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Circular Aperture Antennas in Time Domain

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Abstract

This paper considers the calculation of the time-domain fields that result from a tangential electric field on a plane. For the case of this source field expressed as a time waveform times a spatial distribution, the fields can be conveniently expressed using retarded time. With a uniform field inside a circular aperture the fields on the symmetry axis (z axis) are only transverse and both electric and magnetic fields can be expressed in closed form. The result is extended to the aperture field on the circular disk as any non-singular plane-wave TEM electric field. These exact results are used to explore the approximate impulse (in the far field) resulting from step-function aperture illumination and gain further insight into how to interpret the far field in time domain.

1. Introduction

A previous paper [2] has considered the fields that result from an aperture antenna for which the tangential electric field is specified on a plane. This is formulated in complex-frequency (Laplace-transform) domain in terms of an integral over the tangential electric field with an appropriate Green's function. As a special case the aperture field is chosen in such a way that the signals from every point on the aperture arrive with the same phase at some preselected point $\vec{r}_o$, giving a focused aperture antenna (focused at $\vec{r} = \vec{r}_o$). In this case the integrals for the fields at $\vec{r}_o$ simplify considerably due to the common delay which can be factored out of the integrals. The integrals can also be readily expressed in both frequency and time with integration reduced to frequency/time-independent coefficients. For the case of a circular aperture with uniform spatial illumination (tangential electric field) with focus on the z axis (symmetry axis of the circular aperture) the integrals result in simple analytic expressions.

In this paper let us begin with the aperture antenna with coordinates in fig. 1.1. The aperture plane is S given by $z = 0$ on which is specified the tangential electric field $\vec{E}_t'(x', y'; t)$. This may be zero out of some finite region of S (the aperture) which can be designated S_a. Primed coordinates refer to the aperture plane.

