta_0$. There results

$$Y_t = \frac{-Z_0 L}{Z_c V_0(L)} \frac{1}{\pi} \hat{\xi}^T \hat{b} + \frac{Z_0 L}{Z_c V_0(L)} \frac{1}{\pi} \hat{\psi}^T \hat{\hat{a}} \quad , \tag{94}$$

where $\hat{\psi}$ is a column vector whose components are $j_m(kL)\Gamma_m(\cos\theta_0)$.

In summary, there are four unknowns in the problem: Y_t, $\hat{a}$, $\hat{b}$, and $\hat{\hat{a}}$. These are determined from equations (84), (89), (93), and (94). The simultaneous solution of these matrix equations involves many intermediate steps which for brevity are not presented. The final result is

$$Y_t = \frac{-\alpha\hat{\xi}^T\hat{X}_6 + \alpha\hat{\psi}^T\hat{X}_3 + j\alpha Z_c^{-1}\hat{\psi}^T\tilde{X}_4}{1 + \alpha\hat{\xi}^T\tilde{M}_1\hat{X}_5 - \alpha\hat{\psi}^T X_5} \quad , \tag{95}$$

where

$$\alpha = \frac{Z_0 L}{Z_c} \quad , \tag{a}$$

$$\hat{X}_1 = (\tilde{I} - \tilde{B}\tilde{A})^{-1}\hat{K}Z_c \quad , \tag{b}$$

$$\hat{X}_2 = \tilde{A}X_1 \quad , \tag{c}$$

$$\tilde{M}_1 = (\tilde{I} - \tilde{B}\tilde{A})^{-1}\,(\tilde{\tilde{B}} - \tilde{B}\,\tilde{\Omega}) \quad , \tag{d}$$

$$\tilde{M}_2 = \tilde{A}\tilde{M}_1 - \tilde{\Omega} \quad , \tag{e}$$

$$\hat{X}_3 = (\tilde{I} - \tilde{V}\tilde{M}_2)^{-1}\tilde{V}\hat{X}_2 \quad , \tag{f}$$

$$\hat{X}_4 = (\tilde{I} - \tilde{V}\tilde{M}_2)^{-1}\hat{\lambda} \ , \qquad \text{(g)}$$

$$\hat{X}_5 = (\tilde{I} - \tilde{V}\tilde{M}_2)^{-1}\hat{\eta} \ , \qquad \text{(h)}$$

$$\hat{X}_6 = \hat{X}_1 + \tilde{M}_1\hat{X}_3 + jZ_c^{-1}\tilde{M}_1\hat{X}_4 \ . \qquad \text{(i)} \qquad (96)$$

Once Y_t is computed, the other parameters, $\hat{a}$, $\hat{b}$, and $\hat{\bar{a}}$, are determined from equations (84), (89), (90), and (94).

4. CONCLUSION

In this study we have demonstrated that a methodology for calculating the fields of a biconical antenna with resistive loading is theoretically and computationally feasible. The technique draws on the modal analysis concept originally developed by Schelkunoff for the case without resistive loading. However, the extension to the resistively loaded case is considerably more complex from both a conceptual and computational viewpoint.

The resistively loaded case is found to require the existence of additional terms in the antenna region (compared to the lossless case) so that the interior boundary condition is satisfied. The solution of the interior boundary value equation appears to require the introduction of an orthogonal basis of functions, $\phi_n(r)$, defined in the range $0 \le r \le L$, where r is the radial coordinate and L is the bicone radius. We have not as yet selected the $\phi_n(r)$; however, this does not limit the theoretical analysis. Using the basis functions, $\phi_n(r)$, a matrix formulation is developed. In order to achieve a practical solution to the problem, it will be necessary to assume a finite number of terms. This number has not been determined, but will surely depend on the bicone angle θ_o and radius L, and the magnitude and radial distribution of the resistive loading.

In summary, the implementation of the technique developed in this investigation requires the selection of $\phi_n(r)$ and the evaluation of certain mathematical functions and integrals which depend on $\phi_n(r)$. When this is accomplished it should be a relatively easy matter to predict the near and far fields from a resistively loaded biconical antenna.

5. REFERENCES

1. C.E. Baum, "Resistively Loaded Radiating Dipole Based on a Transmission Model for the Antenna," Sensor and Simulation Note 81, Air Force Weapons Laboratory, Albuquerque, NM, 1969.
2. D.R. Wilton, "Static Analysis of Conical Antenna over a Ground Plane," Sensor and Simulation Note 224, Air Force Weapons Laboratory, Albuquerque, NM, August 1976.
3. D.R. Wilton, "Dynamic Analysis of a Loaded Conical Antenna over a Ground Plane," Sensor and Simulation Note 225, Air Force Weapons Laboratory, Albuquerque, NM, August 1976.
4. S. A. Schelkunoff, "Theory of Antennas of Arbitrary Size and Shape," *Proc. I.R.E.* **29** (1941), 493-521.
5. C. T. Tai, "On the Theory of Biconical Antennas," *J. Appl. Phys.* **19**, (1948) 1155-1160.
6. J. D. Kraus, *Antennas*, McGraw-Hill, New York, N.Y. (1950).
7. P. D. P. Smith, "The Conical Dipole of Wide Angle," *J. Appl. Phys.* **19** (1948), 11-23.
8. C. T. Tai, "Application of a Variational Principal to Biconical Antennas," *J. App. Phys.* **20** (1949), 1076-1084 .
9. S. A. Schelkunoff, *Advanced Antenna Theory*, John Wiley & Sons, New York (1952).
10. J. A. Kong, *Electromagnetic Wave Theory*, John Wiley & Sons, New York (1986).
11. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Appl. Math. Ser. 55, U.S. Government Printing Office, Washington, D.C. (1964).
12. A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York (1953).

APPENDIX: DETERMINATION OF THE ROOTS u_i AND EIGENFUNCTIONS T_{u_i}

FIGURES

TABLES

APPENDIX: DETERMINATION OF THE ROOTS u_i AND EIGENFUNCTIONS T_{u_i}

In this appendix I address the determination of the roots u_i of the equation

$$T_{u_i}(\cos\theta_0) = \frac{1}{2}\left[P_{u_i}(\cos\theta_0) - P_{u_i}(-\cos\theta_0)\right] = 0 \quad , \qquad \text{(A-1)}$$

where P_{u_i} is the Legendre function of order u_i, and θ_0 is the bicone angle of figure 1 in the body of the report. I also discuss the determination of the θ-dependence of the function

$$T_{u_i}(\cos\theta) = \frac{1}{2}\left[P_{u_i}(\cos\theta) - P_{u_i}(-\cos\theta)\right] \quad . \qquad \text{(A-2)}$$

Letting

$$z_0 = \cos\theta_0 \quad , \qquad \text{(A-3)}$$

we write equation (A-1) in the form

$$\frac{1}{2}\left[P_{u_i}(z_0) - P_{u_i}(-z_0)\right] = 0 \quad . \qquad \text{(A-4)}$$

Since u_i is not generallly an integer, the complete series expansion for P_u must be used. This series can be deduced from the hypergeometric function [1] through the relationship

$$P_u(z_0) = F\left(-u, u+1; 1; \frac{1-z_0}{2}\right) \quad , \qquad \text{(A-5)}$$

which is valid in the range $|1 - z_0| < 2$. This latter requirement is satisfied for the angles of interest. The series formula for the hypergeometric function of equation (A-5) can be deduced from the general formula (compare chapter 15 of the reference).

$$F(a, b; c; z_0) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z_0^{\,n}}{n!} \quad , \qquad \text{(A-6)}$$

where $(a)_n$, $(b)_n$, and $(c)_n$ are Pochhammer symbols defined by the equation

$$(\xi)_n = \frac{\Gamma(\xi + n)}{\Gamma(\xi)} \quad , \qquad \text{(A-7)}$$

with Γ being the gamma function.

