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# Extending the Definitions of Antenna Gain and Radiation Pattern Into the Time Domain

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## Abstract

Many of the classical parameters of frequency domain (CW) antennas, such as gain and radiation pattern, are defined only in the frequency domain, and currently have no meaning in the time domain. The purpose of this note is to extend their definitions into the time domain.

We develop here the concept of waveform norms as a mechanism for comparing the radiated field to the driving voltage. The concept of gain that we develop then compares a norm of the radiated field to a norm of the derivative of the driving voltage. The transient gain is therefore a function of both the choice of norm, and of the choice of driving waveform. A key feature of our definitions of gain and pattern factor is that they are equivalent in both transmit and receive modes.

# I. Introduction

When characterizing antennas meant for radiating fast transient pulses, one encounters the problem of how best to describe antenna performance in a compact manner. The parameters normally used to describe antennas, such as gain, antenna pattern, beamwidth, and sidelobe level, are not meaningful in the time domain. Of course, one could express each of these terms as a function of frequency, but they would then lose their usefulness as compact descriptions of the antenna's behavior. What is needed is a small number of parameters that describe the characteristics of an antenna radiating a transient. We will see that such a description is more complicated in the time domain than in the frequency domain. Nevertheless, the problem can be simplified, and we develop a set of expressions that are as simple as possible.

Fast transient antennas are useful in two types of radar problem. The first is a radar based on the Singularity Expansion Method (SEM) [1]. This type of radar attempts to identify an aircraft by exciting its major resonance frequencies. When the resonance frequencies are reradiated, one can identify the aircraft by its aspect-independent resonance frequencies. This type of radar requires a broadband waveform of relatively long duration, with frequency content maximized near the resonant frequencies of interest of the target. The second type of radar one should consider for these antennas is based on detection of scattering centers of the aircraft [2]. In this method, one attempts to detect the location of the major scattering centers on the aircraft. By knowing the distances between the major scattering centers, identification may be possible. Of course, this approach is aspect dependent. This technique is analogous to taking a TDR of an aircraft, and so requires an impulse-like waveform with fast risetime and short duration, i. e., like an impulse function. This last technique is also useful for locating objects in a cluttered environment, such as when looking out over the sea near the horizon.

There are a multitude of ways to define transient gain, because there are a multitude of ways of describing a transient waveform. This stands in stark contrast to frequency domain waveforms which need only magnitude and phase to describe them. For various situations, the peak magnitude of a transient waveform may be important, or the energy in the pulse, or the area of the pulse. In some cases, the magnitude of the frequency domain spectrum will be most important, for example when one wishes to couple a signal into a system with a narrow band transfer function [14]. In other cases, we will wish to maximize the transient radar cross section [6]. The definition of gain we develop must be able to handle all of these cases, and in addition, it must be equally well applicable to transmit mode as receive mode.

A number of other approaches have been used to address this problem. Bounds on the optimum performance of transient antennas antenna have been developed for a number of different configurations, including planar apertures [16], dipoles [17], dipole arrays [19], and arbitrary antennas constrained to fit within a sphere [18]. An interesting approach to antenna optimization [20], although cast in the frequency domain, uses norm concepts and thereby has some relation to the approach we use here. Furthermore, a specific attempt to define transient antenna parameters has appeared previously in [21]. The definitions in [21] are based primarily on peak fields and total energies of a radiated waveform. In this paper we extend upon the work of [21] to define gain in terms of arbitrary measures of performance (norms).

We begin with a review of the meanings of gain and antenna pattern in the frequency domain, quoting from the IEEE standard. Following this, a review of relevant norms for time domain waveforms and functions is provided. This is done in order to have a set of tools with which to characterize the time domain waveforms. Next, we summarize already-known parameters of an antenna, in both the time and frequency domains. Since the time domain antenna parameters are dependent upon the incident field waveshape, we provide a number of waveshapes that are useful as driving functions. Next, definitions for time domain gain and antenna pattern are provided. We then provide examples of gain calculations for a number of common antennas using various specifications of norm and driving waveform. Next, we develop a time domain radar equation as an illustration of how antenna gain would be used as part of a system description. Finally, since the definition of far field is also unclear in the time domain, we provide a definition for it.

## II. Classical Antenna Parameters in the Frequency Domain

In order to extend existing definitions, we begin by reproducing those definitions from the IEEE standard [3]. These definitions apply primarily to continuous wave (CW) systems, although they may also be used for "quasi-CW" systems such as High Power Microwave (HPM) systems, which radiate on the order of 100 cycles of a CW signal, or radars operating in pulsed (gated) CW mode.

The relevant definitions are:

**gain; absolute gain (of an antenna in a given direction).** The ratio of the radiation intensity, in a given direction, to the radiation intensity that would be obtained if the power accepted by the antenna were radiated isotropically.

NOTES: (1) Gain does not include losses arising from impedance and polarization mismatches.

(2) The radiation intensity corresponding to the isotropically radiated power is equal to the power accepted by the antenna divided by $4\pi$.

(3) If an antenna is without dissipative loss, then in any given direction, its gain is equal to its directivity.

(4) If the direction is not specified, the direction of maximum radiation intensity is implied.

(5) The term absolute gain is used in those instances where added emphasis is required to distinguish gain from relative gain; for example, absolute gain measurements.

**gain, partial (of an antenna for a given polarization).** In a given direction, that part of the radiation intensity corresponding to a given polarization divided by the radiation intensity that would be obtained if the power accepted by the antenna were radiated isotropically.

NOTE: The (total) gain of antenna, in a specified direction, is the sum of the partial gains for any two orthogonal polarizations.

**directivity (of an antenna) (in a given direction).** The ratio of the radiation intensity in a given direction from the antenna to the radiation intensity averaged over all directions.

NOTES: (1) The average radiation intensity is equal to the total power radiated by the antenna divided by $4\pi$.

(2) If the direction is not specified, the direction of maximum radiation intensity is implied.

**directivity, partial (of an antenna for a given polarization).** In a given direction, that part of the radiation intensity corresponding to a given polarization divided by the total radiation intensity averaged over all directions.

NOTE: The (total) directivity of an antenna, in a specified direction, is the sum of the partial directivities for any two orthogonal polarizations.

**radiation intensity.** In a given direction, the power radiated from an antenna per unit solid angle.

**radiation pattern; antenna pattern.** The spatial distribution of a quantity which characterizes the electromagnetic field generated by an antenna.

NOTES: (1) The distribution can be expressed as a mathematical function or as a graphical representation.

(2) The quantities which are most often used to characterize the radiation from an antenna are proportional to, or equal to, power flux density, radiation intensity, directivity, phase, polarization, and field strength.

(3) The spatial distribution over any surface or path is also an antenna pattern.

(4) When the amplitude or relative amplitude of a specified component of the electric field vector is plotted graphically, it is called an **amplitude pattern, field pattern, or voltage pattern.** When the square of the amplitude or relative amplitude is plotted, it is called a **power pattern.**

(5) When the quantity is not specified, an amplitude or power pattern is assumed.

**radiation efficiency.** The ratio of total power radiated by the antenna to the net power accepted by the antenna from the connected transmitter.

**far-field region.** That region of the field of an antenna where the angular field distribution is essentially independent of the distance from a specified point in the antenna region.

NOTES: (1) In free space, if the antenna has a maximum overall dimension, D, which is large compared to the wavelength, the far-field region is commonly taken to exist at distances greater than $2D^2/\lambda$ from the antenna, $\lambda$ being the wavelength. The far-field patterns of certain antennas, such as multi-beam reflector antennas, are sensitive to variations in phase over their apertures. For these antennas $2D^2/\lambda$ may be inadequate.

(2) In physical media, if the antenna has a maximum overall dimension, D, which is large compared to $\pi/|\gamma|$, the far-field region can be taken to begin approximately at a distance equal to $|\gamma|\, D^2/\lambda$, $\gamma$ being the propagation constant in the medium.

**standard [reference] directivity.** The maximum directivity from a planar aperture of area A, or from a line source of length L, when excited with a uniform amplitude, equiphase distribution.

NOTES: (1) For planar apertures in which $A >> \lambda^2$. The value of the standard directivity is $4\pi A/\lambda^2$, with $\lambda$ the wavelength and with radiation confined to a half space.

(2) For line sources with $L >> \lambda$, the value of the standard directivity is $2L/\lambda$.

Let us point out some salient features of these definitions. First, we note that gain is independent of source mismatch. In fact, antenna gain is independent of all source parameters with the exception of frequency. In the time domain, we might consider replacing frequency with risetime, peak derivative, or Full Width Half Max (FWHM) of the driving function. Second, we note that the definitions of gain and directivity assume one is looking at the total radiation in a given direction. If one were considering the effects of polarization, one would use a partial gain, or partial directivity. Next, we note that gain is normalized to the "power accepted by the antenna," while directivity is normalized to the "total power radiated by the antenna". Antenna gain takes into account antenna losses, while directivity does not.

It is interesting to note here that all the definitions for terms such as gain and antenna pattern are defined solely for transmit mode--there is no mention of the antenna being used as a receiver. This is another way of stating that the antenna is a reciprocal device. In the frequency domain, the concept of reciprocity is trivial. To convert a transmit pattern to a receive pattern, one merely multiplies by $1/j\omega$, which is a constant in the CW domain. However, in the time domain, to convert from transmit mode to receive mode one must take the integral of the impulse response for transmit mode. Thus, reciprocity is a somewhat more complicated concept in the time domain, and we would like any new definitions we propose to be consistent with it.

Before going on to the time domain definitions, let us associate some equations with the above definitions. From [4, p. 37] we find gain is

$$G(\theta, \phi, \omega) = \frac{4\pi U(\theta, \phi, \omega)}{P_{\text{in}}(\omega)} \tag{2.1}$$

where $U(\theta, \phi, \omega)$ is the radiation intensity in Watts/steradian, and $P_{\text{in}}(\omega)$ is the power accepted by the antenna. Furthermore, one definition of antenna pattern is expressed as

$$F(\theta, \phi, \omega) = \frac{U(\theta, \phi, \omega)}{U(\theta_{\max}, \phi_{\max}, \omega)} \tag{2.2}$$

where $U(\theta_{\max}, \phi_{\max})$ is the radiation intensity in the direction of maximum radiation (boresight).

Note a number of features about the above equations. Gain is expressed as a ratio of power densities, but it also provides information about peak field magnitudes, peak field derivatives, and integrals over a fixed time window of fields or fields squared (energy). This is all true because no matter what form the antenna takes, a sinusoidal driving signal generates a sinusoidal radiated signal. Analogously, a frequency domain antenna pattern is expressed as a ratio of field magnitudes, but there is also information about the ratios of power density, peak derivatives, and energy over a time window.

We now begin to see the difficulty in extending the common antenna parameters into the time domain. Each of these parameters provides more information than is at first readily apparent. In the frequency domain a sinusoidal driving signal must radiate a sinusoidal field. In the time domain, however, there is no guarantee that the same waveshape will be radiated that drives the antenna (although under certain circumstances on boresight an approximate derivative of the driving voltage is radiated). It will become important to compare similar (though not necessarily identical) waveshapes in transmission and reception, in order to satisfy reciprocity relations. If this is true, then one can develop a transient gain that has as much information as its frequency domain counterpart, and still supports reciprocity.

## III. Review of Waveform Norms

In order to describe the waveforms associated with transient antennas, we use the language of norms. This allows us to describe a complicated time domain waveform $f(t)$, in terms of some of its most important features. Later, our definitions of gain and antenna pattern will be based on these norm descriptions of the waveforms. One reason for this norm-based formulation is so we can provide an upper bound on the total response of a system, if its individual components can be described by norms. We discuss this further in Section 10 of this paper.

To begin our review of norms, let us begin with the definition of a norm for continuous time domain waveforms [5]. The three classical criterion necessary for having a norm are

$$\|f(t)\| \begin{cases} = 0 & \text{iff } f(t) \equiv 0 \\ > 0 & \text{otherwise} \end{cases} \tag{3.1}$$

$$\|\alpha f(t)\| = |\alpha| \, \|f(t)\| \tag{3.2}$$

$$\|f(t) + g(t)\| \le \|f(t)\| + \|g(t)\| \tag{3.3}$$

The first equation states that the norm of a function can be zero if and only if the function is zero. The second equation establishes linearity, and the third equation establishes the triangle inequality.