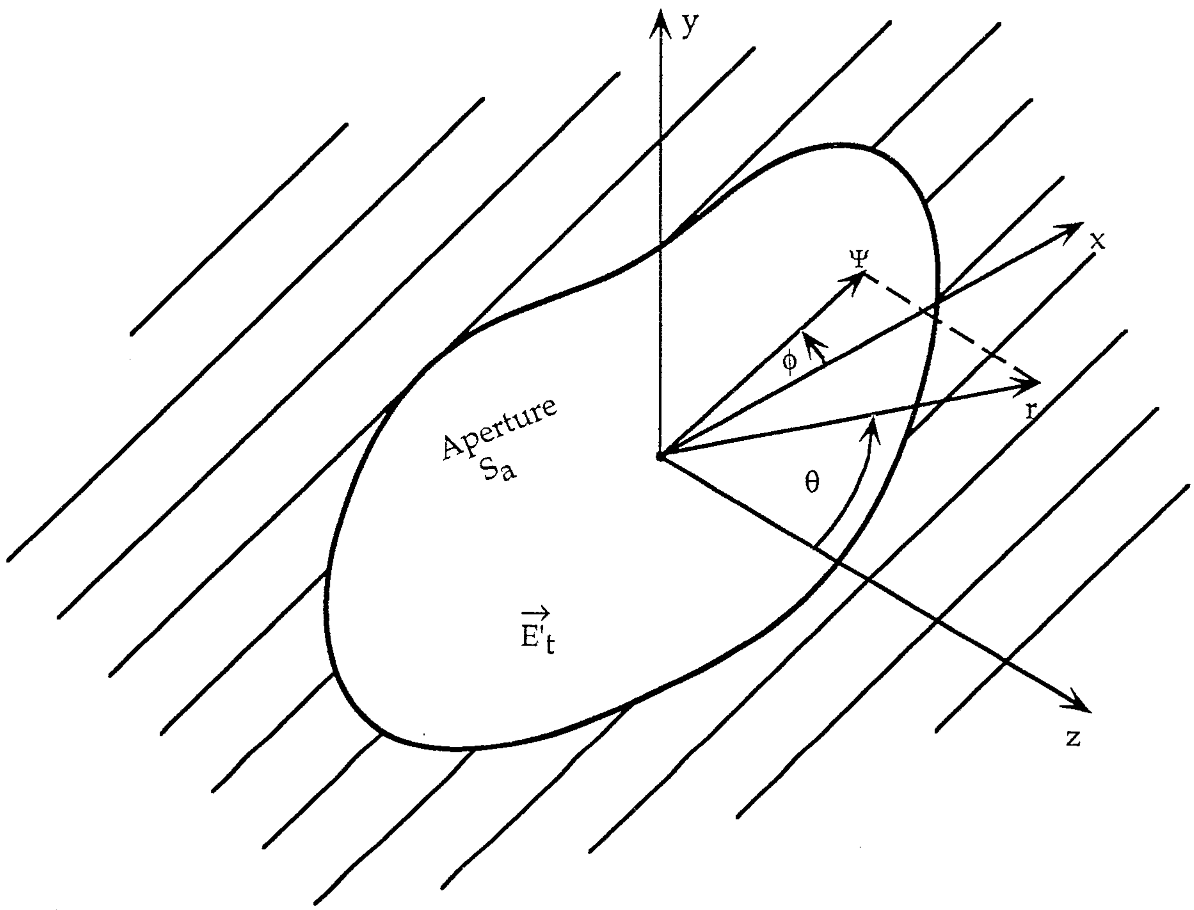
Noting that

$$
\begin{aligned}
&R \equiv |\vec{r} - \vec{r}'| = \left[(x-x')^2 + (y-y')^2 + z^2\right]^{\frac{1}{2}} \\
&\vec{r} \equiv \text{general observation position} \\
&c \equiv [\mu_o \varepsilon_o]^{-\frac{1}{2}} \quad , \quad Z_o \equiv \left[\frac{\mu_o}{\varepsilon_o}\right]^{\frac{1}{2}}
\end{aligned}
\tag{1.1}
$$

the field components from [2 ((2.8) and (2.9))] can be summarized directly in time domain. The electric field is

$$
\begin{aligned}
E_{\substack{x\\y}}(r', t) &= \frac{1}{2\pi}\int_S \frac{z}{R^3}\left[\frac{R}{c}\frac{\partial}{\partial t} + 1\right] E'_{\substack{x\\y}}\left(x', y'; t - \frac{R}{c}\right) dS' \\
E_z(r', t) &= \frac{1}{2\pi}\int_S \frac{x-x'}{R^3}\left[\frac{R}{c}\frac{\partial}{\partial t} + 1\right] E'_x\left(x', y'; t - \frac{R}{c}\right) dS' \\
&\quad + \frac{1}{2\pi}\int_S \frac{y-y'}{R^3}\left[\frac{R}{c}\frac{\partial}{\partial t} + 1\right] E'_y\left(x', y'; t - \frac{R}{c}\right) dS'
\end{aligned}
\tag{1.2}
$$

The magnetic field is



S' is defined by

$\vec{r'} = (x', y', 0)$

Fig. 1.1. Electromagnetic Fields from a Source Plane

$$
\begin{aligned}
Z_o H_x(\bar{r},t) = &-\frac{1}{2\pi}\int_S \frac{(x-x')(y-y')}{R^4}\left[\frac{R}{c}\frac{\partial}{\partial t}+3+\frac{3c}{R}I_t\right]E_x'\left(x',y';t-\frac{R}{c}\right)dS' \\
&+\frac{1}{2\pi}\int_S \frac{1}{R^2}\left\{2+\frac{2c}{R}I_t-\frac{(y-y')^2+z^2}{R^2}\left[\frac{R}{c}\frac{\partial}{\partial t}+3+\frac{3c}{R}I_t\right]\right\}E_y'\left(x',y';t-\frac{R}{c}\right)dS'
\end{aligned}
$$

$$
\begin{aligned}
Z_o H_y(\bar{r},t) = &\frac{1}{2\pi}\int_S \frac{(x-x')(y-y')}{R^4}\left[\frac{R}{c}\frac{\partial}{\partial t}+3+\frac{3c}{R}I_t\right]E_y'\left(x',y';t-\frac{R}{c}\right)dS' \\
&-\frac{1}{2\pi}\int_S \frac{1}{R^2}\left\{2+\frac{2c}{R}I_t-\frac{(x-x')^2+z^2}{R^2}\left[\frac{R}{c}\frac{\partial}{\partial t}+3+\frac{3c}{R}I_t\right]\right\}E_x'\left(x',y';t-\frac{R}{c}\right)dS'
\end{aligned}
\qquad (1.3)
$$

$$
\begin{aligned}
Z_o H_z(\bar{r},t) = &-\frac{1}{2\pi}\int_S \frac{(y-y')z}{R^4}\left[\frac{R}{c}\frac{\partial}{\partial t}+3+\frac{3c}{R}I_t\right]E_x'\left(x',y';t-\frac{R}{c}\right)dS' \\
&+\frac{1}{2\pi}\int_S \frac{(x-x')z}{R^4}\left[\frac{R}{c}\frac{\partial}{\partial t}+3+\frac{3c}{R}I_t\right]E_y'\left(x',y';t-\frac{R}{c}\right)dS'
\end{aligned}
$$

where an integral operator I_t has been introduced as

$$
I_t f(t) \equiv \int_{-\infty}^{t} f(t')dt' \qquad (1.4)
$$

with zero initial conditions assumed. Note the use of the retarded time from points on S. This replaces the exponential delay factor used in complex-frequency domain.