Setting

$$a = -u \ , \qquad \text{(a)}$$
$$b = u + 1 \ , \qquad \text{(b)} \qquad \text{(A-8)}$$
$$c = 1 \ , \qquad \text{(c)}$$

and using the properties of the gamma function gives

$$(a)_n = (-u)(-u+1)(-u+2) \ldots (-u+n-1) \ , \qquad \text{(a)}$$
$$(b)_n = (u+1)(u+2) \ldots (u+n) \ . \qquad \text{(b)} \qquad \text{(A-9)}$$

The resulting series expression for $P_u(z)$ is

$$P_u(z_0) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} \frac{(1-z_0)^n}{2^n} \ . \qquad \text{(A-10)}$$

When equation (A-10) and its counterpart for $P_u(-z_0)$ are inserted in equation (A-4), we have a polynomial of infinite order for the determination of the roots u_i. Since this is impossible to deal with, it is necessary to approximate $P_u(z_0)$ and $P_u(-z_0)$ by a finite number of terms.

In all cases equation (A-10) is a well-behaved representation of $P_u(z_0)$. Figures A-1 and A-2 show the behavior of $P_u(z_0)$ and $P_u(-z_0)$, respectively, for selected values of $z_0 = \cos\,\theta_0$ calculated from equation (A-10) using 128 terms in the summation and including double precision.

The need for double precision in the computation of $P_u(z_0)$ and $P_u(-z_0)$ arises from the oscillatory nature of the individual terms in the series, which can become extremely large for correspondingly large values of u. Thus, in the absence of double precision, we would be faced with large round-off errors resulting from the subtraction of sequences of two extremely large numbers. These uncertainties would be pronounced in the determination of the roots of equation (A-4). The use of double precision combined with 128 terms in the series of $P_u(z_0)$ and $P_u(-z_0)$ appears to circumvent the aforementioned problem.

For very large values of u, even 128 terms may not be sufficient to accurately compute $P_u(\pm z_0)$. Fortunately, in this case the roots of equation (A-4) may be determined using the asymptotic forms of $P_u(\pm z_0)$. The asymptotic expressions [2] for the Legendre functions also provide an excellent starting guess for the ZBRAC and RTBIS root finding algorithms [3] which are used to solve equation (A-6).

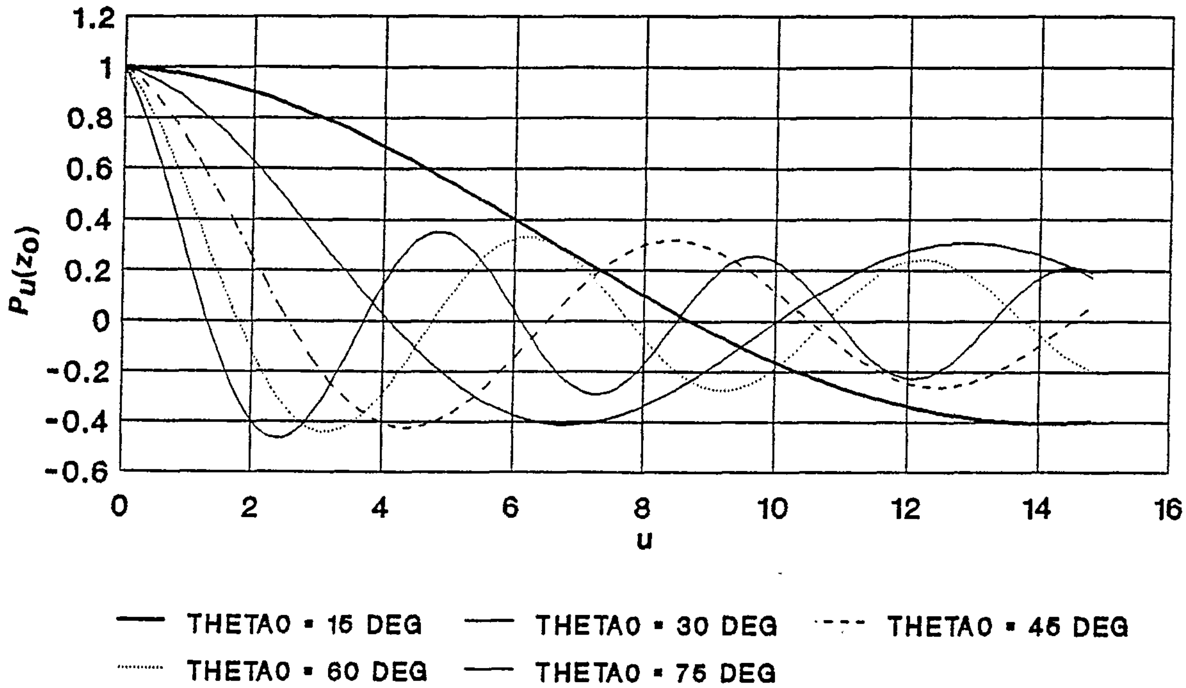

Figure A-1. $P_u(z_0)$ as a function of u with z_0 as a parameter.