In addition to the norms of time domain functions, we are also interested in the norm of time domain operators. The importance of these operator norms will become clear later when we discuss the norm of the convolution operator. Thus, we can define the norm of an operator $\Lambda$ on a time domain function as

$$\|\Lambda(\;)\| = \sup_{f(t)\neq 0} \frac{\|\Lambda(f(t))\|}{\|f(t)\|} \tag{3.4}$$

In order to find this norm, one must search for the time domain function $f(t)$ that maximizes the ratio of the norm of $\Lambda(f(t))$ to the norm of $f(t)$. The same set of requirements apply to operator norms that apply to function norms, analogous to (3.1)-(3.3). Thus,

$$\|\Lambda(\;)\| \begin{cases} = 0 & \text{iff } \Lambda(\;) \equiv 0 \\ > 0 & \text{otherwise} \end{cases} \tag{3.5}$$

$$\|\alpha \Lambda(\;)\| = |\alpha| \, \|\Lambda(\;)\| \tag{3.6}$$

$$\|\Lambda(\;) + \mathrm{T}(\;)\| \le \|\Lambda(\;)\| + \|\mathrm{T}(\;)\| \tag{3.7}$$

$$\|\Lambda(\mathrm{T}(\;))\| \le \|\Lambda(\;)\| \, \|\mathrm{T}(\;)\| \tag{3.8}$$

where the last equation is Schwartz's inequality. Finally, we note that equation (3.4) implies

$$\|\Lambda(f(t))\| \le \|\Lambda(\;)\| \, \|f(t)\| \tag{3.9}$$

A typical example of this equation might be if the operator is the convolution of a second function (say $g(t)$) with the first function, in this case $f(t)$. In this context, the above equation becomes

$$\left\| g(t) \circ f(t) \right\| \le \left\| g(t) \circ \right\| \left\| f(t) \right\| \tag{3.10}$$

One can think of $g(t)$ as the impulse response of a system, and $f(t)$ as the driving function. Thus, the above equation leads to an upper bound on the output of a system.

Now that we have defined the concept of norms, we can describe some common norms. Perhaps the most common norm one can think of is the class of $p$-norms, which are defined as

$$\left\| f(t) \right\|_p \equiv \left( \int_{-\infty}^{\infty} \left| f(t) \right|^p dt \right)^{1/p}, \qquad \left\| f(t) \right\|_\infty \equiv \sup_t \left| f(t) \right| \tag{3.11}$$

We can identify three $p$-norms of possible interest to the current problem, the 1-norm, which is the area of the rectified waveform, the 2-norm, which is the area of the waveform squared (proportional to the square root of the total energy in the waveform), and the $\infty$-norm, which identifies the peak magnitude of the waveform. We will see later that $p$-norms have some properties that make them particularly easy to work with. These properties allow us to set bounds on the norms of the measured response in a cascade of system components which are each describable in terms of norms.

One of the special properties we will need later deals specifically with the application of a $p$-norm to a convolution operator. It is shown in [5] that the $p$-norm of the convolution operator has the property

$$\left\| g(t) \circ \right\|_p \le \left\| g(t) \right\|_1 \tag{3.12}$$

Thus, any $p$-norm of a convolution operator is bounded by the 1-norm (or rectified integral) of the corresponding impulse response. This is a result that will be useful to us later when the system to which we are referring is a transient antenna. It turns out that for the 1-norm and the $\infty$–norm the above equation is actually an equality. Thus,

$$\left\| g(t) \circ \right\|_1 = \left\| g(t) \right\|_1 \tag{3.13}$$

$$\left\| g(t) \circ \right\|_\infty = \left\| g(t) \right\|_1 \tag{3.14}$$

These properties are shown in [5], and will come in handy later.

Another type of norm of interest to us is the $m$-norm. This is applied to vectors, and identifies the peak magnitude of a vector over all time. This is applied, for example, to plane waves and far fields (two components) in a polarization-independent and coordinate-independent way. Thus,

$$\left\| \vec{f}(t) \right\|_m = \sup_t \left| \vec{f}(t) \right|, \qquad \left\| \vec{f} \right\|_m = \left| f \right| \tag{3.15}$$

This in some sense is a "natural" norm of a vector [6], or a norm that uses symmetry to best advantage. The 2-norm of such a vector also retains this symmetry and has

$$\left\|\vec{f}(t)\right\|_2 = \left(\int_{-\infty}^{\infty}\left|\vec{f}(t)\right|^2 dt\right)^{1/2} = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}\tilde{\vec{f}}(j\omega)\cdot\tilde{\vec{f}}(-j\omega)\,d\omega\right)^{1/2}$$

$$s = \Omega + j\omega \tag{3.16}$$

$$\sim \;=\; \text{Laplace transform (2-sided)}$$

In these definitions, it is assumed that $\vec{f}(t)$ is real. Similar expressions are available for scalar $f(t)$ as well.

Since we have mentioned the concept of natural norms, it may be of benefit to review their properties. By a natural norm we mean one that incorporates the symmetries of the physical problem at hand to the extent feasible [6]. First, it should be invariant under time translation, i.e.,

$$\left\|\vec{f}(t-t_o)\right\| = \left\|\vec{f}(t)\right\| \tag{3.17}$$

so that we get the same answer when we repeat an experiment at a later time. All $p$-norms have this property. As discussed in [6], when we are dealing with two-component vectors as in TEM plane waves or antenna far fields, the norm should be invariant to coordinate rotation about the propagation direction (axis). Thus in a vector sense, this leads to a 2-norm as the only acceptable form. In addition, it turns out to be convenient to have the norm invariant under time reversal,

$$\left\|\vec{f}(-t)\right\| = \left\|\vec{f}(t)\right\| \tag{3.18}$$

and $p$-norms do this, but the physical problem does not in general have this symmetry, so we should be willing to sacrifice it if desired in future considerations. These two criteria are necessary, but not sufficient, for a norm to be a natural norm. These definitions lead to the 2-norm and the $m$-norm as being practical choices for vectors and dyads. For scalars, any of the $p$-norms satisfy the above conditions, and are therefore natural. The reason we make a distinction for vectors and dyads, is that one can think of many norms that do not satisfy these basic properties.

So our general $p$-like norm for real transverse plane waves has the general form

$$\left\|\vec{f}(t)\right\| = \left\{\int_{-\infty}^{\infty}\left|\vec{f}(t)\right|^p dt\right\}^{1/p} \tag{3.19}$$

$$= \left\|\,\left|\vec{f}(t)\right|\,\right\|_{pf} \qquad = \left\|\,\left\|\vec{f}(t)\right\|_{2v}\,\right\|_{pf}$$

Note that one *first* takes the 2-norm in vector ($v$) sense, followed by the $p$-norm in function ($f$) sense. These two operations do not in general commute unless the same $p$-value is used for both (in which case it is a true $p$-norm). Examples include the 2-norm as above, and the $m$-norm as

$$\left\|\vec{f}(t)\right\|_m = \sup_t\left|\vec{f}(t)\right| = \left\|\,\left\|\vec{f}(t)\right\|_{2v}\,\right\|_{\infty f} \tag{3.20}$$

Another norm of interest we call the $A$-norm, and it identifies the maximum area of a lobe of a waveform segment bounded by either a zero crossing or infinity. Thus, if a waveform $f(t)$ has

$N$-1 zero crossings, it is divided into $N$ sections of $f_i(t)$ where $1 \le i \le N$, each of which is bounded by either a zero crossing or $\pm\infty$. The $A$-norm is defined by

$$\|f(t)\|_A = \max_i \left[ \|f_i(t)\|_1 \right] \tag{3.21}$$

This norm is essentially the area of the largest "lobe" of the waveform. It is particularly useful when quantifying the area under an approximate delta function, which occurs when an Impulse Radiating Antenna is excited by a step function.

Finally, we develop two sets of norms that extend the class of $p$-norms. The first of these is the $D$-$p$-norm, which is the $p$-norm of the derivative of a waveform. It is specified as

$$\|f(t)\|_{Dp} = \left\| \frac{df(t)}{dt} \right\|_p \tag{3.22}$$

If $p = \infty$ then this identifies the peak derivative of $f(t)$, which may be useful as an alternative to the 10-90% risetime of the waveform. Another method of extending the class of $p$-norms is by adding an integration. Thus we define

$$\|f(t)\|_{Ip} = \left\| \int_{-\infty}^{t} f(t')\, dt' \right\|_p \tag{3.23}$$

where $f(t)$ is constrained to be zero at $t = -\infty$. The idea here is that if a particular $p$-norm of a waveform is ill-behaved, we can replace it with either a $D$-$p$-norm or (more likely) an $I$-$p$-norm. This is useful, for example, when one wishes to take the 2-norm of a system function containing a Dirac delta function, generating an ill-behaved (infinite) norm. Instead, we can use the $I$-2 norm, which is well-defined for our problem. The practical advantage we obtain here is that we do not have to express a set of equations twice (in the original form and the integrated form); we simply switch to the $I$-$p$-norm to include the integrated form.

These $I$-$p$ and $D$-$p$ norms can be extended to vector $\vec{f}(t)$ as well. A discussion of the norms of vector functions is provided in [12]. Thus, we define the $I$-2 and $I$-$m$ norms as

$$\|\vec{f}(t)\|_{I2} = \left\| \|\vec{f}(t)\|_{2v} \right\|_{I2f} = \left\| |\vec{f}(t)| \right\|_{I2} \tag{3.24}$$

$$\|\vec{f}(t)\|_{Im} = \left\| \|\vec{f}(t)\|_{2v} \right\|_{I\infty f} = \left\| |\vec{f}(t)| \right\|_{I\infty} \tag{3.25}$$

As usual, we first take the 2-norm in a vector sense, and then take either the $I$-2-norm or the $I$-$\infty$–norm of the resulting scalar function. The 2-norm in a vector sense is just the magnitude of the vector at a given time. Analogously, we have for the $D$-2 and $D$-$m$ norms of a vector function

$$\|\vec{f}(t)\|_{D2} = \left\| \|\vec{f}(t)\|_{2v} \right\|_{D2f} = \left\| |\vec{f}(t)| \right\|_{D2} \tag{3.26}$$

$$\left\|\vec{f}(t)\right\|_{Dm} = \left\| \left\|\vec{f}(t)\right\|_{2v} \right\|_{D\infty f} = \left\| \left|\vec{f}(t)\right| \right\|_{D\infty} \tag{3.27}$$

Having reviewed a wide variety of possible norms we can apply to time domain functions, let us now consider how to apply these to antennas.

## IV. Norms of the Antenna/Waveform

We now apply the norms of the preceding section to the combination of the antenna+waveform. This will include properties of the source (pulser) for the case of transmission, and the incident field, for the case of reception.

### A. The Antenna/Waveform Norm in Transmission

Consider the arrangement shown in Figure 4.1. A pulser drives an antenna with a waveform of some shape, and a field is radiated. How does one describe the radiated field and the gain of the system?

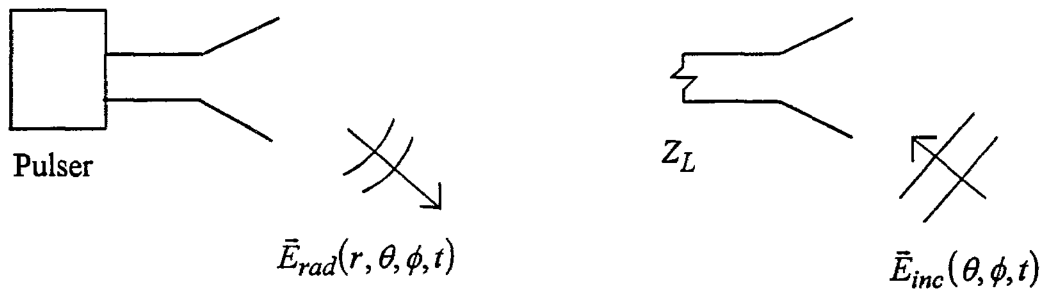


Figure 4.1. The antenna and pulser in transmission, and the antenna and load in reception.