For later use we have cylindrical coordinates (Ψ,ϕ,z)

$$
x = \Psi\cos(\phi) \quad , \quad y = \Psi\sin(\phi) \qquad (1.5)
$$

and spherical coordinates (r,θ,ϕ)

$$
r = z\cos(\theta) = \Psi\sin(\theta) \qquad (1.6)
$$

Using primes these coordinates also apply to the source plane.

II. Uniform Tangential Electric Field on Circular Aperture

Now let us choose the special aperture field

$$\vec{E}_t'(x',y';t) \equiv \begin{cases} E_o f(t)\vec{1}_y & \text{for } 0 \le \Psi' < \alpha \\ \vec{0} & \text{for } \Psi' > \alpha \end{cases}$$

$$f(t) \equiv \text{waveform} \tag{2.1}$$

This is a uniform field on a circular aperture of radius a. The waveform can be general (subject to zero initial conditions), although later it will be made a step function.

Let the observer be located on the +z axis for which

$$R = \Psi'^2 + z^2 \qquad \text{(not a function of } \phi'\text{)} \tag{2.2}$$

Note that there are two symmetry planes, yz and xz, with fields respectively symmetric and antisymmetric [1,6,7]. This implies that the only non-zero field components on the z axis are E_y and H_x. Noting that

$$dS' = \Psi' d\Psi' d\phi' \tag{2.3}$$

we have

$$\begin{aligned} E_y\left(z\vec{1}_z,t\right) &= \frac{E_o}{2\pi}\int_o^a \int_o^{2\pi} \frac{z}{R^3}\left[\frac{R}{c}\frac{\partial}{\partial t}+1\right] f\left(t-\frac{R}{c}\right)\Psi' d\phi' d\Psi' \\ &= E_o \int_o^a \frac{z}{R^3}\left[\frac{R}{c}\frac{\partial}{\partial t}+1\right] f\left(t-\frac{R}{c}\right)\Psi' d\Psi' \end{aligned} \tag{2.4}$$

$$\begin{aligned} Z_o H_x\left(z\vec{1}_z,t\right) &= \frac{E_o}{2\pi}\int_o^a \int_o^{2\pi} \frac{1}{R^2}\left\{2+\frac{2c}{R}I_t - \frac{\Psi'^2\sin^2(\phi')+z^2}{R^2}\left[\frac{R}{c}\frac{\partial}{\partial t}+3+\frac{3c}{R}I_t\right]\right\} f\left(t-\frac{R}{c}\right)\Psi' d\phi' d\Psi' \\ &= E_o \int_o^a \frac{1}{R^2}\left\{2+\frac{2c}{R}I_t - \frac{\frac{1}{2}\Psi'^2+z^2}{R^2}\left[\frac{R}{c}\frac{\partial}{\partial t}+3+\frac{3c}{R}I_t\right]\right\} f\left(t-\frac{R}{c}\right)\Psi' d\Psi' \end{aligned}$$

Now change the retarded time inside the integral as

$$t' \equiv t - \frac{R}{c}, \frac{\partial}{\partial t} = \frac{\partial}{\partial t'}, I_t = I_{t'}$$

$$t_1 \equiv t'|_{\Psi'=0} = t - \frac{z}{c} \equiv \text{ retarded time from center of aperture} \tag{2.5}$$

$$t_2 \equiv t'|_{\Psi'=a} = t - \frac{1}{c}\left[z^2 + a^2\right]^{\frac{1}{2}} \equiv \text{ retarded time from edge of aperture}$$

Note further that for a given z,t then t' is a function of Ψ' and we can write

$$t' = t - \frac{R}{c} = t - \frac{1}{c}\left[\Psi'^2 + z^2\right]^{\frac{1}{2}} \quad , \quad R = c[t - t']$$

$$(t - t')^2 = \frac{R^2}{c^2} = \frac{1}{c^2}\left[\Psi'^2 + z^2\right] \tag{2.6}$$

$$(t - t')dt' = \frac{\Psi'}{c^2}d\Psi'$$

$$\Psi' d\Psi' = c^2(t' - t)dt' = -cRdt'$$

With these substitutions (2.4) becomes

$$\frac{1}{E_0}E_y\left(z\vec{1}_z, t\right) = -c\int_{t_1}^{t_2} \frac{z}{R^2}\left[\frac{R}{c}\frac{\partial}{\partial t'} + 1\right]f(t')dt'$$

$$-\frac{Z_0}{E_0}H_x\left(z\vec{1}_z, t\right) = c\int_{t_1}^{t_2} \frac{1}{R}\left\{2 + \frac{2c}{R}I_{t'} - \frac{R^2 + z^2}{2R^2}\left[\frac{R}{c}\frac{\partial}{\partial t'} + 3 + \frac{3c}{R}I_{t'}\right]f(t')dt'\right. \tag{2.7}$$

which is in normalized form.