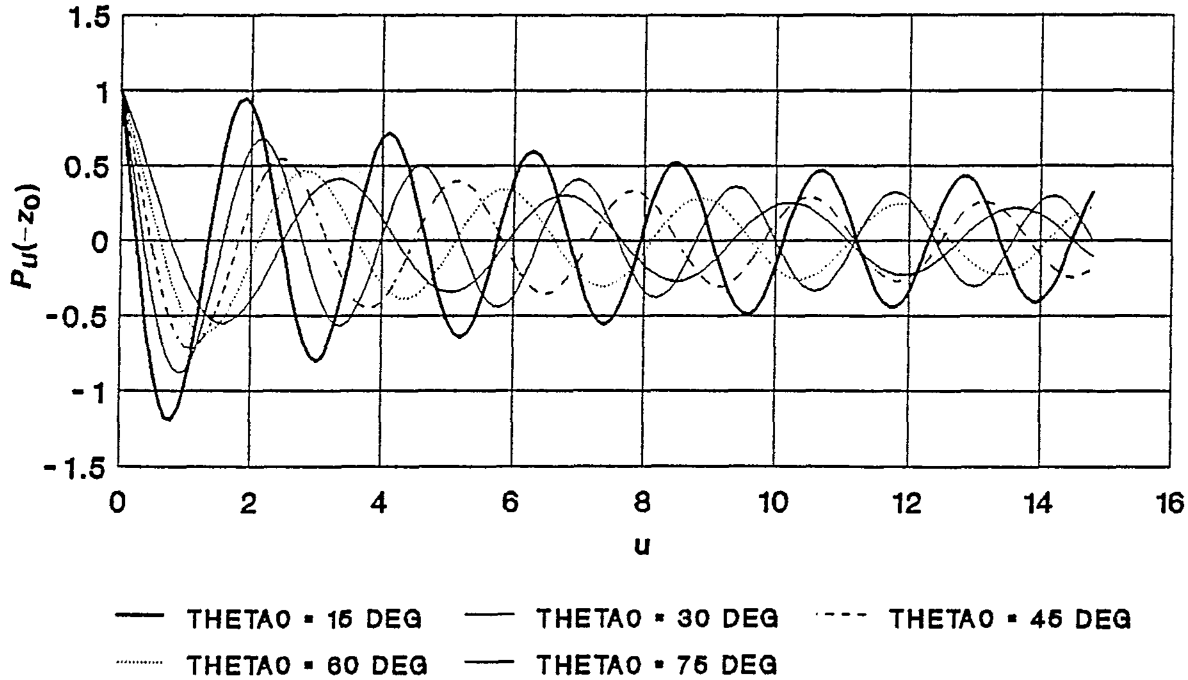

Figure A-2. $P_u(-z_0)$ as a function of u with z_0 as a parameter.

Several asymptotic expressions were examined, and the one which seems most appropriate for this investigation is that rendered by Magnus and Oberhettinger [2]. The asymptotic expression for the Legendre function of order u is denoted by $\bar{P}_u(\cos\theta)$ and is given by

$$\bar{P}_{u_i}(\cos\theta) = \sqrt{\frac{2}{\pi\bar{u}_i \sin\theta}}\ \cos\left[(u_i + \tfrac{1}{2})\ \theta - \tfrac{\pi}{4}\right];\ (\varepsilon \leqq \theta \leqq \pi - \varepsilon,\ \varepsilon > 0,\ |\,u_i\,| >> \tfrac{1}{\varepsilon})\ . \quad \text{(A-11)}$$

Using equation (A-11) in equation (A-4) and recalling that

$$-z_0 = \cos(\pi - \theta_0) \qquad \text{(A-12)}$$

gives the following equation for the asymptotic roots $\bar{u}_i$:

$$\cos\left[(\bar{u}_i + \tfrac{1}{2})\,\theta_0 - \tfrac{\pi}{4}\right] - \cos\left[(\bar{u}_i + \tfrac{1}{2})(\pi - \theta_0) - \tfrac{\pi}{4}\right] = 0\ . \qquad \text{(A-13)}$$

Using the formula

$$\cos a - \cos b = -2\sin\left[\tfrac{1}{2}(a+b)\right]\sin\left[\tfrac{1}{2}(a-b)\right] \qquad \text{(A-14)}$$

and letting

$$a = v_i\theta_0 - \frac{\pi}{4}\ , \qquad \text{(a)}$$

$$b = v_i(\pi - \theta_0) - \frac{\pi}{4}\ , \qquad \text{(b)} \qquad \text{(A-15)}$$

$$v_i = \bar{u}_i + \frac{1}{2}\ , \qquad \text{(c)}$$

reduces equation (A-13) to the form

$$\sin\left(\bar{u}_i\frac{\pi}{2}\right)\ \sin\left[v_i\left(\frac{\pi}{2} - \theta_0\right)\right] = 0\ . \qquad \text{(A-16)}$$

Since we are looking for solutions where $\bar{u}_i$ is not an integer, the solution of equation (A-16) is given by

$$v_i\left(\frac{\pi}{2} - \theta_0\right) = m\pi\ , \qquad \text{(A-17)}$$

where m is an integer. Letting

$$\phi_0 = \frac{\pi}{2} - \theta_0 \tag{A-18}$$

gives

$$\bar{u}_i = \frac{m\pi}{\phi_0} - \frac{1}{2} \ . \tag{A-19}$$

As observed from equation (A-19), every angle θ_0 generates an infinite set of roots with magnitudes increasing with the index m. Table A-1 compares the asymptotic roots given by equation (A-19) and the exact roots determined from the numerical solution of equation (A-4).

The asymptotic values appear to provide a good approximation at the smaller values of u_i with excellent agreement occurring at the larger ones. This is consistent with the theoretical expectations.

Table A-1. Comparison between exact and asymptotic roots

θ_o (degrees)	Root no.	Asymptotic value, $\bar{u}_i$	Exact value u_i
15.0	1	1.90000	1.80156
15.0	2	4.30000	4.25625
15.0	3	6.70000	6.67405
15.0	4	9.10000	9.04312
15.0	5	11.5000	11.4272
30.0	1	2.50000	2.44524
30.0	2	5.50000	5.45228
30.0	3	8.50000	8.45644
30.0	4	11.5000	11.4999
30.0	5	14.5000	14.4999
45.0	1	3.50000	3.45397
45.0	2	7.50000	7.48620
45.0	3	11.5000	11.4812
45.0	4	15.5000	15.4990
45.0	5	19.5000	19.4997
60.0	1	5.50000	5.48263
60.0	2	11.5000	11.4999
60.0	3	17.5000	17.4993
60.0	4	23.5000	23.5000
60.0	5	29.5000	29.5001
75.0	1	11.5000	11.4870
75.0	2	23.5000	23.4998
75.0	3	35.5000	35.5005
75.0	4	47.5000	47.5000
75.0	5	59.5000	59.4999

It should be noted in passing that Schelkunoff also obtained estimates for the asymptotic roots [4] which agree with equation (A-19) only in the limit where $m\pi/\phi_0 >> 1/2$. He did not attempt to generate numerical solutions relevant to our range of interest.

After the roots are determined it is also necessary to calculate the angular dependence of the function

$$T_{u_i}(\cos\theta) = \frac{1}{2}\left[P_{u_i}(\cos\theta) - P_{u_i}(-\cos\theta)\right] \qquad \text{(A-20)}$$

which from equation (A-4) is observed to satisfy the condition

$$T_{u_i}(\cos\theta_0) = T_{u_i}\left[\cos(\pi-\theta_0)\right] = T_{u_i}(-\cos\theta_0) = 0 \quad . \qquad \text{(A-21)}$$

Figures A-3 to A-7 show plots of $T_{u_i}(z = \cos\theta)$, $P_{u_i}(z = \cos\theta)$, and $P_{u_i}(-z = -\cos\theta)$ as a function of θ for the first five roots when $\theta_0 = 30°$. Using equation (A-10), we can compute the angular dependence of $T_{u_i}(z)$ from the expression

$$T_{u_i}(\cos\theta) = \frac{1}{2}\sum_{n=1}^{128} \frac{(a)_n (b)_n}{(n!)^2} \frac{1}{2^n}\left[(1-\cos\theta)^n - (1+\cos\theta)^n\right] \quad , \qquad \text{(A-22)}$$

where the $(a)_n$ and $(b)_n$ are calculated from equation (A-9) with the u_i's given in table A-1 for $\theta_0 = 30°$.

It is also of interest to examine the sensitivity of the behavior of $T_{u_i}(\cos\theta)$ to the choice of the asymptotic versus exact root as a function of θ_0. Figures A-8 to A-12 show the comparison between T_{u_i} and $T_{\bar{u}_i}$ for the first five modes when $\theta_0 = 30°$. As expected, the differences become smaller as m increases from 1 to 5.