The radiated field is easily expressed as a norm of the radiated field. We call this the transmission norm. Thus, we have

$$\text{transmission norm}(\theta, \phi) = \left\| r\, \vec{E}(r, \theta, \phi, t) \right\| \tag{4.1}$$

where the norm used could be any of the norms of vector functions described in Section 3. We can optimize over all angles, if we wish. So in order to quantify the radiated field, one must (for a given radiation direction) take the magnitude of the electric field, and then apply some scalar norm to the resulting scalar function. Note that the transmission norm is dependent upon the waveshape of the radiated field.

One can also identify a pulser norm, or a norm associated with some characteristic of the source. This is not as easy to write down in mathematical terms as the transmission norm, but it is simple to describe. One can consider a pulser norm to be any of a number of different quantities such as the square root of energy stored in the pulser before firing, its peak voltage, its peak derivative voltage, peak current, etc. Any of these quantities might be something to which one might want to normalize the transmission norm, in order to define a gain of the antenna+pulser. Note that the pulser norms are a function of waveform shape and source impedance.

Now we can put the transmission norm and the pulser norm together to form a gain of sorts for a transient antenna in transmission. The quantity defined is an antenna/pulser gain

$$\text{antenna / pulser gain}(\theta,\phi) = \frac{\text{transmission norm}(\theta,\phi)}{\text{pulser norm}} \tag{4.2}$$

As we have seen, the pulser norm can be defined in a wide variety of ways, so the antenna/pulser gain also has a multitude of definitions. We will see later that one very interesting pulser norm is some norm of the derivative of the driving voltage. In order to sort through all these norms, it is necessary to look at the norms generated for the receive mode, and see if we can establish consistency between the transmit and receive modes.

### B. The Antenna/Waveform Norm in Reception

For the case of reception, we now define various norms and gains as we did for transmission. In this case, it is the incident quantity (field) that is easy to identify, and the received quantity that can be defined in numerous ways. Thus the field norm is now defined as

$$\text{field norm}(\theta,\phi,) = \left\| \vec{E}_{inc}(r,\theta,\phi,t) \right\| \tag{4.3}$$

where we take the norm of a vector quantity in the usual manner.

The received quantity can be defined in a variety of different ways. This may include the square root of the received energy, the peak voltage received, the peak current, or a number of others. Each of these are dependent upon not only the waveshape, but also the load impedance on the antenna. Any of these may be useful quantities to measure the performance of the antenna.

We now put the two norms together to find a gain in reception

$$\text{reception gain}(\theta,\phi) = \frac{\text{reception norm}}{\text{field norm}(\theta,\phi)} \tag{4.4}$$

As before, we must somehow sort through which of all the possible reception norms are the most meaningful. The ultimate answer will be one that has some consistency in both transmission and reception.

### C. Unifying the Transmission and Reception Gains

In the frequency domain, it is simple to unify the gains in reception and transmission. The gain is defined in terms of transmission (see Section II), and for reception, one has an effective area defined as $A_e = G\ \lambda^2 / (4\pi)$, where $G$ is a power gain. Thus, for reception, the received voltage when measured with a matched load is

$$V_{rec}(\omega) = \sqrt{f_g A_e}\ E_{inc}(\omega) = \sqrt{f_g G \lambda^2 / 4\pi}\ E_{inc}(\omega) \tag{4.5}$$

where $f_g$ is the ratio of the load impedance to the impedance of free space. Thus, the concept of gain in the frequency domain applies equally well in transmission and reception.

In the time domain, however, we have seen a multitude of definitions of the gains in both the transmission and reception. How does one establish consistency between transmit and receive modes in the time domain? In order to do so, one must first review the basic theory of these antennas. Once the theory is in place, the choice for a gain definition that is consistent in transmit and receive modes will become more clear.

## V. The Antenna as a Convolution Operator

In order to unify the concept of transient gain for both transmission and reception, we consider the radiation characteristics of an antenna in more detail. We will see that the radiation from an antenna can be thought of as the convolution of a system impulse response with a driving function. In a similar manner, the received voltage can be thought of as the convolution of the antenna impulse response with the incident field. Once we have established this system function, the norm of this system function (convolution operator) represents the best possible response the antenna can achieve for a given norm. In order to achieve this optimal radiation, one must drive the antenna with "ideal" functions such as step functions and impulse functions. Since this is not in general practical, we extend the theory to "nonideal" excitation functions, with finite risetimes and bandwidths.

The first step in developing a gain that is consistent in transmit and receive mode is to define the pulser and load impedances. Consider the antenna and feed system shown in Figure 5.1 for transmission, and Figure 5.2 for reception. Our gain definition will consist of a ratio of norms of two waveforms. Consider the case of reception. If we have an open circuit or short circuit at the receiving end, then the measured waveform is changed by the load. By having a matched load, however, the received voltage $V_r(t)$ is independent of the load. If we use the same condition for the case of transmission, i.e., a matched source, then a symmetry is preserved, and one can develop a gain that is consistent between transmission and reception. This is analogous to the use of scattering parameters in circuit theory.

Let us now develop the equations that describe these antennas in both the frequency and time domains.

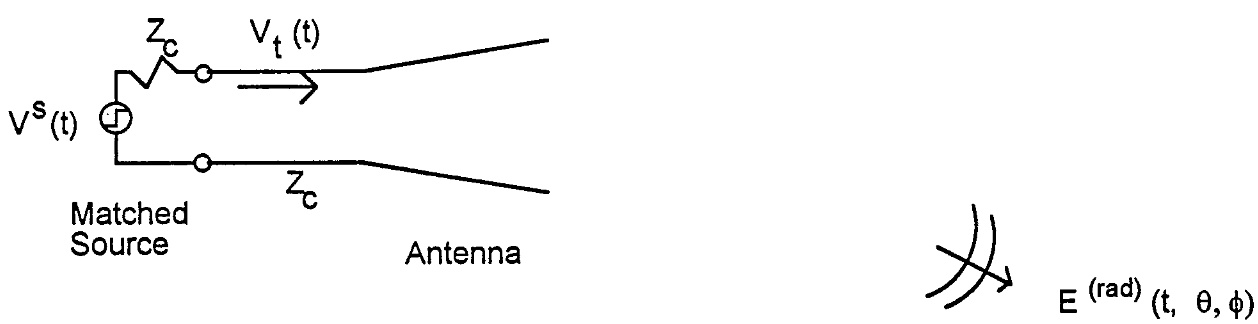


Figure 5.1. A transient antenna in transmit mode.

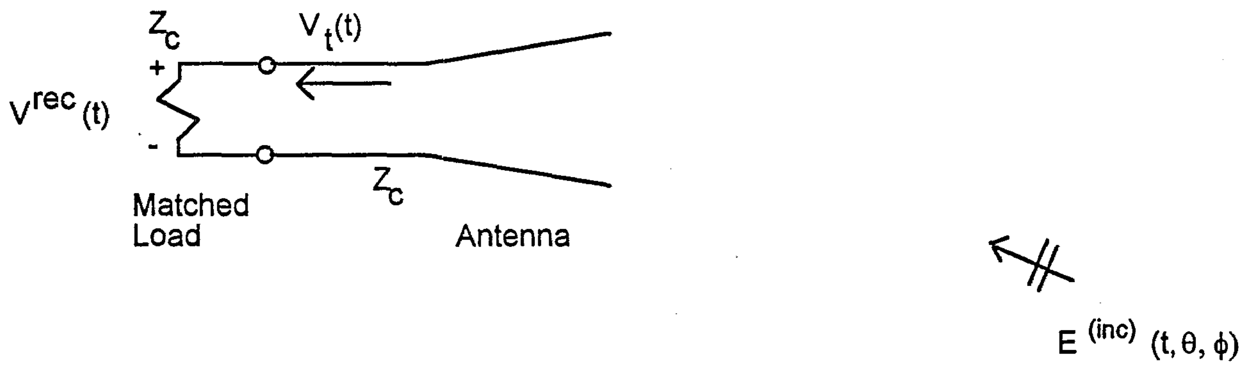


Figure 5.2. A transient antenna in receive mode.

**A. Frequency Domain Expressions**

We develop first the relevant equations in the frequency domain. The equations for the time domain are somewhat more involved, and the next three subsections are devoted to them. Consider the antenna/feed system shown in Figure 5.1. An incident voltage wave $V_t(t)$approaches the antenna on a transmission line of impedance $Z_C$, where $Z_C$ is assumed to be real. As shown in [7], the radiated field is

$$\tilde{\bar{E}}(\bar{r},s) = \frac{e^{-\gamma r}}{r}\tilde{\bar{F}}_t(\bar{1}_r,s)\tilde{V}_t(s) \tag{5.1.1}$$

where $\bar{1}_r$ is the direction of radiation. Alternatively, if we are only interested in the portion of the radiated field that is polarized in the $\bar{1}_e$ direction, we find

$$\tilde{\bar{E}}(\bar{r},s)\cdot\bar{1}_e = \frac{e^{-\gamma r}}{r}\left[\tilde{\bar{F}}_t(\bar{1}_r,s)\cdot\bar{1}_e\right]\tilde{V}_t(s) \tag{5.1.2}$$

We can also consider what happens when a planar electric field is incident upon the antenna. Consider the configuration of Figure 5.2, where an incoming plane wave has the form $\tilde{\bar{E}}_o(s)$ at the origin. It is shown in [7] that

$$\tilde{V}_t(s) = \tilde{\bar{h}}_t(\bar{1}_i,s)\cdot\tilde{\bar{E}}_o(s) \tag{5.1.3}$$

where $\tilde{\bar{E}}_o(s)$ is the incident field at the origin and $\bar{1}_i$ is the direction of incidence for the incoming field. If an incoming plane wave is polarized in the $\bar{1}_P$ direction, then this simplifies to

$$\tilde{V}_t(s) = \left[\tilde{\bar{h}}(\bar{1}_i,s)\cdot\bar{1}_P\right]\tilde{E}_o(s) \tag{5.1.4}$$

where $\tilde{\bar{E}}_o(t) = \bar{1}_P\, E_o(t)$.

There is now a simple relationship between the pattern factor $\tilde{\vec{F}}_t(\vec{1}_r, s)$ and the effective height $\tilde{\vec{h}}_t(-\vec{1}_r, s)$. In general,

$$\tilde{\vec{F}}_t(\vec{1}_r, s) = -\frac{s\mu_o}{2\pi Z_c} \vec{1}_r \cdot \tilde{\vec{h}}_t(-\vec{1}_r, s) \tag{5.1.5}$$

where

$$\vec{1}_r = \vec{1} - \vec{1}_r\vec{1}_r, \qquad \vec{1} = \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y + \vec{1}_z\vec{1}_z, \qquad \vec{1}_e \cdot \vec{1}_r = 0 \tag{5.1.6}$$

Once again, this simplifies if we express this in terms of a single polarization with $\vec{1}_P = \vec{1}_e$, so

$$\tilde{\vec{F}}_t(\vec{1}_r, s) \cdot \vec{1}_e = -\frac{s\mu_o}{2\pi Z_c} \tilde{\vec{h}}_t(-\vec{1}_r, s) \cdot \vec{1}_e \tag{5.1.7}$$

These equations are important because they allow us to establish reciprocity between the radiated field and the received field in the frequency domain.

Note that these definitions are all set up for an antenna with a matched load. This may be of concern to some, since the IEEE definitions of gain are set up in such a manner that the load is irrelevant--gain is simply normalized to the power accepted by the antenna in transmission. When working in the time domain, however, we will be comparing the characteristics of one waveshape to another. Since the waveshape is clearly going to be affected by the load type, we have to settle on some standard. Most of the antennas of interest, such as the Impulse Radiating Antenna [8] or the TEM horn [9], normally require an approximately matched resistive load (frequency independent) anyway, so this seems like the natural choice.