Considering first the electric field we have the integration-by-parts formula

$$-c\int_{t_1}^{t_2} \frac{z}{R^2}f(t')dt' = -\frac{z}{c}\int_{t_1}^{t_2} \frac{f(t')dt'}{(t' - t)^2}$$

$$= \frac{z}{c}\left.\frac{f(t')}{t' - t}\right|_{t_1}^{t_2} - \frac{z}{c}\int_{t_1}^{t_2} \frac{1}{t' - t}\frac{\partial f(t')}{\partial t'}dt' \tag{2.8}$$

$$= f(t_1) - \frac{z}{\left[z^2 + a^a\right]^{\frac{1}{2}}}f(t_2) + \int_{t_1}^{t_2} \frac{z}{R}\frac{\partial f(t')}{\partial t'}dt'$$

Substituting in the electric-field integral we have

$$\frac{1}{E_o} E_y\left(z\overline{1}_z, t\right) = f(t_1) - \frac{z}{\left[z^2 + a^2\right]^{\frac{1}{2}}} f(t_2)$$

$$= f\left(t - \frac{z}{c}\right) - \frac{z}{\left[z^2 + a^2\right]^{\frac{1}{2}}} f\left(t - \frac{1}{c}\left[z^2 + a^2\right]^{\frac{1}{2}}\right) \tag{2.9}$$

This agrees with the results in [8,9,11,12]. Here the result has now been derived directly in time domain.

Considering second the magnetic field let us first group the terms according to the time derivative (or integration) order of the waveform in the integrand as

$$\begin{aligned} -\frac{Z_o}{E_o} H_x\left(z\overline{1}_2, t\right) &= \int_{t_1}^{t_2} \left[-\frac{1}{2} - \frac{z^2}{2R^2}\right] \frac{\partial f(t')}{\partial t'} dt' \\ &+ c\int_{t_1}^{t_2} \left[\frac{1}{2R} - \frac{3z^2}{2R^2}\right] f(t')dt' \\ &+ c^2 \int_{t_1}^{t_2} \left[\frac{1}{2R^2} - \frac{3z^2}{2R^4}\right] I_{t'} f(t')dt' \end{aligned} \tag{2.10}$$

Taking the first integral we have first the integration-by-parts formula

$$\begin{aligned} z^2 \int_{t_1}^{t_2} \frac{1}{R^2} \frac{\partial f(t')}{\partial t'} dt' &= \frac{z^2}{c^2} \int_{t_1}^{t_2} \frac{1}{(t'-t)^2} \frac{\partial f(t')}{\partial t'} dt' \\ &= \frac{z^2}{c^2} \frac{1}{(t'-t)^2} f(t') \Bigg|_{t_1}^{t_2} + \frac{z^2}{c^2} \int_{t_1}^{t_2} \frac{2}{(t'-t)^3} f(t')dt' \\ &= \frac{z^2}{z^2 + a^2} f(t_2) - f(t_1) - c\int_{t_1}^{t_2} \frac{2z^2}{R^3} f(t')dt' \end{aligned} \tag{2.11}$$

So we have

$$\int_{t_1}^{t_2}\left[-\frac{1}{2}-\frac{z^2}{2R^2}\right]\frac{\partial f(t')}{\partial t'}dt'$$

$$=\frac{1}{2}f(t_1)-\frac{1}{2}f(t_2)+\frac{1}{2}f(t_1)-\frac{1}{2}\frac{z^2}{z^2+a^2}f(t_2)+c\int_{t_1}^{t_2}\frac{2z^2}{R^3}f(t')dt' \tag{2.12}$$

$$=f(t_1)-\frac{1}{2}\frac{2z^2+a^2}{z^2+a^2}f(t_2)+c\int_{t_1}^{t_2}\frac{z^2}{R^3}f(t')dt'$$

Substituting (2.12) into (2.10) gives a simpler expression for the magnetic field as

$$\begin{aligned}-\frac{Z_o}{E_o}H_x\left(z\bar{1}_z,t\right)&=f(t_1)-\frac{1}{2}\frac{2z^2+a^2}{z^2+a^2}f(t_2)\\&+c\int_{t_1}^{t_2}\left[\frac{1}{2R}-\frac{z^2}{2R^3}\right]f(t')dt'\\&+c^2\int_{t_1}^{t_2}\left[\frac{1}{2R^2}-\frac{3z^2}{2R^4}\right]I_{t'}f(t')dt'\end{aligned} \tag{2.13}$$

where the time derivative term has now been removed.