Figure A-13 shows the comparison between the T_{u_i}'s for the smallest root, which occurs in the $\theta_0 = 15°$, $m = 1$ case. For this situation it is observed that the difference can become quite large, indicating that the asymptotic approximation is not especially good. On the other hand, figure A-14 shows the $\theta_0 = 75°$, $m = 1$ case, which corresponds to a relatively large root of 11.50 (compare table A-1). As observed from figure A-14 the differences in this case are imperceptible.

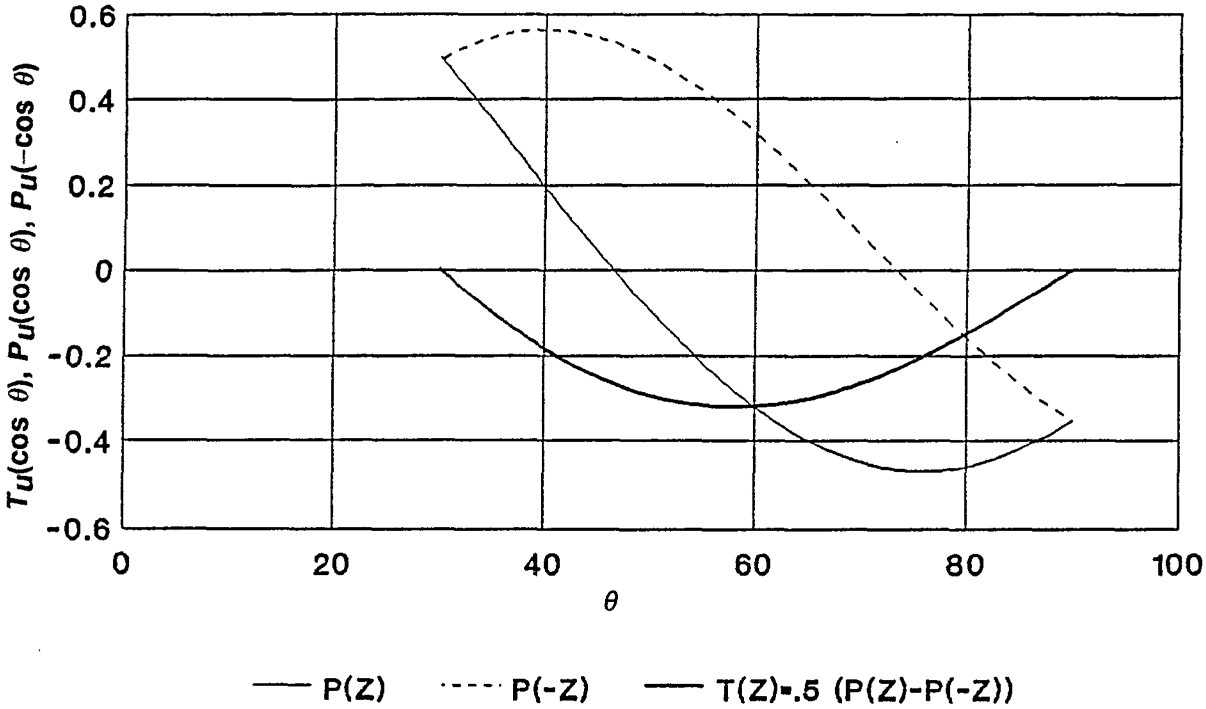

Figure A-3. $T_U(\cos\theta)$, $P_U(\cos\theta)$, and $P_U(-\cos\theta)$ as a function of θ for $\theta_0 = 30°$ and $m = 1$.

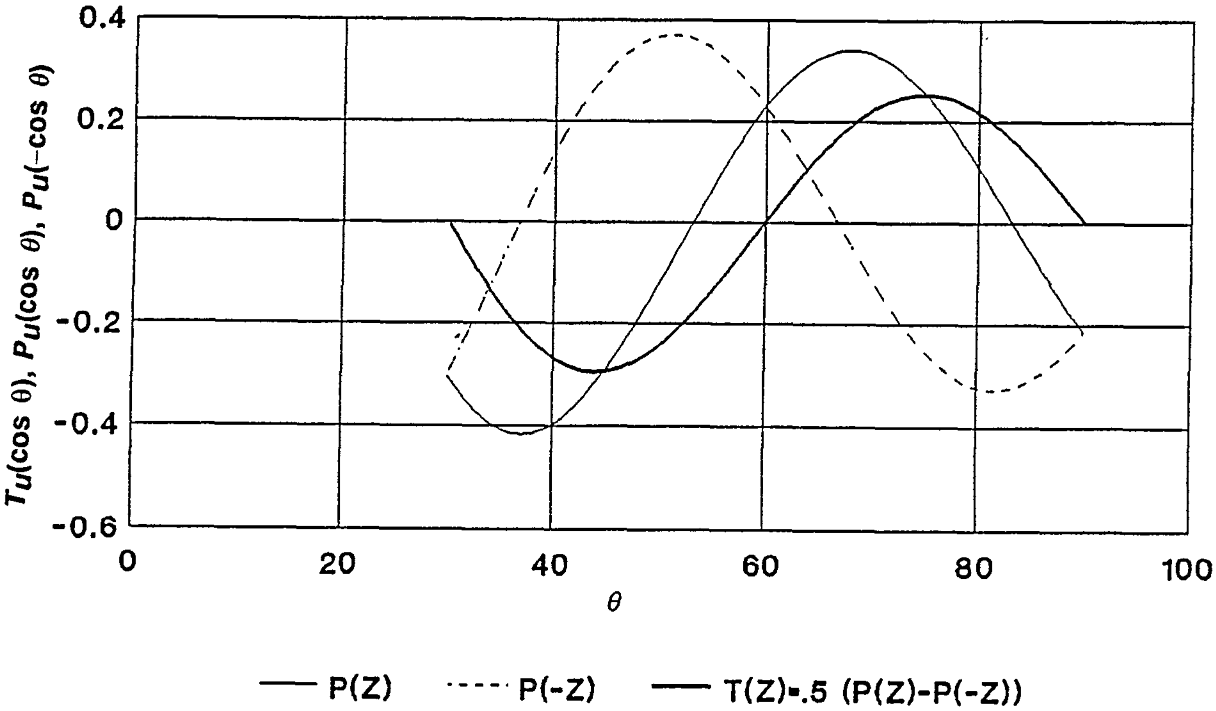

Figure A-4. $T_U(\cos\theta)$, $P_U(\cos\theta)$, and $P_U(-\cos\theta)$ as a function of θ for $\theta_0 = 30°$ and $m = 2$.