### B. Time Domain Expressions -- Receive Mode

Now that we have expressed all of the relevant equations in the frequency domain, we repeat the process in the time domain. In doing so, we apply the norm relationships to bound the time domain output. We begin by converting the equation for the received voltage (5.1.3) to the time domain. Thus we find

$$V_t(t) = \vec{h}_t(\vec{1}_i, t) \overset{\circ}{\cdot} \vec{E}_o(t) \tag{5.2.1}$$

where the dot product convolution operator $\overset{\circ}{\cdot}$ implies the convolution of each component of the vectors. In terms of a single polarization we have

$$V_t(t) = \left[\vec{h}_t(\vec{1}_i, t) \cdot \vec{1}_P\right] \circ E_o(t) \tag{5.2.2}$$

Now, the quantity in square brackets is just a simple time domain function (impulse response). Note that the units of $\vec{h}(t)$ are meters/second. At first glance, this may be surprising, since $\vec{h}(t)$

seems to represent a height, i.e., with units of meters. In fact, it is the *operator* $\vec{h}(t)\circ$ that has the units of meters. Taking now the norm of both sides we have

$$\|V_t(t)\| = \left\| \vec{h}(\vec{1}_i,t) \overset{\circ}{\cdot} \vec{E}_o(t) \right\| \tag{5.2.3}$$

or for a single polarization

$$\|V_t(t)\| = \left\| \left[\vec{h}(\vec{1}_i,t)\cdot\vec{1}_P\right]\circ E_o(t) \right\| \tag{5.2.4}$$

The above equations simply state that the norm of the received voltage is equal to the norm of the convolution of the impulse response of the antenna and the incident field. If we group the convolution operator with the effective height and apply the definition of the norm of an operator, we find an upper bound on the received voltage. Thus,

$$\|V_t(t)\| = \left\| \left[\vec{h}(\vec{1}_i,t) \overset{\circ}{\cdot} \right]\vec{E}_o(t) \right\| \le \left\| \vec{h}(\vec{1}_i,t) \overset{\circ}{\cdot} \right\| \left\|\vec{E}_o(t)\right\| \tag{5.2.5}$$

where $\left\| \vec{h}(\vec{1}_i,t) \overset{\circ}{\cdot} \right\|$ has to be interpreted in the sense of a norm of a dyadic (matrix) convolution. This is much simpler to understand one polarization at a time, so for a single polarization,

$$\|V_t(t)\| = \left\| \left[\vec{h}(\vec{1}_i,t)\cdot\vec{1}_P\circ\right]E_o(t) \right\| \le \left\| \vec{h}(\vec{1}_i,t)\cdot\vec{1}_P\circ \right\| \left\|E_o(t)\right\| \tag{5.2.6}$$

Thus, if some norm is applied to the incident field and to the antenna system response, then one can place an upper bound on the norm of the received voltage.

Finally, we invoke the property that the $p$-norm of a convolution operator is less than or equal to the 1-norm of the impulse response. Thus we find, for a single polarization,

$$\|V_t(t)\|_p \le \left\| \vec{h}(\vec{1}_i,t)\cdot\vec{1}_P \right\|_1 \|E_o(t)\|_p \tag{5.2.7}$$

This establishes a bound on the norm of the received voltage for a given incident field polarization. Note that in general we will not want to constrict ourselves to using $p$-norms, but if one chooses to one can invoke a nice simplification.

Consider now the boundedness of the received voltage. Invoking the properties of $p$-norms, we find

$$\left\|\vec{h}_t(\vec{1}_i,t)\cdot\vec{1}_P\circ\right\|_p \le \left\|\vec{h}_t(\vec{1}_i,t)\cdot\vec{1}_P\right\|_1 = \int_0^\infty \left|\vec{h}_t(\vec{1}_i,t)\cdot\vec{1}_P\right| dt \tag{5.2.8}$$

where we have begun the integration at $t = 0$ because the operator is causal. Thus, the received voltage is bounded if the impulse response is integrable. It seems this must always be true for antennas of finite size, or else the antenna would ring forever when excited by an impulse-like incident field. Therefore, the impulse response is bounded for all $p$-norms. At some point it would be worthwhile to extend this to include norms other than the $p$-norm.

## C. Time Domain Expressions -- Transmit Mode

Let us consider now the problem of the transmitting antenna in the time domain. If we launch a voltage $V_t(t)$ onto the line toward the antenna, we find the radiated field in the time domain is

$$\vec{E}(\vec{r},t) = \left[\frac{e^{-\gamma r}}{r}\vec{F}_t\left(\vec{1}_r,t\right)\circ\right]V_t(t) \tag{5.3.1}$$

or for a single polarization,

$$\vec{E}(\vec{r},t)\cdot\vec{1}_e = \left[\frac{e^{-\gamma r}}{r}\vec{F}_t\left(\vec{1}_r,t\right)\cdot\vec{1}_e\right]\circ V_t(t) \tag{5.3.2}$$

where $\vec{1}_e$ is the polarization vector, and $\vec{1}_e\cdot\vec{1}_r = 0$. Note that the units of $\vec{F}\left(\vec{1}_r,t\right)$ are 1/seconds, and the convolution operator $\vec{F}_t\left(\vec{1}_r,t\right)\circ$ is unitless. We now proceed in a manner analogous to the receive mode. We move the $1/r$ factor to the left side to balance the units, and take the norm of both sides. Finally, we invoke time translation invariance to find

$$\left\| r\vec{E}(\vec{r},t)\right\| = \left\|\left[\vec{F}_t\left(\vec{1}_r,t\right)\circ\right]V_t(t)\right\| \le \left\|\vec{F}_t\left(\vec{1}_r,t\right)\circ\right\|\left\|V_t(t)\right\| \tag{5.3.3}$$

again with due care to the meaning of $\left\|\vec{F}_t\left(\vec{1}_r,t\right)\circ\right\|$. For a single polarization we have

$$\left\|r\vec{E}(\vec{r},t)\cdot\vec{1}_e\right\| = \left\|\left[\vec{F}_t\left(\vec{1}_r,t\right)\cdot\vec{1}_e\circ\right]V_t(t)\right\| \le \left\|\vec{F}_t\left(\vec{1}_r,t\right)\cdot\vec{1}_e\circ\right\|\left\|V_t(t)\right\| \tag{5.3.4}$$

We run into a problem now, since the norm of the convolution operator $\vec{F}_t\left(\vec{1}_r,t\right)\circ$ is not bounded in general for arbitrary $p$-norms. One can demonstrate this by converting (5.1.5) to the time domain to find

$$\left\|\vec{F}_t\left(\vec{1}_r,t\right)\circ\right\| = -\frac{\mu_o}{2\pi Z_c}\left\|\vec{1}_r\cdot\frac{d}{dt}\vec{h}_t\left(-\vec{1}_r,t\right)\circ\right\| \tag{5.3.5}$$

We suggested in the previous subsection that any $p$-norm of the convolution operator $\vec{h}_t\left(-\vec{1}_r,t\right)\circ$ on the right side is bounded, however, with the derivative it seems unlikely that the operator norm is bounded. A counterexample might be the case of an ideal IRA, whose ideal effective height function contains an impulse function. Thus, the 1-norm of the derivative of such an effective height function is unbounded (infinite). In order to remedy this, we multiply the driving voltage by the complex frequency $s$, and the pattern factor by $1/s$ in (5.3.3) and (5.3.4). In the time domain, this leads to

$$r\vec{E}(\vec{r},t) = \left[\int_0^t \vec{F}_t\left(\vec{1}_r,t'\right)dt'\right]\circ\frac{dV_t(t)}{dt} \tag{5.3.6}$$

or for a single polarization

$$r\,\vec{E}(\vec{r},t)\cdot\vec{1}_e \;=\; \left[\int_0^t\left[\vec{F}_t(\vec{1}_r,t')\cdot\vec{1}_e\right]dt'\right]\circ\frac{dV_t(t)}{dt} \tag{5.3.7}$$

Note that we can start the integration at $t = 0$ because the operator is causal. We can now safely take the norms of the above two equations to find the bounds

$$\left\| r\,\vec{E}(\vec{r},t) \right\| = \left\| \left[\int_0^t \vec{F}_t(\vec{1}_r,t')\,dt'\right]\circ\frac{dV_t(t)}{dt} \right\| \le \left\| \left[\int_0^t \vec{F}_t(\vec{1}_r,t')\,dt'\right]\circ \right\| \left\| \frac{dV_t(t)}{dt} \right\| \tag{5.3.8}$$

or for a single polarization

$$\left\| r\,\vec{E}(\vec{r},t)\cdot\vec{1}_e \right\| = \left\| \left[\int_0^t\left[\vec{F}_t(\vec{1}_r,t')\cdot\vec{1}_e\right]dt'\right]\circ\frac{dV_t(t)}{dt} \right\| \le \left\| \left[\int_0^t\left[\vec{F}_t(\vec{1}_r,t')\cdot\vec{1}_e\right]dt'\right]\circ \right\| \left\| \frac{dV_t(t)}{dt} \right\| \tag{5.3.9}$$

The integrated pattern function is now well-behaved. The integral of $\vec{F}(\vec{1}_r,t)$ is unitless, so the convolution operator above has the units of seconds. Thus, in order to establish a bound on the radiated field it is necessary to find the norm of the *derivative* of the input voltage. Perhaps this may not be surprising, since the radiated field on boresight tends to be the derivative of the input signal.

We can introduce now the usual simplification for $p$-norms of a convolution operator to get, for a single polarization

$$\left\| r\,\vec{E}(\vec{r},t)\cdot\vec{1}_e \right\|_p \le \left\| \left[\int_0^t\left[\vec{F}_t(\vec{1}_r,t')\cdot\vec{1}_e\right]dt'\right] \right\|_1 \left\| \frac{dV_t(t)}{dt} \right\|_p \tag{5.3.10}$$

This establishes a simple upper bound on the $p$-norm of the radiated field, although as we stated earlier, we do not necessarily wish to confine ourselves to $p$-norms.

### D. Correlation of Transmit Mode to Receive Mode

We now relate the effective height convolution operator, generated for receive mode, to the pattern factor convolution operator, which was developed for the transmit mode. We begin by expressing (5.1.5) in the time domain. Thus,

$$\vec{F}_t(\vec{1}_r,t)\circ \;=\; -\frac{\mu_o}{2\pi Z_c}\;\vec{1}_r\cdot\frac{d}{dt}\vec{h}_t(-\vec{1}_r,t)\circ \tag{5.4.1}$$

or for a single polarization

$$\vec{F}_t(\vec{1}_r,t)\cdot\vec{1}_e \circ = -\frac{\mu_o}{2\pi Z_c}\frac{d}{dt}\left[\vec{h}_t(-\vec{1}_r,t)\cdot\vec{1}_e\right]\circ \tag{5.4.2}$$

We include the convolution operator in the above equations since it is the properties of the operator we are concerned with. If we now take the norm of both sides, we find

$$\left\|\vec{F}_t(\vec{1}_r,t)\circ\right\| = -\frac{\mu_o}{2\pi Z_c}\left\|\frac{d}{dt}\vec{h}_t(-\vec{1}_r,t)\circ\right\| \tag{5.4.3}$$

or for a single polarization

$$\left\|\vec{F}_t(\vec{1}_r,t)\cdot\vec{1}_e\circ\right\| = -\frac{\mu_o}{2\pi Z_c}\left\|\frac{d}{dt}\left[\vec{h}_t(-\vec{1}_r,t)\cdot\vec{1}_e\right]\circ\right\| \tag{5.4.4}$$

Since the above expression for the time derivative of $h_t(t)$ is not bounded for all $p$-norms, a better formulation would have put the $s$ factor on the other side, thus in the time domain,

$$\left[\int_0^t \vec{F}_t(\vec{1}_r,t')\,dt'\right]\circ = -\frac{\mu_o}{2\pi Z_c}\vec{1}_r\cdot\vec{h}_t(-\vec{1}_r,t)\circ \tag{5.4.5}$$

or for a single polarization

$$\left[\int_0^t \vec{F}_t(\vec{1}_r,t')\cdot\vec{1}_e\,dt'\right]\circ = -\frac{\mu_o}{2\pi Z_c}\vec{h}_t(-\vec{1}_r,t)\cdot\vec{1}_e\circ \tag{5.4.6}$$

If we apply norms to the above two equations, we get

$$\left\|\left[\int_0^t \vec{F}_t(\vec{1}_r,t')\,dt'\right]\circ\right\| = -\frac{\mu_o}{2\pi Z_c}\left\|\vec{1}_r\cdot\vec{h}_t(-\vec{1}_r,t)\circ\right\| \tag{5.4.7}$$

or for a single polarization

$$\left\|\left[\int_0^t \vec{F}_t(\vec{1}_r,t')\cdot\vec{1}_e\,dt'\right]\circ\right\| = -\frac{\mu_o}{2\pi Z_c}\left\|\vec{h}_t(-\vec{1}_r,t)\cdot\vec{1}_e\circ\right\| \tag{5.4.8}$$

Alternatively, we could have switched to the class of $I$-$p$-norms, which already include the integral. Finally, we have established reciprocity. With these expressions, we understand a simple relationship between the transmit and receive parameters. Furthermore, if we specify the norm used in the above equation to the 1-norm, then we already know that the above expressions are the bounds we already established for transmit and receive modes. Thus, we have satisfied a central criterion for "reasonableness" of the gain function.