Considering the remaining integrals we have the integration-by-parts formula

$$\begin{aligned}c\int_{t_1}^{t_2}\frac{f(t')}{R}dt'&=-\int_{t_1}^{t_2}\frac{f(t')}{t'-t}dt'\\&=-\frac{1}{t'-t}I_{t'}f(t')\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{1}{(t'-t)^2}I_{t'}f(t')dt'\\&=\frac{c}{\left[z^2+a^2\right]^{\frac{1}{2}}}I_{t_2}f(t_2)-\frac{c}{z}I_{t_1}f(t_1)-c^2\int_{t_1}^{t_2}\frac{1}{R^2}I_{t'}f(t')dt'\end{aligned} \tag{2.14}$$

and a second formula

$$
\begin{aligned}
cz^2\int_{t_1}^{t_2}\frac{f(t')}{R^3}dt' &= -\frac{z^2}{c^2}\int_{t_1}^{t_2}\frac{f(t')}{(t'-t)^3}dt' \\
&= -\frac{z^2}{c^2}\frac{1}{(t'-t)^3}I_{t'}f(t')\Bigg|_{t_1}^{t_2} - \frac{3z^2}{c^2}\int_{t_1}^{t_2}\frac{1}{(t'-t)^4}I_{t'}f(t')dt' \\
&= \frac{cz^2}{\left[z^2+a^2\right]^{\frac{3}{2}}}I_{t_2}f(t_2) - \frac{c}{z}I_{t_1}f(t_1) - 3c^2z^2\int_{t_1}^{t_2}\frac{1}{R^4}I_{t'}f(t')dt'
\end{aligned}
\tag{2.15}
$$

Substituting these in (2.13) gives

$$
\begin{aligned}
-\frac{Z_o}{E_o}H_x\left(z\bar{1}_z,t\right) &= f(t_1) - \frac{1}{2}\frac{2z^2+a^2}{z^2+a^2}f(t_2) \\
&\quad + \frac{1}{2}\frac{c}{\left[z^2+a^2\right]^{\frac{1}{2}}}I_{t_2}f(t_2) - \frac{1}{2}\frac{c}{z}I_{t_1}f(t_1) \\
&\quad - \frac{1}{2}\frac{cz^2}{\left[z^2+a^2\right]^{\frac{3}{2}}}I_{t_2}f(t_2) - \frac{1}{2}\frac{c}{z}I_{t_1}f(t_1) \\
&= f(t_1) - \frac{1}{2}\frac{2z^2+a^2}{z^2+a^2}f(t_2) + \frac{1}{2}\frac{ca^2}{\left[z^2+a^2\right]^{\frac{3}{2}}}I_{t_2}f(t_2) \\
&= f\left(t-\frac{z}{c}\right) + \left[-\frac{1}{2}\frac{2z^2+a^2}{z^2+a^2}f(t_2) + \frac{1}{2}\frac{ca^2}{\left[z^2+a^2\right]^{\frac{3}{2}}}I_{t_2}f(t_2)\right]\Bigg|_{t_2=t-\frac{1}{c}\left[z^2+a^2\right]^{\frac{1}{2}}}
\end{aligned}
\tag{2.16}
$$

Having now the exact solution for the magnetic field as well as the electric field on the z axis they can be compared. Note that both normalized solutions have a leading term $f(t_1)$ which gives the early-retarded-time solution. As discussed in [3] this is just the solution for a uniform field on an infinite aperture, i.e., for $a \to \infty$ in (2.1). This solution applies for times before the aperture radius can be noticed at the observer. This time difference is reflected in $t_2 - t_1$ which becomes arbitrarily small as $z \to \infty$. The t_2 terms for the electric and magnetic fields are somewhat different, the magnetic field having an extra near-field (z^{-3}) term proportional to the time integral of the excitation waveform on the aperture.