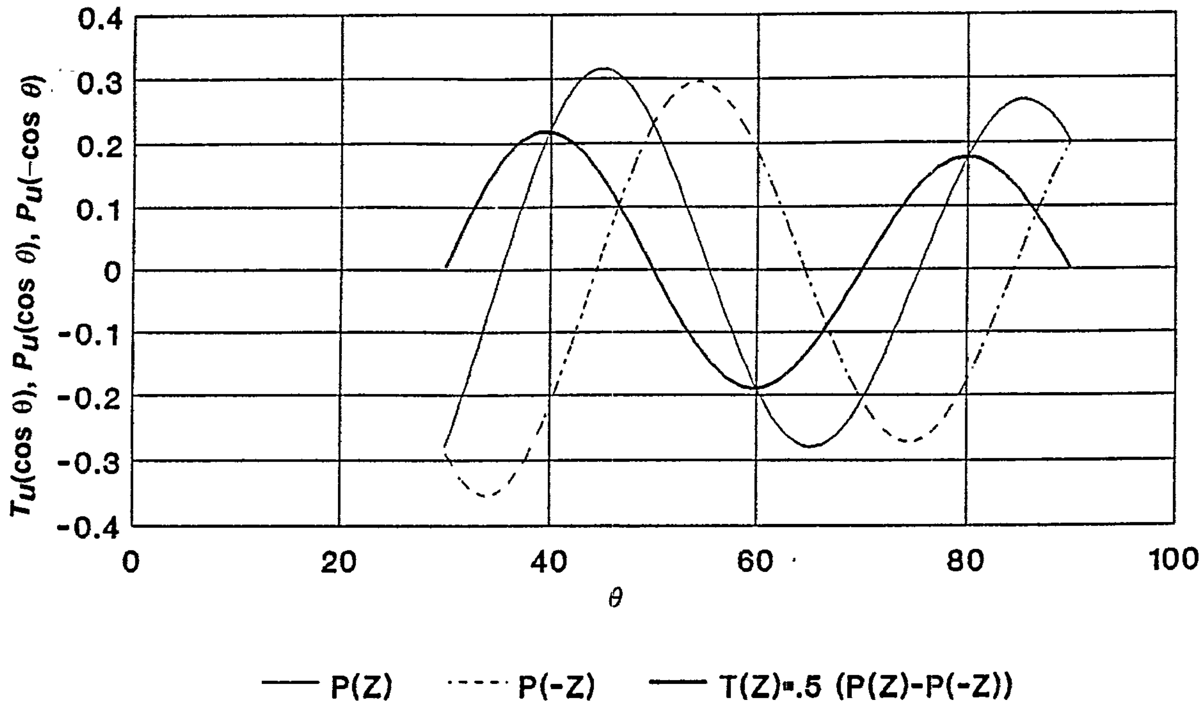

Figure A-5. $T_u(\cos\theta)$, $P_u(\cos\theta)$, and $P_u(-\cos\theta)$ as a function of θ for $\theta_0 = 30°$ and $m = 3$.

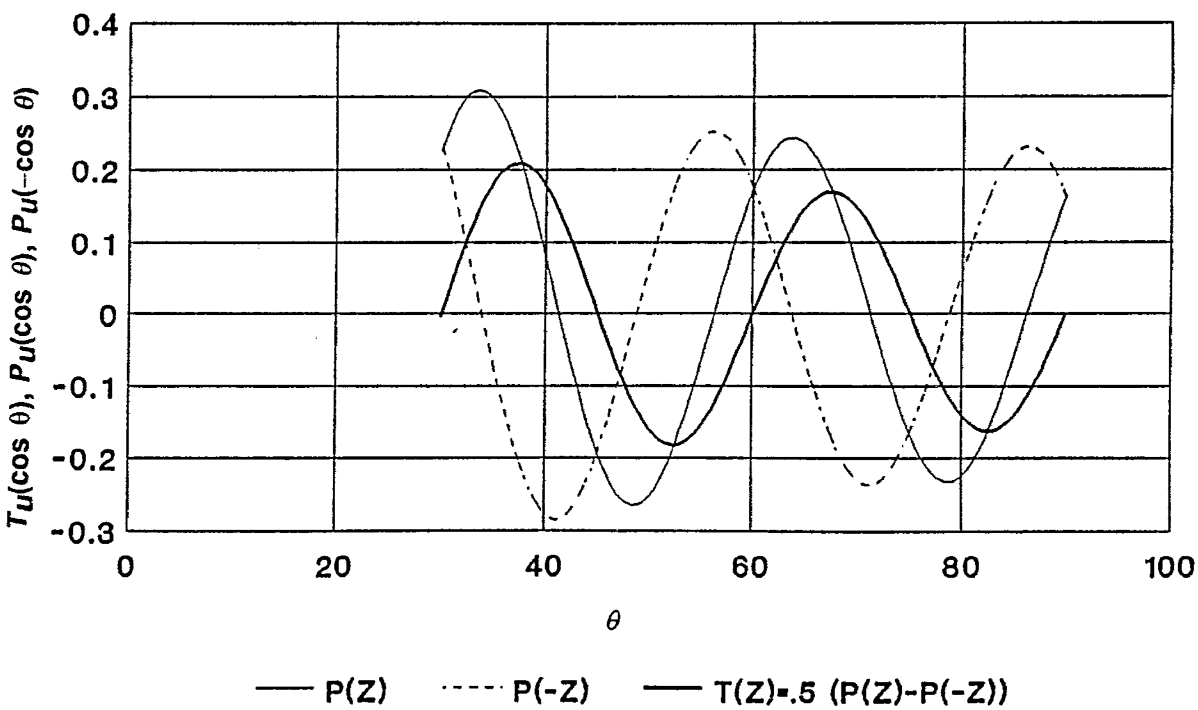

Figure A-6. $T_u(\cos\theta)$, $P_u(\cos\theta)$, and $P_u(-\cos\theta)$ as a function of θ for $\theta_0 = 30°$ and $m = 4$.

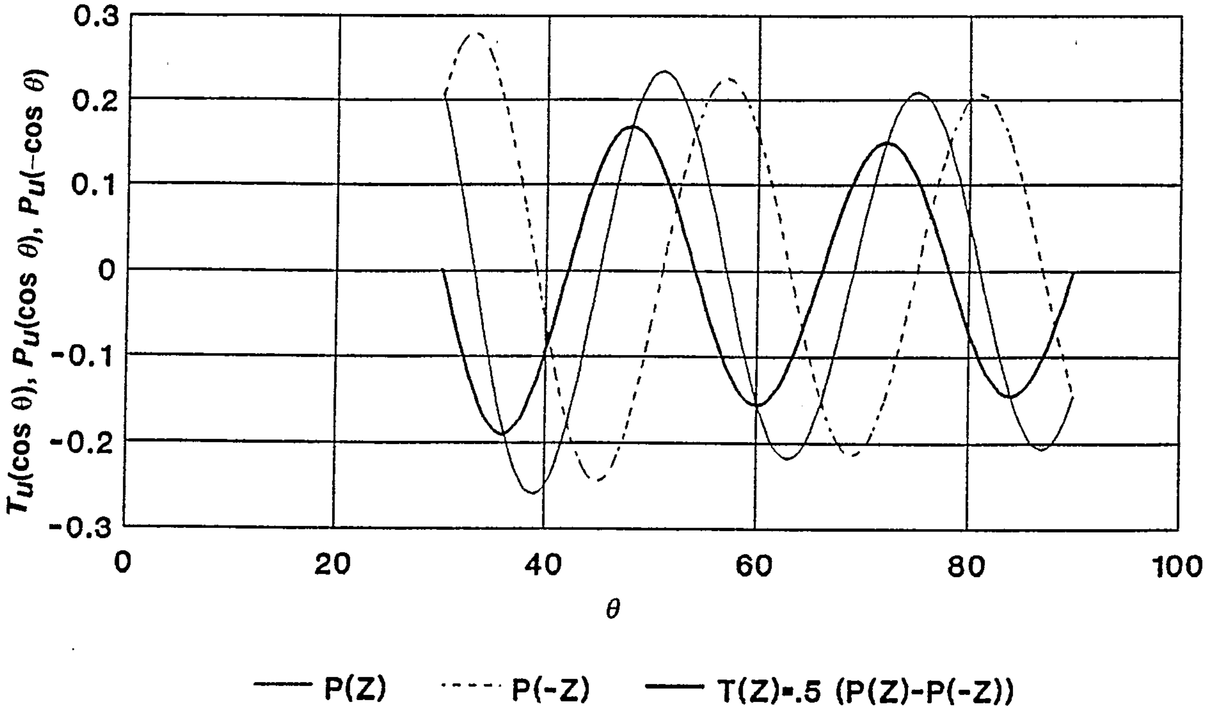

Figure A-7. $T_u(\cos\theta)$, $P_u(\cos\theta)$, and $P_u(-\cos\theta)$ as a function of θ for $\theta_0 = 30°$ and $m = 5$.