It is worthwhile now to point out a physical meaning to the effective height function. The effective height $\vec{h}_t(\vec{1}_r,t)$ is proportional to the radiated field when the driving voltage is a step function. It is also proportional to the received voltage when the incoming plane wave is an impulse function. Thus, we can say that in transmit mode $\vec{h}_t(\vec{1}_r,t)$ is the step response, and in receive mode, it is the impulse response. We already know the form of the step response in transmission of various antennas, e.g., the step response of an IRA is described in [8], and that of a TEM horn is described in [9]. Thus, these effective height functions, when applied to the boresight cases of the IRA and TEM horn, look very familiar.

## VI. Transient Antenna Gain: A Practical Perspective

Now that we have developed the equations of a transient antenna in detail, let us apply this to the practical situation of specifying antenna gain in the time domain. To do so, we first recall the definition in the frequency domain, "*the ratio of the radiation intensity, in a given direction, to the radiation intensity that would be obtained if the power accepted by the antenna were radiated isotropically.*" It would be nice if we could use for transient gain something like the following, (replacing power by 2 norm and appropriate impedances)

$$G(\theta,\phi) = \frac{\sqrt{U_{rad}(\theta,\phi)}}{\sqrt{P_{in}/4\pi}} = \frac{\left\| r\,\vec{E}(\theta,\phi,t)\cdot\vec{1}_e/\sqrt{Z_o}\right\|_2}{\left\|V_t(t)/\sqrt{Z_c}\right\|_2/\sqrt{4\pi}} = \frac{\sqrt{4\pi f_g}\left\| r\,\vec{E}(\theta,\phi,t)\cdot\vec{1}_e\right\|_2}{\left\|V_t(t)/\sqrt{Z_c}\right\|_2} \tag{6.1}$$

This is the simplest expression we can think of that seems to satisfy all of the conditions of the IEEE standard. Furthermore, this expression is linear in the norm sense (not quadratic). One could assume a standard shape for the incident voltage $V_t(t)=V_o f(t)$, and calculate the radiated field as a function of the risetime of $f(t)$. In this way, we would have a gain that is dependent upon risetime, analogous to being dependent on frequency in the frequency domain. Furthermore, note that the above expression is dimensionless.

Unfortunately, the above expression is not very useful, because it is not consistent with reciprocity. That is, the above expression has no particular meaning when the antenna is in receive mode. As we stated earlier, we would like antenna gain to be equivalent in transmit and receive modes. One can adapt the above definition slightly, however, to get a definition that is indeed consistent with reciprocity. In order to simplify the notation, we consider only one polarization at a time. Thus, we propose

$$G(\theta,\phi) = \lim_{r\to\infty} \frac{2\pi c\left\| r\,\vec{E}^{rad}(\theta,\phi,t)\cdot\vec{1}_e/\sqrt{Z_o}\right\|}{\left\| d\left(V^{inc}(t)/\sqrt{Z_c}\right)/dt\right\|} = \lim_{r\to\infty} \frac{2\pi c\sqrt{f_g}\left\| r\,\vec{E}^{rad}(\theta,\phi,t)\cdot\vec{1}_e\right\|}{\left\| dV^{inc}(t)/dt\right\|}$$

$$= \lim_{r\to\infty} \frac{2\pi c\sqrt{f_g}\left\|\int r\,\vec{E}^{rad}(\theta,\phi,t)\cdot\vec{1}_e\,dt\right\|}{\left\| V^{inc}(t)\right\|} \tag{6.2}$$

where $V^{inc}(t) = V_t(t)$. We have had to include the factor of $2\pi c$ in order to maintain consistency with the definition for receive mode, as we shall see shortly. Note the inclusion of $f_g^{1/2}$, which puts fields and voltages in a form normalized to power. The use of an ideal transformer at the antenna terminals allows one to change the load impedance $Z_c$ arbitrarily. One could ignore the $f_g^{1/2}$, in which case the gain would be normalized to voltage rather than power. Since the IEEE standard specifies power, this seems to be less advantageous. In addition, the inclusion of the $f_g^{1/2}$ factor will allow a symmetry in $f_g^{1/2}$ when compared to the definition for the receive mode. Note also that all three of the above forms are equivalent. The limits are necessary, since in general the shape of the radiated field varies slightly with $r$, i.e., the radiation characteristic of an IRA has an "approximate delta function" [15]. In the receive mode, an alternative definition is

$$G(\theta,\phi) = \frac{\left\| V^{rec}(t)/\sqrt{Z_c} \right\|}{\left\| \vec{E}^{inc}(\theta,\phi,t)\cdot\vec{1}_e/\sqrt{Z_o} \right\|} = \frac{1}{\sqrt{f_g}} \frac{\left\| V^{rec}(t) \right\|}{\left\| \vec{E}^{inc}(\theta,\phi,t)\cdot\vec{1}_e \right\|} \tag{6.3}$$

All of the above definitions in (6.2) and (6.3) are equivalent by the relationships established in the previous section, as long as the incident field waveshape in receive mode is the derivative of the incident voltage waveshape in transmit mode. Furthermore, the best value of gain for a given norm at a given polarization angle is just the norm of the convolution operator, $f_g^{-1/2}\left\| \vec{h}_t(t)\cdot\vec{1}_e \circ \right\|$.

In order to apply the above definitions, we assume the same norm is used in the numerator and denominator. Furthermore, we assume a time domain waveshape such as a Gaussian for an incident field, or an integrated Gaussian for an incident voltage. In addition, there is a certain parameter associated with the waveshape, such as FWHM or 10-90% risetime. A disadvantage of this definition is that it is not unitless, but has the units of meters. Nevertheless, this definition is consistent with reciprocity.

One might think of normalizing this newly defined transient gain to some characteristic dimension of the antenna, such as the square root of the aperture area. That would be disadvantageous, however, for two reasons. First, it can be difficult to specify the aperture area of these antennas. For example, it is unclear if one should include the fringe fields of a TEM horn in an aperture area calculation. The second, more compelling reason for not normalizing to antenna size is that doing so gives an incorrect result. If the size of an antenna is increased, then one expects that the gain should increase. However, if the gain is normalized to the aperture size the gain would remain constant.

We can attach bounds to the gain as defined above with the usual methods. From (5.2.6) (5.3.9) and (5.4.6), we have

$$G(\vec{1}_r) \le \frac{1}{\sqrt{f_g}}\left\| \vec{1}_r \cdot \vec{h}_t(\vec{1}_r,t)\circ \right\| = 2\pi c\sqrt{f_g}\left\| \left[ \int_0^t \vec{F}_t(\vec{1}_r,t')\,dt' \right]\circ \right\| \tag{6.4}$$

or for a single polarization

$$G(\vec{1}_r) \le \frac{1}{\sqrt{f_g}}\left\| \vec{h}_t(\vec{1}_r,t)\cdot\vec{1}_e \circ \right\| = 2\pi c\sqrt{f_g}\left\| \left[ \int_0^t \vec{F}_t(\vec{1}_r,t')\cdot\vec{1}_e\,dt' \right]\circ \right\| \tag{6.5}$$

Note once again that the units of $\vec{h}_t(t)$ are meters/second, so the units of the convolution operator $\vec{h}_t(t)\circ$ are meters. In addition, the units of the norm of the convolution operator are also meters, since the units of the norm of an operator are always the same as those of the operator by itself. It is simple to demonstrate this by considering the definition of the norm of an operator in (3.4). Finally, if the incident field time history is optimal we get a further simplification. Recall that the optimal incident field is the field that optimizes the ratio of $\|h_t(t) \circ E(t)\| \,/\, \|E(t)\|$. This is nothing more than the norm of the convolution operator function $\vec{h}_t(t)\circ$. Therefore when an antenna is excited by its optimal waveform, such as for an IRA a delta-function in reception or a step function in transmission, the inequalities in the above two equations become equalities. If a $p$-norm is used in the ratio in (6.2) or (6.3), we have the further simplifications

$$\left\| \vec{h}_t(t) \cdot \vec{1}_e \circ \right\|_p \le \left\| \vec{h}_t(t) \cdot \vec{1}_e \right\|_1 \tag{6.6}$$

$$\left\| \left[ \int_0^t \vec{F}_t\left(\vec{1}_r, t'\right) \cdot \vec{1}_p \, dt' \right] \circ \right\|_p \le \left\| \left[ \int_0^t \vec{F}_t\left(\vec{1}_r, t'\right) \cdot \vec{1}_p \, dt' \right] \circ \right\|_1 \tag{6.7}$$

where the equalities hold if $p$=1 or $\infty$. Furthermore if $p = 2$, we have the exact bounds

$$\left\| \vec{h}_t(t) \cdot \vec{1}_e \circ \right\|_2 = \sup_{\omega} \left| \tilde{\vec{h}}(j\omega) \cdot \vec{1}_e \right| \tag{6.8}$$

$$\left\| \left[ \int_0^t \vec{F}_t\left(\vec{1}_r, t'\right) \cdot \vec{1}_e \, dt' \right] \circ \right\|_2 = \sup_{\omega > 0} \frac{1}{\omega} \left| \tilde{\vec{F}}_t\left(\vec{1}_r, j\omega\right) \right| \tag{6.9}$$

We now pause to consider what we have learned. In order to provide a definition of gain that takes advantage of reciprocity, we have had to establish a somewhat unconventional definition. Actually, the definition we use is not really so unusual, since in the frequency domain it is analogous to the effective height of the antenna. One of the characteristics of our definition is that an input waveshape must be specified. One would normally want to specify a function that approximates a step function for the incident voltage in transmit mode, and an impulse function in receive mode. In order to establish reciprocity, the driving function for receive mode must be the derivative of that for transmit mode. Normally, gain improves with faster risetime as does the gain of a frequency-domain reflector antenna with increasing frequency. Therefore, for IRA's and TEM horns, the optimal driving functions are the step function in transmit mode and the impulse function in receive mode. With these optimal driving functions, the gain is simply $\left\| \vec{h}_t(t) \cdot \vec{1}_e \circ \right\| / \sqrt{f_g}$. Hence, we see a simple interpretation of what the norm of a operator is, and we now understand why our antenna gain is bounded by it. In the time domain, gain must be specified for a given waveform, analogous to specifying gain for a specific frequency in the frequency domain. Furthermore, there is an optimal driving function that provides an optimal antenna gain, which is just the norm of the effective height convolution operator for a given norm.

## VII. Transient Antenna Pattern

Let us now consider the problem of antenna pattern in the time domain. Recalling from the IEEE definition the directivity in the frequency domain is "*The spatial distribution of a quantity which characterizes the electromagnetic field generated by an antenna*". Thus, we have for transient electric field antenna pattern,

$$F(\theta,\phi) = \lim_{r\to\infty} \frac{\left\| r\vec{E}^{rad}(r,\theta,\phi,t)\cdot\vec{1}_e \right\|}{\left\| r\vec{E}^{rad}(r,\theta_{\max},\phi_{\max},t)\cdot\vec{1}_e \right\|} \tag{7.1}$$

where $\theta_{\max}$ and $\phi_{\max}$ point to the direction where the norm of the signal is maximized. Once an antenna pattern is established, then such terms as side lobe level and half power beam width have simple direct analogies to their frequency domain definitions. Again, note that the norms are linear (not quadratic) in field strength.

A feature of the above definition is that it is just as valid in transmit mode as in receive mode. Thus, for the receive mode, it can be expressed as

$$F(\theta,\phi) = \frac{\left\| V^{rec}(\theta,\phi,\vec{1}_P,t) \right\|}{\left\| V^{rec}(r,\theta_{\max},\phi_{\max},\vec{1}_P,t) \right\|} \tag{7.2}$$

where $\theta$ and $\phi$ denote the angle of incidence and $\vec{1}_P$ denotes the polarization angle of the received signal. It is simple to show that the above two definitions are equivalent, based on the theory of Section V of this paper. This is true provided that the same norm is used in both equations, and as long as the shape of the driving waveform for receive mode is the derivative of the driving waveform for transmit mode.