III. Step-Function Aperture Field

Now take the limiting case of a step-function aperture waveform as

$$f(t) = u(t) \tag{3.1}$$

noting that this is a mathematical idealization since any realistic waveform will have a non-zero (but perhaps small) rise time. Then our solutions for the fields on the +z axis become

$$\begin{aligned}
\frac{1}{E_o} E_y\left(z\bar{1}_z, t\right) &= u(t_1) - \frac{z}{\left[z^2 + a^2\right]^{\frac{1}{2}}} u(t_2) \\
-\frac{Z_o}{E_o} H_x\left(z\bar{1}_z, t\right) &= u(t_1) - \frac{1}{2}\frac{2z^2 + a^2}{z^2 + a^2} u(t_2) + \frac{1}{2}\frac{a^2}{\left[z^2 + a^2\right]^{\frac{3}{2}}} ct_2 u(t_2) \\
t_1 \equiv t - \frac{z}{c} \quad &, \quad t_2 \equiv t - \frac{1}{c}\left[z^2 + a^2\right]^{\frac{1}{2}}
\end{aligned} \tag{3.2}$$

As discussed in [3,4] the early-time portion of this waveform can be thought of as an approximate impulse which can be written as

$$\begin{aligned}
\delta_a(t) &= \frac{2cz}{a^2}\left[u(t) - u\left(t - \frac{a^2}{2cz}\right)\right] \\
\int_{-\infty}^{\infty} \delta_a &= 1
\end{aligned} \tag{3.3}$$

As $z \to \infty$ the pulse amplitude is proportional to z and the width is proportional to z^{-1}, thereby approaching a delta function in the limit. In terms of this (3.2) can be written (for early time) as

$$\begin{aligned}
\frac{1}{E_o} E_y\left(z\bar{1}_z, t\right) &\simeq \frac{a^2}{2cz}\delta_a\left(t - \frac{z}{c}\right) \\
-\frac{Z_o}{E_o} H_x\left(z\bar{1}_z, t\right) &\simeq \frac{a^2}{2cz}\delta_a\left(t - \frac{z}{c}\right)
\end{aligned} \tag{3.4}$$

This takes the form of a far field with the z^{-1} dependence when we consider δ_a as a delta function. Actually this applies in a more strict sense if one convolutes this with a pulse with fast changes (e.g., rise) over times slow compared to $a^2/(2cz)$. A true delta function is not square integrable (has infinite energy). The form in (3.4) has energy flux density $\left(J/m^2\right)$ which falls off as z^{-1}, part of a larger class of mathematical functions which have energy flux density falling off slower than z^{-1} as $z \to \infty$ [10]. However, this is basically a question of the proper definition of the far field. A real pulse with a finite maximum rate of rise does not have this behavior. Eventually, after some finite z, the radiated pulse amplitude decreases proportional to z^{-1}. However, these formulas do give a useful mathematical idealization.

With the exact solution (3.2) one can now see how well (3.4) approximates this. Note that

$$\begin{aligned} t_2 &= t - \frac{1}{c}\left[z^2 + a^2\right]^{\frac{1}{2}} = t - \frac{z}{c}\left[1 + \left(\frac{a}{z}\right)^2\right]^{\frac{1}{2}} \\ &= t - \frac{z}{c}\left[1 + \frac{1}{2}\left(\frac{a}{z}\right)^2 - \frac{1}{8}\left(\frac{a}{z}\right)^4 + O\left(z^{-6}\right)\right] \\ &= t - \frac{z}{c} - \frac{a^2}{2cz} + \frac{a^4}{8cz^3} + O\left(z^{-5}\right) \\ t_1 - t_2 &= \frac{a^2}{2cz} - \frac{a^4}{8cz^3} + O\left(z^{-5}\right) \qquad \text{as } z \to \infty \end{aligned} \tag{3.5}$$

the first term of which has been used as an approximation to the width of the initial narrow pulse. This approximation is not exact but has a relative error

$$\frac{t_1 - t_2}{\left(\frac{a^2}{2cz}\right)} - 1 = -\frac{a^2}{4z^2} \qquad \text{as } z \to \infty \tag{3.6}$$

which is negligible for $z >> a$. Since (3.4) gives the exact amplitude of unity, then (3.6) gives the error in the time integral (area) of the initial pulse. Then (3.6) gives an estimate of how far away one should be for (3.4) to apply (in sense as above and in [3]).