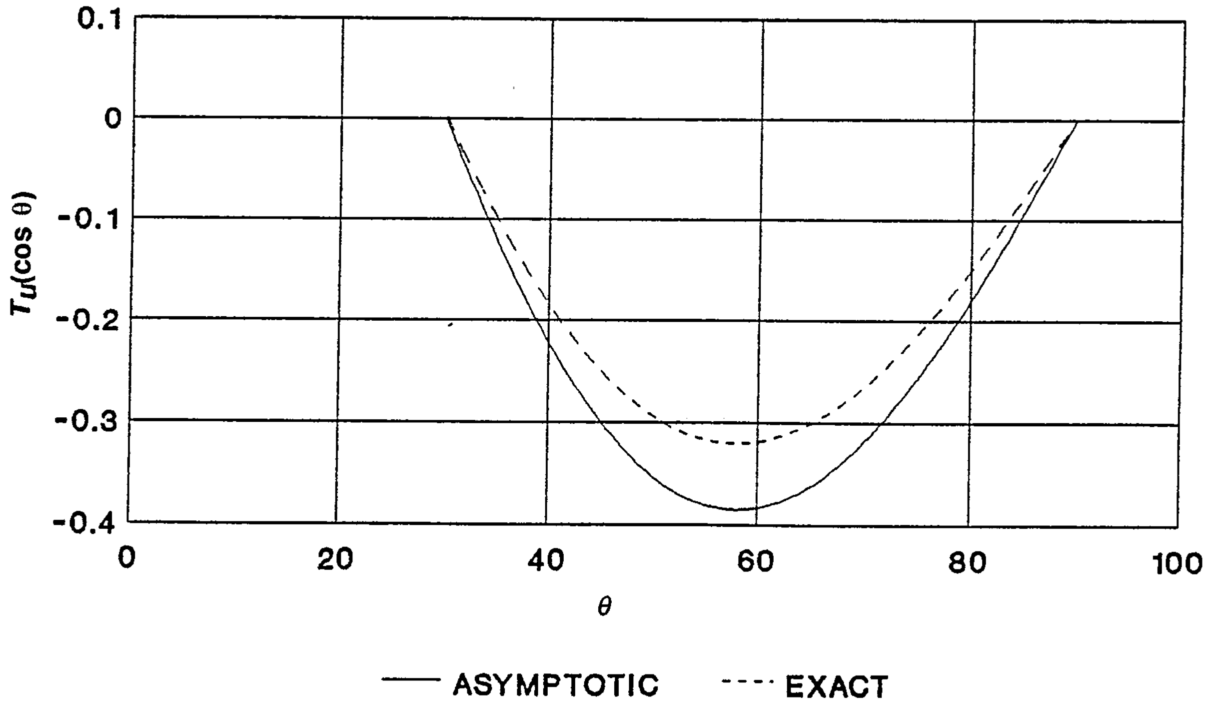

Figure A-8. Comparison between asymptotic and exact T_u for $\theta_0 = 30°$ and $m = 1$.

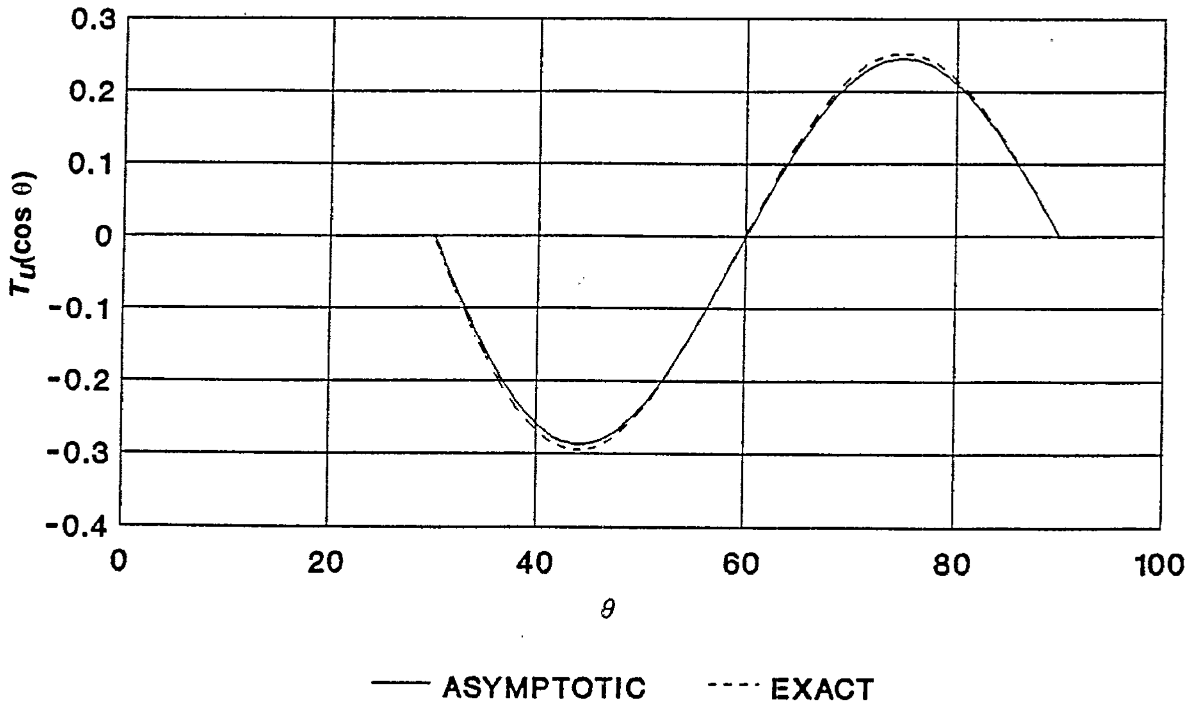

Figure A-9. Comparison between asymptotic and exact T_u for $\theta_0 = 30°$ and $m = 2$.