## VIII. Candidate Input Waveforms

The gain and antenna pattern of a transient antenna are all dependent upon the input shape, whether in transmission or reception. One can get around this by always specifying that gain will be calculated for a step function in transmission and an impulse function in reception. As we have seen, this represents for a common class of antennas the best gain one can achieve for a given norm, and it is proportional to the norm of the effective height convolution operator.

There are a number of problems, however, with insisting on ideal driving functions that yield optimal waveforms. First, and most obviously, there are no sources available to provide a perfect step function, so the waveform is nonphysical. Second, as we shall see later, it is impossible to get into an antenna's far field to make measurements when it radiates a perfect impulse function. But perhaps the most compelling reason for allowing some variability in the shape of the driving function is because we do so in the frequency domain, when we specify the frequency. Thus, it seems there should be some analogous parameter one could vary in the time domain, such as risetime or FWHM. By insisting that one specify gain in terms of step and impulse functions, it seems one is specifying the analogue of gain at infinite frequency in the frequency domain. (Actually, a better analogy in the frequency domain is effective height, which is bounded at high frequencies.) In any case, it seems we need to allow ourselves some flexibility in specifying the shape of the input waveform.

We consider now some desirable characteristics for a driving waveform. This discussion is limited to the class of antennas like the IRA that are driven with an approximate step function, and radiate an approximate impulse function. For these antennas one must consider both impulse-like waveforms for reception, and step-like waveforms for transmission. Furthermore, the waveforms should be considered in pairs of impulse-like waveforms and their integrals, in order to satisfy reciprocity relations. Desirable properties for these waveforms are as follows. First, the functions and at least their first derivative should be continuous. Furthermore, they should have an analytic Fourier transform, in order to facilitate the numerics. Finally, it would be preferable if they were zero before some specific time. None of the waveforms we discuss here satisfy all these criterion, but a few satisfy almost all of them. For this reason, it is still premature to pick a standard waveform. Nevertheless, we can talk about the candidates.

Before suggesting specific waveforms, let us define the parameters we will use to describe them. For impulse-like waveforms, the Full Width Half Max time $t_{FWHM}$ is already clear. For step-like waveforms, we can describe a risetime in two ways. The first is a 10-90% risetime, $t_{10\text{-}90}$, which is also probably familiar. The second we call the derivative risetime, or $t_d$, and it is defined for a step-like $g(t)$ by

$$t_d = \frac{\sup_t |g(t)|}{\sup_t \left| \frac{dg(t)}{dt} \right|} \tag{8.1}$$

This is just the peak of the waveform divided by its peak derivative. This definition has appeared previously in [22], where it was called $t_{mr}$, since it identifies the maximum rate of rise of the waveform. Alternatively, it can be redefined in terms of the corresponding impulse-like function $f(t)$. Thus, if $f(t) = \frac{dg(t)}{dt}$, we also have

$$t_d = \frac{\sup_t \left| \int f(t)\,dt \right|}{\sup_t |f(t)|} \tag{8.2}$$

This is just the area of the impulse-like waveform divided by its peak. The reason we define $t_d$ is that many antennas radiate a derivative of an approximate step function, so the magnitude of the radiated impulse-like field is directly proportional to the peak derivative of the driving function, and inversely proportional to $t_d$. It has been customary in some circles to characterize risetime by $t_{10\text{-}90}$, but the magnitude of the radiated field is much more closely proportional to $1/t_d$ than to $1/t_{10\text{-}90}$.

Let us now identify specific candidate waveforms. First, we consider a normal or Gaussian distribution, which has the form [10, p.931]

$$f(t) = \frac{1}{t_o\sqrt{2\pi}} e^{-\frac{1}{2}(t/t_o)^2}, \qquad t_{FWHM} = 2.355\,t_o \tag{8.3}$$

$$g(t) = \int f(t)\,dt = P(t/t_o), \qquad t_d = \sqrt{2\pi}\,t_o = 2.507\,t_o \tag{8.4}$$

$$t_{10-90} = 2.563\,t_o$$

This waveform is shown in Figure 8.1 for $t_o = 1$, and it satisfies all the criterion except that the function should be zero before some specified time. Nevertheless, it may be the most useful of the waveforms because the Fourier transform of $f(t)$ is of the same form as $f(t)$. Note that the maximum of $f(t)$ at $t = 0$ is $0.3989/t_o$ and $t_{10\text{-}90} = 1.08\,t_{FWHM}$. Note also that for this waveform the $t_d$ is very close to $t_{10\text{-}90}$.

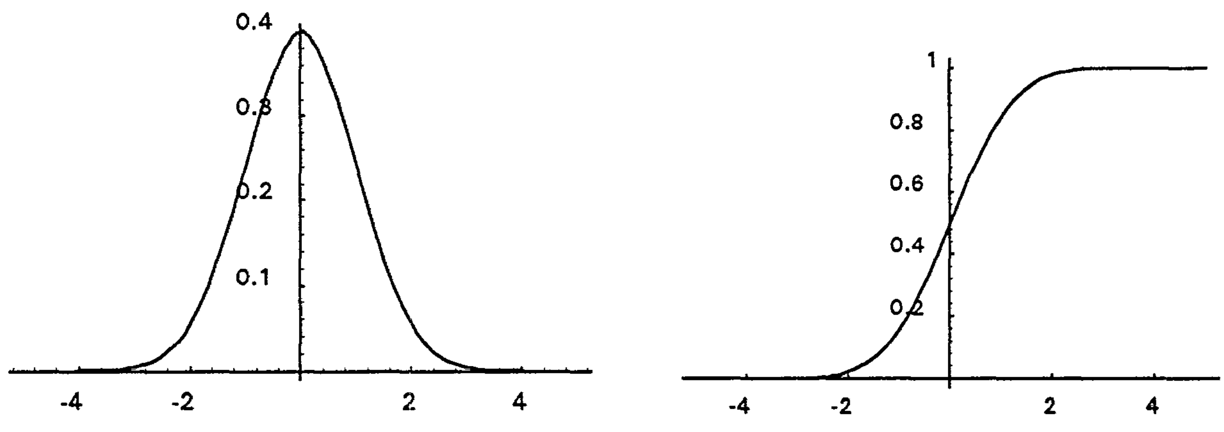


Figure 8.1. Normal distribution function (left), and its integral (right).

Another waveform of interest is a simple exponential function. It has the form

$$f(t) = \begin{cases} ae^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases} \quad , \qquad t_{FWHM} = \ln(2)/a = 0.6931/a \quad (8.5)$$

$$g(t) = \int f(t)\,dt = 1 - e^{-at} \; , \qquad t_d = 1/a \quad (8.6)$$

$$t_{10-90} = \ln(9)/a = 2.197/a$$

and it is plotted in Figure 8.2 for $a$=1. Although this has an extremely simple form, $f(t)$ is discontinuous at $t = 0$, a nonphysical situation. Furthermore, if one is interested in the peak derivative of the integrated (step-like) waveform, it will be difficult to measure, since it occurs at $t = 0$, where the integral of $f(t)$ is zero, or buried in the noise. One would prefer the peak derivative to occur above the noise level. Thus, there are some clear disadvantages to using this waveform. Note also that for this waveform $t_{10\text{-}90} = 2.2\ t_d$, so if one specifies this waveform just by its $t_{10\text{-}90}$, then one would be misled. Recall that for the class of antennas we are discussing in this section, the radiated field is determined more by its $t_d$ than its $t_{10\text{-}90}$.

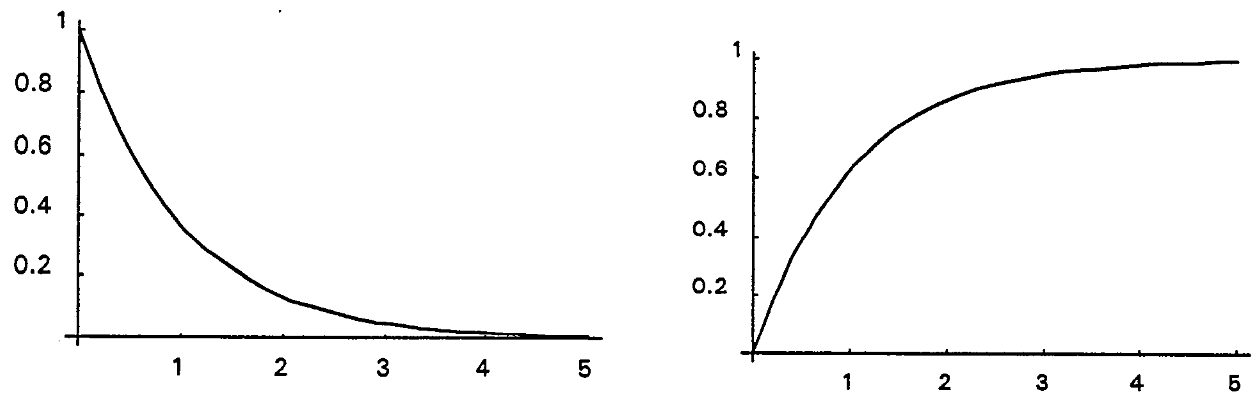


Figure 8.2. Exponential function (left), and its integral (right).

One method we might use to get around the discontinuity at $t$=0 in the simple exponential function is to use the so-called "smooth exponential" function. This is specified as

$$f(t) = \begin{cases} a^2\, t\, e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases} \quad , \qquad t_{FWHM} = 2.446/a \quad (8.7)$$

$$g(t) = \int f(t)\,dt = 1 - [1 + at]e^{-at} \; , \qquad t_d = 2.718/a \quad (8.8)$$

$$t_{10-90} = 3.358/a$$

and it is plotted in Figure 8.3 for $a$=1. Again, this has a simple form and is analytically Fourier transformable, however the derivative of the impulse-like function $f(t)$ is still not continuous at $t$=0. Note that the maximum of $f(t)$ occurs at $t$=1/$a$ and has a value of $a/e$.

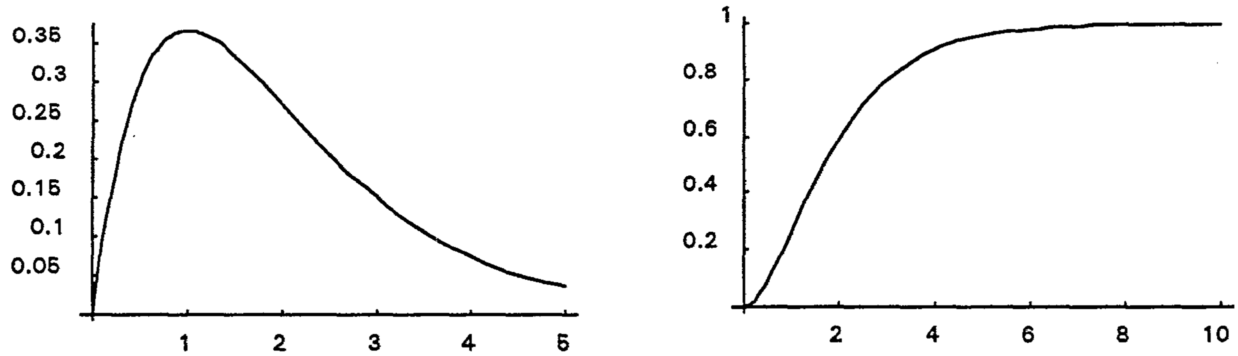


Figure 8.3. Smooth exponential (left) and its integral (right).

Finally, if we wish to force all the relevant functions to have continuous derivatives, we can try a second-order smoothed exponential

$$f(t) = \begin{cases} \tfrac{1}{2} a^3 t^2 e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases} \quad , \qquad t_{FWHM} = 3.395/a \tag{8.9}$$

$$g(t) = \int f(t)\,dt = 1 - \tfrac{1}{2}\gamma(3, at) \ , \qquad t_d = e^2/2a = 3.694/a \tag{8.10}$$

$$t_{10-90} = 4.220/a$$

as shown in Figure 8.4 for $a$=1. In the above equation $\gamma(3,t)$ is the incomplete gamma function defined by [10, p. 260, 6.5.2]

$$\gamma(b,x) = \int_0^x e^{-q}\, q^{b-1}\, dq \tag{8.11}$$

Note that $f(t)$ peaks at $t$=2/$a$, with a peak value of $2/e^2$. Note also that as we use successively smoother and smoother functions that the derivative risetime approaches the 10-90% risetime. This function has the advantage that both the step-like and the impulse-like functions are continuous and have continuous derivatives. Unfortunately, it is a bit cumbersome to calculate.