After $t_2 = 0$ another term enters the electric field in (3.2). The resulting amplitude is

$$\begin{aligned} u(t_1) - \frac{z}{\left[z^2 + a^2\right]^{\frac{1}{2}}} u(t_2) &= 1 - \frac{z}{\left[z^2 + a^2\right]^{\frac{1}{2}}} \\ &= 1 - \left[1 + \left(\frac{a}{z}\right)^2\right]^{-\frac{1}{2}} \\ &= \frac{1}{2}\left(\frac{a}{z}\right)^2 + O\left(z^{-4}\right) \qquad \text{as } z \to \infty \end{aligned} \tag{3.7}$$

In an amplitude sense this is a relative error, also negligible for $z >> a$. In an area (time-integral) sense, however, the "error" is more significant. Integrating from $t_2 = 0$ to some other later time we have

$$\text{relative "error"} = \frac{t_2\left[\frac{1}{2}\left(\frac{a}{z}\right)^2 + 0\left(z^{-4}\right)\right]}{\left(\frac{a^2}{2cz}\right)}$$

$$\begin{aligned} &= \frac{c}{z} t_2\left[1 + 0\left(z^{-2}\right)\right] \\ &= \frac{c}{z}\left[t - \frac{1}{c}\left[z^2 + a^2\right]^{\frac{1}{2}}\right]\left[1 + 0\left(z^{-2}\right)\right] \\ &= \left[\frac{ct}{z} - 1\right]\left[1 + 0\left(z^{-2}\right)\right] \qquad \text{as } z \to \infty \end{aligned} \tag{3.8}$$

So for this to be small we need

$$0 < t - \frac{z}{c} << \frac{z}{c} \tag{3.9}$$

As one goes to larger and larger z the retarded time is allowed to also get larger with (3.4) still approximating the electric-field pulse in a time-integral sense. This is analogous to the usual far-field criterion for electrically small antennas (i.e., $w > c / r$). So this is a late-time or low-frequency limitation.

Similarly after $t_2 = 0$ another term enters the magnetic field in (3.2). This has two parts, one a constant and the second a ramp function of time. The constant part is

$$\begin{aligned} u(t_1) - \frac{1}{2}\frac{2z^2 + a^2}{z^2 + a^2} u(t_2) &= 1 - \frac{1}{2}\frac{2z^2 + a^2}{z^2 + a^2} \\ &= \frac{1}{2}\frac{a^2}{z^2 + a^2} \\ &= \frac{1}{2}\left(\frac{a}{z}\right)^2 + 0\left(z^{-4}\right) \qquad \text{as } z \to \infty \end{aligned} \tag{3.10}$$

This is the same asymptotic result as in (3.7) and produces the same kind of "errors". In an amplitude sense this (3.10) is the relative error and in a time integral sense (3.8) and (3.9) directly apply.

The ramp-function term in the magnetic field has an amplitude (after $t_2 = 0$)

$$\begin{aligned} \frac{1}{2}\frac{a^2}{\left[z^2 + a^2\right]^{\frac{3}{2}}} ct_2 &= \frac{a^2 ct_2}{2z^3}\left[1 + \left(\frac{a}{z}\right)^2\right]^{-\frac{3}{2}} \\ &= \frac{a^2 ct_2}{2z^3}\left[1 + \left(\frac{a}{z}\right)^2\right] \qquad \text{as } z \to \infty \end{aligned} \tag{3.11}$$

Comparing this to the initial unit amplitude gives the requirement

$$0 < t - \frac{z}{c} << \frac{2z^3}{ca^2} \tag{3.12}$$

which for $z > a$ is somewhat less stringent than (3.9). In an area (time-integral) sense we have

$$\begin{aligned}
\text{relative "error"} &= \frac{\frac{a^2 ct_2}{4z^3}\left[1+0\left(z^{-2}\right)\right]}{\left(\frac{a^2}{2cz}\right)} \\
&= \frac{1}{2}\left(\frac{ct_2}{z}\right)^2\left[1+0\left(z^{-2}\right)\right] \\
&= \frac{1}{2}\left[\frac{ct}{z}-1\right]^2\left[1+0\left(z^{-2}\right)\right] \qquad \text{as } z \to \infty
\end{aligned} \tag{3.13}$$

For this to be small we need

$$0 < t - \frac{z}{c} << \sqrt{2}\,\frac{z}{c} \tag{3.14}$$

which is similar to (3.9). So these additional terms for the magnetic field are near-field effects.

IV. TEM Plane Wave on Circular Aperture

The solution obtained thus far has been for the case of a uniform tangential electric field on a circular aperture as in (2.1). A more general type of aperture distribution is given by [4]

$$\begin{aligned}
\vec{E}_t(x',y';t) &\equiv \begin{cases} E_0\vec{g}(x',y')f(t) & \text{for } 0 \le \Psi' < a \\ 0 & \text{for } \Psi' > a \end{cases} \\
\vec{g}(x',y') &= -\nabla_t'\Phi(x',y') \\
\nabla_t' \bullet \vec{g}(x',y') &= -\nabla_t'^2\Phi(x',y') = 0
\end{aligned} \tag{4.1}$$

i.e., the spatial distribution is derived from a scalar potential satisfying the two-dimensional Laplace equation in the transverse (x',y') coordinates. This is an important type of field distribution associated with a TEM plane wave (propagating in the z direction).