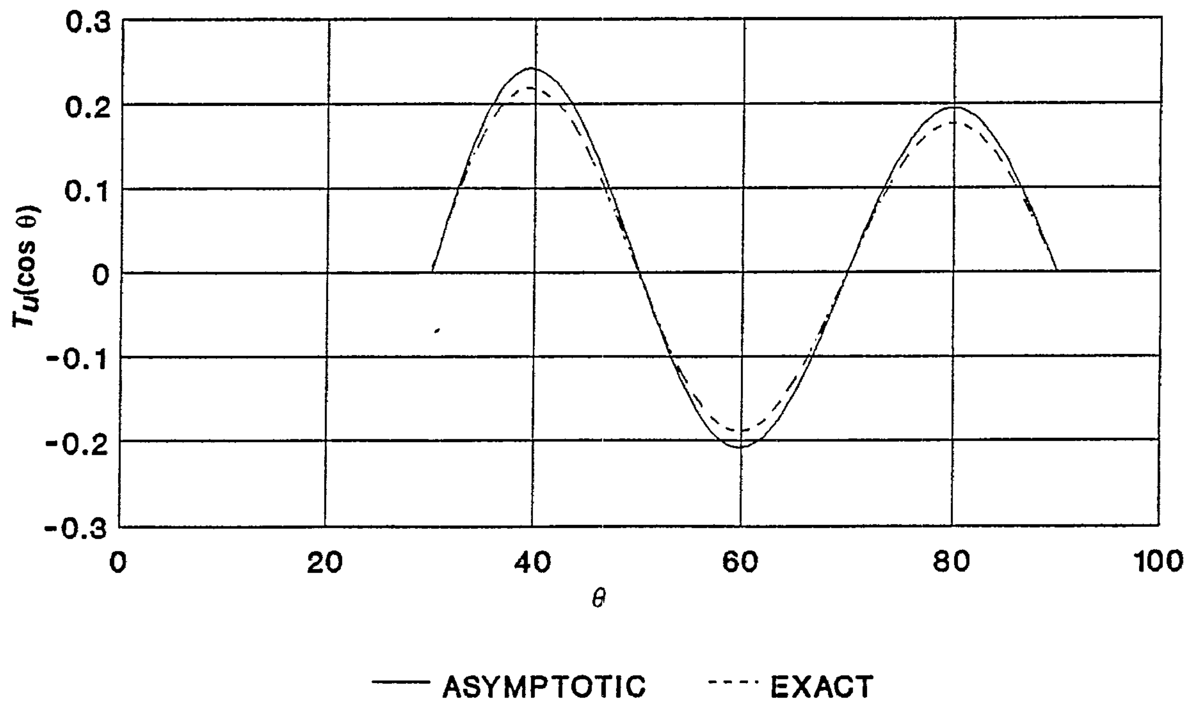

Figure A-10. Comparison between asymptotic and exact T_u for $\theta_0 = 30°$ and $m = 3$.

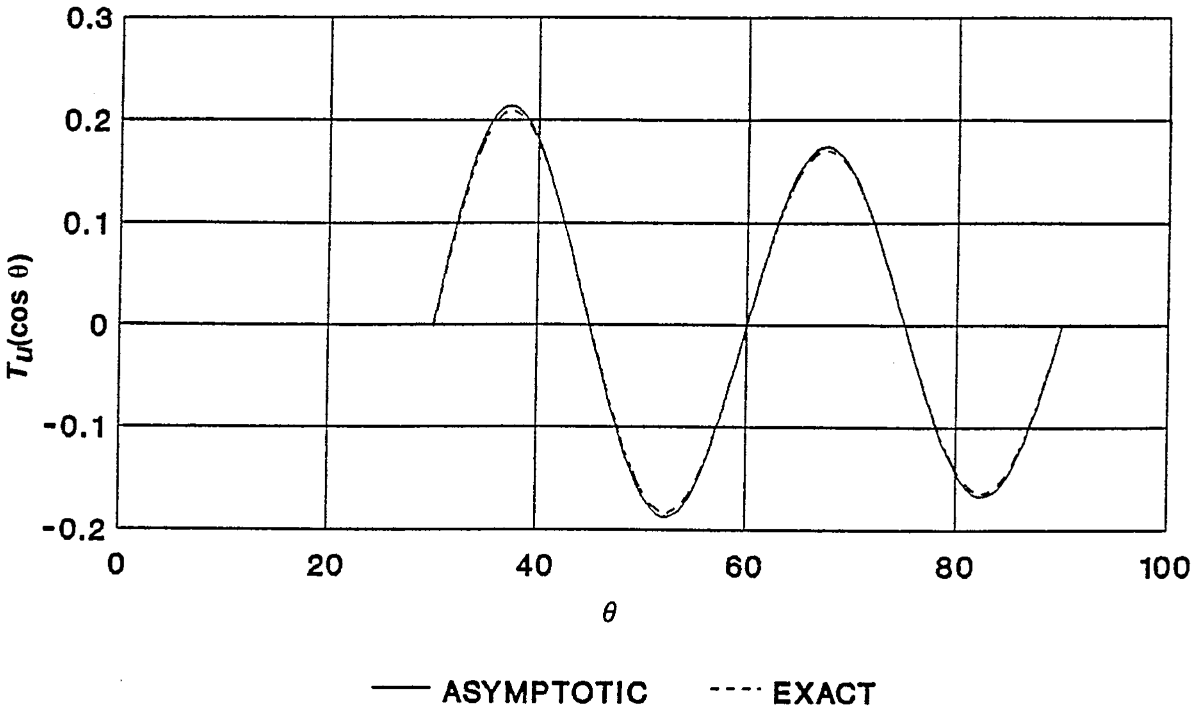

Figure A-11. Comparison between asymptotic and exact T_u for θ_0= 30° and m = 4.

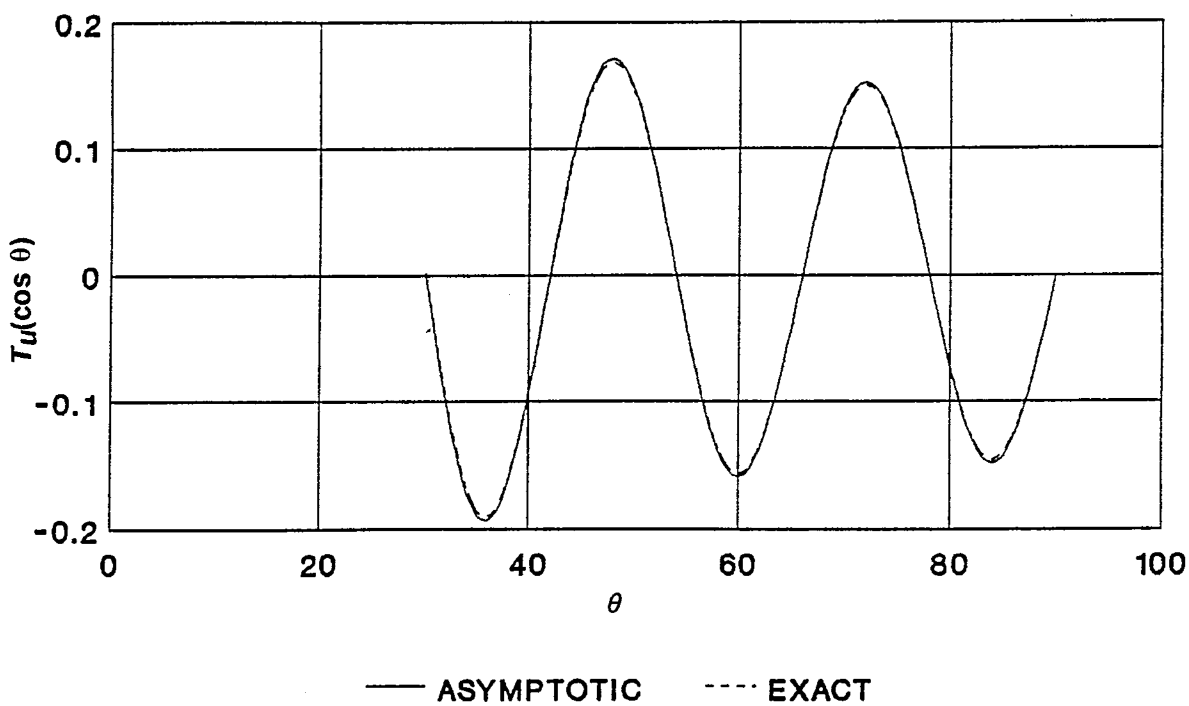

Figure A-12. Comparison between asymptotic and exact T_u for θ_0= 30° and m = 5.