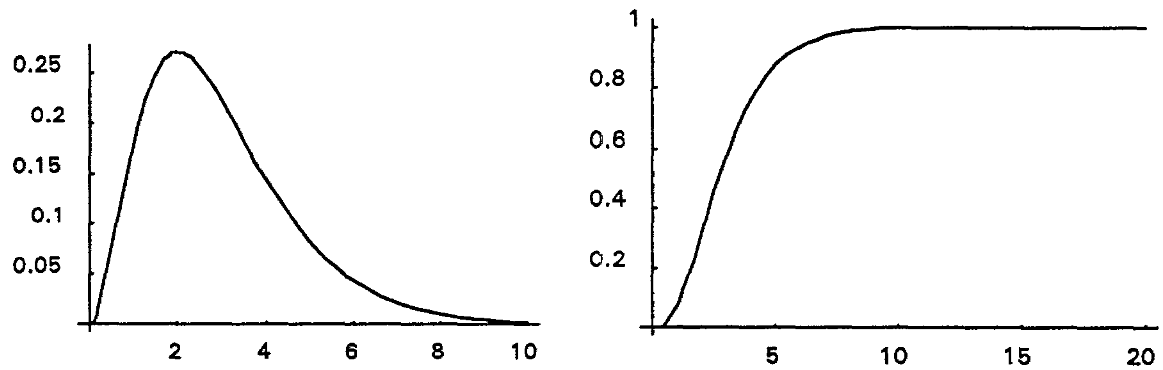


Figure 8.4. Second-order smoothed exponential and its integral.

There are many other waveshapes one might consider as well. Other waveforms that come to mind are the double exponential and inverse double exponential function that are sometimes used to describe electromagnetic pulse waveforms [2, 13]. These waveforms have two parameters, rather that one (a fall time in addition to a risetime), so in that sense they may be less desirable than those waveforms provided here. Our intent here has not been to be exhaustive, but rather to suggest some of the possibilities. As we can see, none of the suggested waveforms satisfy all of our criteria, suggesting that there is no one ideal waveshape.

## IX. The Gains of Some Typical Antennas

Let us consider now the gains of three antennas for comparison. We consider a TEM horn and two sizes of Impulse Radiating Antenna (IRA) (Figure 9.1). The small IRA has the same aperture area as the TEM horn, and the medium IRA has the same largest dimension as the TEM horn. The feed impedance of the TEM horn is about 116 Ω, and that of the two IRAs is 400 Ω. Methods for calculating the radiated field on boresight have already been developed for the IRA [8] and for the TEM horn [9]. The purpose here is to assign a gain (in meters) to the three antennas for both the optimal driving functions and realistic driving functions [11]. The length of the TEM horn is $\ell$, its height at the aperture is $h$, its width at the aperture is $w$. The diameter of the IRA is $D$.

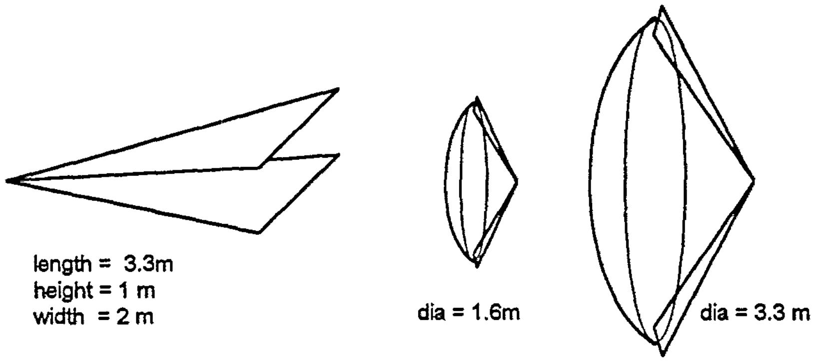


Figure 9.1. Three antennas for comparing transient gains.

First, we compare optimal gains, meaning we compare the three radiated fields under the assumption that the three antennas are being driven by a step function $V_O\, u(t)$. We use the $A$-norm as defined in Section III of this paper. Furthermore, the definition of gain is taken from (6.2) of this paper, where boresight is on the intersection of the two symmetry planes

$$G^{(A)} = \lim_{r\to\infty} 2\pi c\sqrt{f_g}\,\frac{\left\| r\,E^{rad}(t)\right\|_A}{\left\| dV^{inc}(t)/dt\right\|_A} \tag{9.1}$$

and with polarization well-defined on boresight. The $A$-norm in the denominator is calculated trivially as $V_O$. Consider first the IRAs. The impulsive portion of the radiated field is

$$E(t) = \frac{V_o}{r}\frac{D}{4\pi c f_g}\,\delta(t), \qquad \left\| rE(t)\right\|_A = \frac{V_o D}{4\pi c f_g} \tag{9.2}$$

so for an IRA the optimal gain under the $A$-norm is

$$G_{opt}^{(A)} = D\big/\left(2\sqrt{f_g}\right) \tag{9.3}$$

a surprisingly simple result. Based on this result, a preference for low feed impedances is clear. Thus, we see that the small IRA has a gain of 0.78 m, the medium IRA has a gain of 1.6 m.

Next, we consider the optimal gain of the TEM horn. Since we are going to take the A-norm of the radiated field, we need only use the low-frequency approximation to the radiated field as described in [9]. Thus, the impulsive portion of the radiated field is

$$E(t) = \frac{h}{4\pi c f_g}\,\delta(t) \quad , \qquad \| r\, E(t) \|_A = \frac{h}{4\pi c f_g} \tag{9.4}$$

and the optimal gain for the TEM horn under the $A$-norm is

$$G_{opt}^{(A)} = h \big/ \left( 2\sqrt{f_g} \right) \tag{9.5}$$

Again, there is a preference for lower impedance feeds. Thus, we see that the TEM horn has a gain of 0.9 m, compared with .78 m and 1.6 m for the two IRAs. All these are calculated under optimal conditions, a situation that provides a useful theoretical bound but not a practical quantity. Under these particular definitions, the TEM horn can hold its own against the IRAs. however, we will see that under more realistic circumstances, the TEM horn's gain diminishes.

In order to calculate a more practical quantity, we consider now the gain of the three antennas when they are driven by the integral of the Gaussian distribution, as shown in Equations (8.3-8.4) and Figure 8.1 of this paper. In particular, we use the normal distribution whose integral has a derivative risetime $t_d$ of 196 ps and a $t_{10\text{-}90}$ of 200 ps. Also, we use the ∞-norm (or peak) as defined in Section III of this paper. The gain definition we use here is the same as (9.1), except that the ∞-norm is used instead of the $A$-norm, so

$$G^{(\infty)} = \lim_{r \to \infty} 2\pi c \sqrt{f_g}\, \frac{\left\| r\, E^{rad}(t) \right\|_\infty}{\left\| dV^{inc}(t)/dt \right\|_\infty} \tag{9.6}$$

Let us begin with the IRAs. The denominator is trivial to calculate as

$$\left\| \frac{dV}{dt} \right\|_\infty = V_o\, 0.3989 \,/\, t_o \tag{9.7}$$

where $V_o$ is a scale factor for the driving voltage. The peak of the radiated field of the IRA is once again governed by the impulse-like portion of the waveform, so

$$E(t) = \frac{D}{4\pi c f_g}\frac{dV(t)}{dt}, \qquad \| r\, E(t) \|_\infty = \frac{D}{4\pi c f_g}\left\| \frac{dV}{dt} \right\|_\infty = \frac{V_o D}{4\pi c f_g}\frac{0.3989}{t_o} \tag{9.8}$$

Thus, applying (9.6) we find the gain of the IRA is

$$G^{(\infty)} = D \big/ \left( 2\sqrt{f_g} \right) \tag{9.9}$$

This is the same gain as under the ideal circumstances. This is a characteristic of all antennas whose radiated field is dominated by a clean derivative of the driving voltage.

Next, we consider the transient gain for the TEM horn. We can no longer make a series of simplifications in order to generate a closed-form solution. It is now necessary to numerically convolve the derivative of the driving voltage with the step response of the TEM horn. The step response is available in [9]. After doing the numerical convolution for our configuration and driving voltage, we find

$$G^{(\infty)} = 0.61\text{ m} \tag{9.10}$$

Thus, we see that with the TEM horn we have lost almost 40 percent of the gain by using a driving voltage with a finite risetime.

Finally, we compare the gains based on the ∞-norm (peak) for a perfect step function driving voltage. For the TEM horn, the denominator is infinite, since the peak magnitude of the derivative of the step function is infinite. The numerator is finite, since the radiated field is well-defined. Thus, for the TEM horn, the gain is zero! For the IRAs, we get the ratio of the magnitude of the two delta functions, so we get the same expression we have been getting all along. All of the results are summarized in Table 9.1. Note that with zero risetime we can only use the approximate delta function from [15]. Actually, there is no rigorous far field here except in an $A$-norm sense.

**Table 9.1. Gain of three antennas as a function of driving waveform and norm.**

| Driving Waveform | Norm | TEM Horn | Small IRA | Medium IRA |
|---|---|---|---|---|
| Step function | A-norm | 0.90 m | 0.78 m | 1.6 m |
| Step function | ∞-norm | 0 m | 0.78 m | 1.6 m |
| Integrated Gaussian<br>$t_d$ = 196 ps<br>$t_{10\text{-}90}$ = 200 ps | ∞-norm | 0.61 m | 0.78 m | 1.6 m |

Note that with the IRAs there is no change (under our simple models) of the gain when moving between the various gain definitions. This is characteristic of all antennas that radiate a clean delta function in response to a step input voltage. However, the TEM horn is much more sensitive to the choice of norm and the waveform shape. If an ideal lens were placed in front of the TEM horn, the gain of the TEM horn would be stable like that of the IRA. However, for large antennas, lenses can be quite heavy.

Finally, we compare the results above with a comparison of radiated fields calculated in [11]. In [11] the same three antennas were considered, also driven by a $t_{10\text{-}90}$=200 ps risetime pulse. However, the waveform used there was a double exponential function with a long decay time. This is equivalent to the simple exponential function described in Equations (8.5-8.6) of this paper. It was found in [11] that the radiated field of the TEM horn, when normalized for equal power, was down by a factor of about 2.5, when compared to the small IRA. In Table 8.1 of this paper, we find when we use an integrated Gaussian, that the gain of the TEM horn is only down by a factor of about 1.3 from the gain of the small IRA. Why the discrepancy? The integrated Gaussian used in this paper has a derivative risetime of $t_d$=196 ps, and the exponential function used in [11] has a derivative risetime of $t_d$=91ps. It is the derivative risetime that matters, as we have suggested earlier, so in [11] we drove the antennas with a waveform that was about twice as fast. This points out why it is important to use the correct (derivative) definition of risetime.

## X. Cascading of Norms to Bound a System Response: The Time Domain Radar Equation

One of the advantages of having expressed antenna gain and field patterns in terms of norms is that we now have a mechanism for bounding the output in a cascade of linear responses. In order to illustrate this, we develop a form of the radar equation in the time domain. Consider the arrangement of Figure 10.1. A transmit antenna with an effective height function $h_{trans}(t)$ radiates out to a scatterer of effective length $\Lambda(t)$. This reradiates and is received by an antenna of $h_{rec}(t)$.