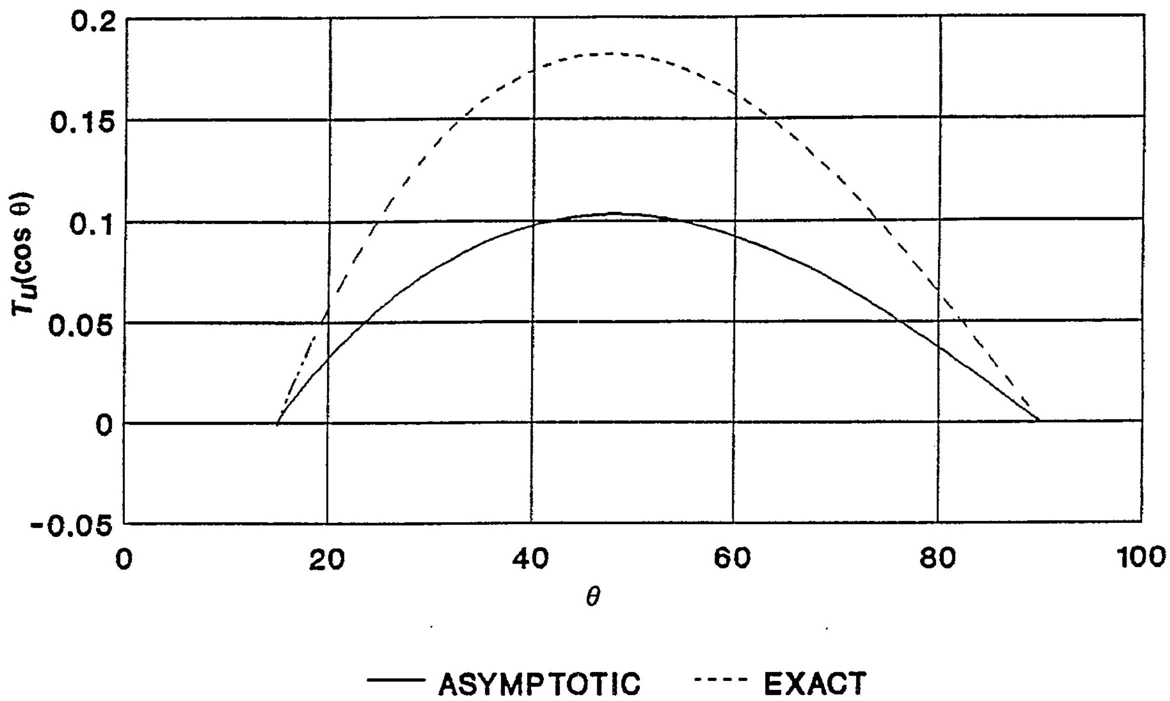

Figure A-13. Comparison between asymptotic and exact T_u for θ_0 = 15° and m = 1.

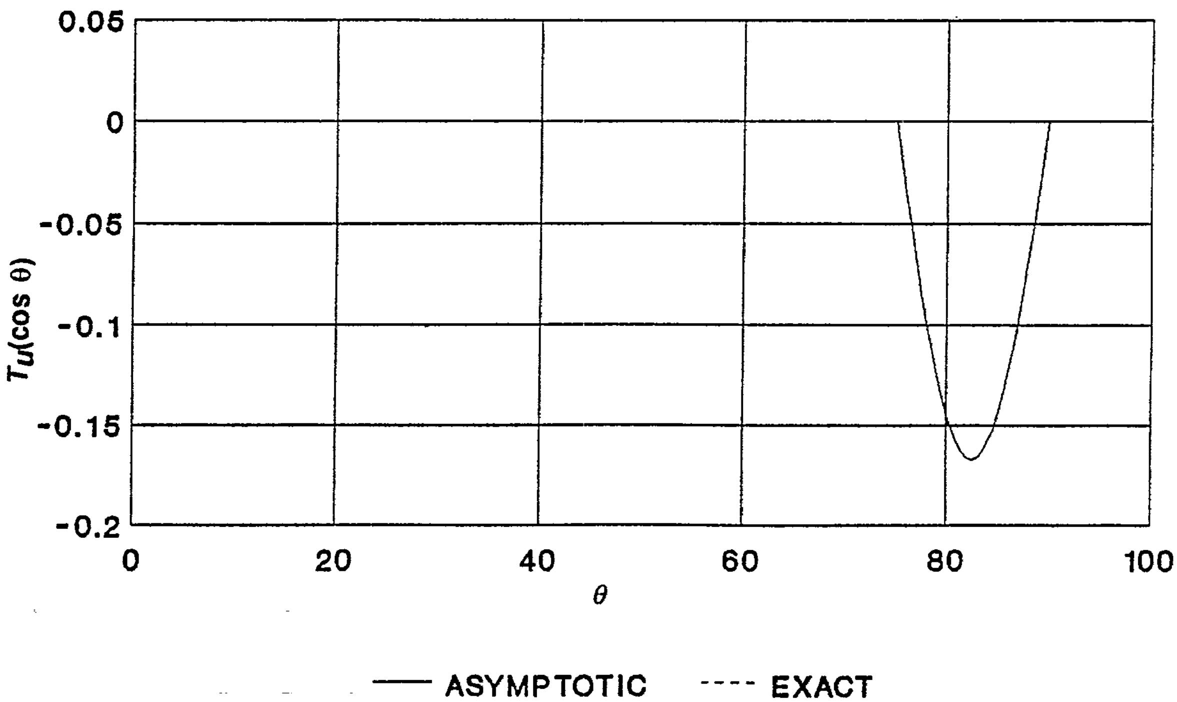

Figure A-14. Comparison between asymptotic and exact T_u for θ_0 = 75° and m = 1.

REFERENCES

1. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions,* Appl. Math. Ser. 55, U.S. Government Printing Office, Washington, D.C. (1964).

2. W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics,* Chelsea Publishing Company, New York (1949).

3. W. Press, B.P. Flannery, S.A. Teukolsky, and W.T. Vetterling, *Numerical Recipes: The Art of Scientific Computing,* Cambridge University Press (1986).

4. S.A. Schelkunoff, *Advanced Antenna Theory,* John Wiley & Sons, New York (1952).