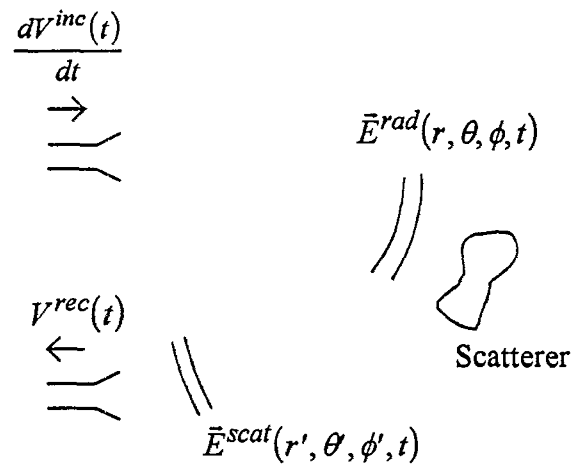


Figure 10.1. A case study of cascaded norms.

One can calculate the radiated field as

$$\vec{E}^{rad}(r,\theta,\phi,t) = \frac{\mu_o}{2\pi r Z_c}\vec{1}_r \cdot \vec{h}_{trans}(t) \circ \frac{dV^{inc}(t-r/c)}{dt} \tag{10.1}$$

and it can be bounded for a single polarization as

$$\left\| \vec{E}^{rad}(r,\theta,\phi,t)\cdot\vec{1}_e \right\| \le \frac{\mu_o}{2\pi r Z_c}\left\| \vec{1}_e\cdot\vec{h}_{trans}(t)\circ\right\| \left\|\frac{dV^{inc}(t)}{dt}\right\| \tag{10.2}$$

The scattered field is

$$\vec{E}^{scat}(r',\theta',\phi',t) = \frac{1}{4\pi r}\tilde{\Lambda}(t) \overset{\circ}{\cdot}\ \vec{E}^{rad}(t'-(r+r')/c) \tag{10.3}$$

where $\tilde{\Lambda}(t)$ is the scattering length dyad [6]. The scattered field can now be bounded as

$$\left\| \vec{E}^{scat}(r',\theta',\phi',t)\cdot\vec{1}_e \right\| \le \frac{1}{4\pi r}\left\| \tilde{\Lambda}(t')\circ\right\| \left\|\vec{E}^{rad}(t')\cdot\vec{1}_e\right\| \tag{10.4}$$

Finally, the received voltage is

$$V^{rec}(t) = \vec{h}_{rec}(t) \overset{\circ}{\cdot} \vec{E}^{scat}(r', \theta', \phi', t - (r + r')/c) \quad (10.5)$$

and it can be bounded for a single component by

$$\left\| V^{rec}(t) \right\| \le \left\| \vec{h}_{rec}(t) \cdot \vec{1}_e \circ \right\| \left\| \vec{E}^{scat}(r', \theta', \phi', t) \cdot \vec{1}_e \right\| \quad (10.6)$$

Putting it all together, we find a total response of

$$V^{rec}(t) = \frac{e^{-\gamma(r+r')/c}}{8\pi^2 r^2 c f_g} \vec{h}_{rec}(t) \overset{\circ}{\cdot} \vec{\Lambda}(t) \overset{\circ}{\cdot} \vec{h}_{trans}(t) \circ \frac{dV^{inc}(t - 2(r + r')/c)}{dt} \quad (10.7)$$

where the three time convolutions may commute, if one wishes. Finally, the above equation may be bounded by

$$\left\| V^{rec}(t) \right\| \le \frac{1}{8\pi^2 r^2 c f_g} \left\| \vec{h}_{rec}(t) \cdot \vec{1}_P \right\| \left\| \vec{\Lambda}(t) \overset{\circ}{\cdot} \right\| \left\| \vec{h}_{trans}(t) \cdot \vec{1}_e \right\| \left\| \frac{dV^{inc}(t)}{dt} \right\| \quad (10.8)$$

These last two equations may be considered a time domain analogue of the standard radar equation in the frequency domain.

Finally, let us consider the interesting case of when the transmit and receive antennas are the same, and in the same location (backscatter). We can then express transmit polarization $m$ and receive polarization $n$ by either $h$ or $v$, representing horizontal or vertical polarizations. Thus, the transient radar equation simplifies to

$$V^{rec}(t) = \frac{1}{8\pi^2 r^2 c f_g} h_n(t) \circ \Lambda_{nm}(t) \circ h_m(t) \circ \frac{dV^{inc}(t - 2r/c)}{dt} \quad (10.9)$$

$$= \frac{1}{8\pi^2 r^2 c f_g} h_n(t) \circ h_m(t) \circ \Lambda_{nm}(t) \circ \frac{dV^{inc}(t - 2r/c)}{dt}$$

where the operator $h_n(t) \circ h_m(t) \circ$ is an antenna operator for backscatter radar. Note that $h_n$ includes the polarization vector $\vec{1}_n$ in general (for a tighter bound) in the event that $\vec{h}$ is not transverse to $\vec{1}_r$ (to the target). Choosing $\vec{h}$ to be transverse to $\vec{1}_r$ of course maximizes the backscattering. The bounds on this equation can be expressed as

$$\left\|V^{rec}(t)\right\| \le \frac{1}{8\pi^2 r^2 c f_g}\left\| h_n(t)\circ h_m(t)\circ \Lambda_{nm}(t)\circ \right\| \left\| \frac{dV^{inc}(t)}{dt} \right\|$$

$$\le \frac{1}{8\pi^2 r^2 c f_g}\left\| h_n(t)\circ h_m(t)\circ \right\| \left\| \Lambda_{nm}(t)\circ \right\| \left\| \frac{dV^{inc}(t)}{dt} \right\| \tag{10.10}$$

$$\le \frac{1}{8\pi^2 r^2 c f_g}\left\| h_n(t)\circ \right\| \left\| h_m(t)\circ \right\| \left\| \Lambda_{nm}(t)\circ \right\| \left\| \frac{dV^{inc}(t)}{dt} \right\|$$

noting the commutation of the scalar convolution operator. Thus, there are three cases of interest for the polarizations -- *hh*, *vv*, and the two cross polarization terms *hv* and *vh* which are equivalent.

Let us assume for the moment that the vertical polarization is dominant, so we can limit our discussion to the *v* polarization for both transmit and receive modes. With this assumption, the above equation simplifies further to

$$\left\|V^{rec}(t)\right\| \le \frac{1}{8\pi^2 r^2 c f_g}\left\| h_v(t)\circ h_v(t)\circ \right\| \left\| \Lambda_{vv}(t)\circ \right\| \left\| \frac{dV^{inc}(t)}{dt} \right\| \tag{10.11}$$

where $\left\| h_v(t)\circ h_v(t)\circ \right\|$ is now an important characterization of the radar antenna. In a bound sense we have

$$\left\| h_v(t)\circ h_v(t)\circ \right\| \le \left\| h_v(t)\circ \right\|^2 \tag{10.12}$$

so the left side provides a tighter bound. If we now express the above in terms of the gain of the antenna using (6.5), we have

$$\left\|V^{rec}(t)\right\| \le \frac{G^2}{8\pi^2 r^2 c}\left\| \Lambda_{vv}(t)\circ \right\| \left\| \frac{dV^{inc}(t)}{dt} \right\| \tag{10.13}$$

This is a remarkably simple expression for such a complicated process. Note that the gain $G$ is a function of the choice of norm as well as the shape of the driving waveform $V^{inc}(t)$. Furthermore, recall that the units of gain are meters. Note also that the factor $\left\| \Lambda_{vv}(t)\circ \right\|$ is a time domain equivalent of the square root of the radar cross section of the scatterer, and also has units of meters.

Thus, we see that there are various ways of expressing the classical radar equation in the time domain. Furthermore, we can see that one of the benefits of using a norm-based definition of antenna gain is that one can cascade effects, developing bounds on overall system performance based on the performance of its component parts. This example also points out the flexibility one can achieve with our present definition of gain. That is, different radar systems will measure performance based on different norms of the received signal. A transient radar that takes a "TDR" of an aircraft may wish to measure the peak magnitude or the energy in the received signal. A so-called SEM radar, which identifies objects based on excitation of the object's fundamental resonances, might use as a norm the magnitude of the received voltage at the resonance frequency of interest. The radar equation developed here handles both cases equally well.

## XI. Transient Far Field

Another quantity of interest in transient antennas is how far out one has to go in order to be in the antenna's far field. This is of practical importance because when one builds an antenna range to measure a transient antenna, one needs to know how large to make the range.

Consider the diagram of Figure 11.1, which shows a reflector aperture of largest radius $a$ from the center.

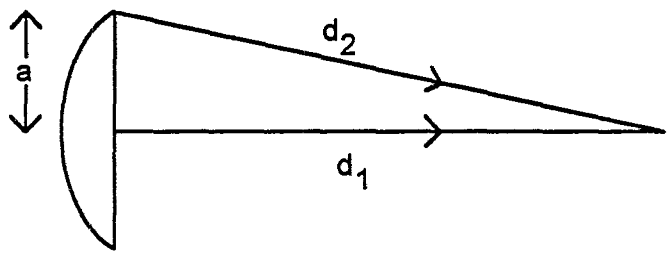


Figure 11.1. Far field calculation

To be in the far field, the clear time between the arrival of the closest ray and the arrival of the outermost ray should be small compared with the full-width-half max of the radiated signal (or 10-90% risetime of the driving voltage). In other words

$$(d_2 - d_1)/c \leq t_{\mathrm{FWHM}} / \nu \qquad (11.1)$$

where $\nu$ is a constant that identifies how much larger the $t_{FWHM}$ is than the clear time. For large $d_1$, $d_2 - d_1 \approx a^2 / d_1$, so to be in the far field

$$d_1 \geq \frac{\nu a^2}{c t_{FWHM}} \qquad (11.2)$$

Another way of expressing this is that one must be far enough away that the difference in time of arrival must be less than $t_{FWHM}/\nu$. The question then becomes how large should one have to make $\nu$. Intuition suggests that $\nu$ should be somewhere in the neighborhood of three to five, however, it is unclear how one would pin this down more accurately.

Note that if we drive an antenna of the IRA class with a step function, the radiated field contains an impulse function on boresight. By the above definition, it is therefore impossible to be far enough away to be in the far field. This may be expected, since an analogous problem occurs in the frequency domain when considering the distance out to the far field at infinite frequency. Recall that in the frequency domain, the distance to the far field is $2D^2 / \lambda$ . At infinite frequency, the wavelength is zero, so one has to be infinitely far away. This points out the problem of using the "optimal" waveforms in the gain calculations, since one cannot make practical use of them. In this case, one cannot be far enough away from them to measure their far-field characteristics.

## XII. Conclusions

We have seen that in the time domain, the gain of an antenna is somewhat difficult to characterize. A direct extension of the frequency domain definition of gain leads to a quantity that does not satisfy reciprocity conditions, and is therefore not useful. A quantity does exist, however, that is similar to gain, and does satisfy reciprocity. This quantity is analogous to the effective height in the frequency domain, and has the dimensions of meters.

A drawback of our new definition of gain is that it requires one to specify both an input waveform shape and a norm for comparing waveforms. One could avoid this problem by specifying that the incident waveshape is always a step function in transmission, and a delta function in reception. For many antennas this provides the optimal gain for any given norm, but the optimal waveform is not achievable in most practical cases. The optimal gain represents merely the best that one can achieve with a very fast risetime.

Since there is some flexibility in gain definitions, how is one to assign a definition for one's application? The choice of norm has an analogy in the area of electromagnetic pulse. In this area, one might be interested in the peak of a waveform, its total energy, its peak derivative, or its rectified area, depending on the physical effect one is testing for, such as burnout or upset of a solid state device. Analogously, one might be interested in either the peak or the area of the radiated impulse as it strikes an object that is being detected by radar. Thus, it should not bother us that there is some flexibility in the norm specification.

The specification of waveform shape, however, is a bit more problematic, since different waveshapes with the same 10-90% risetimes can have very different peak derivatives. It is clear that if any single parameter of a waveform is most important for determining radiated far field, it is not the 10-90% risetime. Instead, one should use the derivative risetime, as developed in Section VIII of this paper. The peak of the radiated far field is inversely proportional to the derivative risetime, unlike the 10-90% risetime. However, even if we standardized on the use of the derivative risetime, we still expect to get different results for different waveshapes. One solution might be to encourage the standardized usage of the Gaussian distribution. One might specify an alternative waveshape if it is important to their application, but unless otherwise specified, a Gaussian would be assumed. This allows a specification of derivative risetime to have a rigorous meaning. It is readily conceded that a Gaussian has the disadvantage that it does not fall to zero at early times, however in a practical system, it falls into the noise.

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