In [5] it is shown that a spherical TEM wave (as on a conical wave launcher) can be converted into a planar TEM wave by a suitably positioned paraboloidal reflector. It is shown that besides a uniform arrival time on the aperture plane the TEM plane wave has its field components given by the reflector transform which is just the usual stereographic transform [14] with a minus sign. Furthermore, this solution is exact up to the time at which scattering from the reflector edge or feed arms (blockage) reaches the observer. Not only the reflector type of IRA (impulse radiating antenna) has this property, but also the lens type of IRA using appropriate leases in [15] which convert spherical TEM waves into planar TEM waves.

In [4] the form in (4.1) with $f(t)$ as $u(t)$ is used to find the coefficient of the approximate delta function δ_a in the "far field." Here the result can be considered in greater detail on the z axis in a more exact sense. So let us consider the fields on the z axis with the distribution in (4.1) substituted into the integral in (1.2) and (1.3).

Expand the two-dimensional potential in cylindrical (Ψ',ϕ') coordinates [14] on the aperture plane as

$$\Phi(x',y') = \sum_{n=0}^{\infty} \Psi'^n\left[a_n\cos(n\phi') + b_n\sin(n\phi')\right] \tag{4.2}$$

where only terms with no singularities (in $\Psi' < a$) are included and the coefficients a_n and b_n are real since Φ is assumed real. Then we have

$$\vec{g}(x',y') = -\sum_{n=1}^{\infty}\left\{\vec{1}_{\Psi'} n\Psi'^{n-1}\left[a_n\cos(n\phi') + b_n\sin(n\phi')\right]\right.$$
$$\left. + \vec{1}_{\phi'} n\Psi'^{n-1}\left[-a_n\sin(n\phi') + b_n\cos(n\phi')\right]\right\} \tag{4.3}$$

where the unit vectors are related to the Cartesian components as

$$\begin{aligned}\vec{1}_{\Psi'} &= \cos(\phi')\vec{1}_x + \sin(\phi')\vec{1}_y \\ \vec{1}_{\phi'} &= -\sin(\phi')\vec{1}_x + \cos(\phi')\vec{1}_y\end{aligned} \tag{4.4}$$

Note in what follows that only $n \geq 1$ are included.

In evaluating the integrals over the aperture field note the use of the Cartesian field components. The incremental area is $\Psi' d\Psi' d\phi'$ from (2.3) and R are not functions of ϕ' as in (2.2). Since the aperture boundary is on constant $\Psi'(=a)$ let us consider the integral of the various terms in (4.3) with (4.4) over $0 \leq \phi' < 2\pi$.

For the electric field in (1.2) with the observer on the z axis we have for $n \neq 1$ integrals of the form for E_x and E_y.

$$\begin{aligned}\int_0^{2\pi}\cos(\phi')\cos(n\phi')d\phi' = 0 \quad &, \quad \int_0^{2\pi}\sin(\phi')\sin(n\phi')d\phi' = 0 \\ \int_0^{2\pi}\cos(\phi')\sin(n\phi')d\phi' = 0 \quad &, \quad \int_0^{2\pi}\sin(\phi')\cos(n\phi')d\phi' = 0\end{aligned} \tag{4.5}$$

which can be found using standard formulas (e.g., [13]). So only $n = 1$ contributes to these field components through integrals of $\cos^2(\phi')$ and $\sin^2(\phi')$ which give π. For E_z, noting the inclusion of

$$x' = \Psi'\cos(\phi') \quad , \quad y' = \Psi'\sin(\phi') \tag{4.6}$$

then we have for all n the integrals which are grouped (for cancellation) as the coefficients of the a_n and b_n as

$$a_n\int_0^{2\pi}\left[\cos^2(\phi')\cos(n\phi') + \cos(\phi')\sin(\phi')\sin(n\phi')\right.$$
$$\left. + \sin^2(\phi')\cos(n\phi') - \cos(\phi')\sin(\phi')\sin(n\phi')\right]d\phi'$$
$$= a_n\int_0^{2\pi}\cos(n\phi')d\phi' = 0$$

$$
\begin{aligned}
b_n\int_0^{2\pi}&\Big[\cos^2(\phi')\sin(n\phi')-\cos(\phi')\sin(\phi')\cos(n\phi') \\
&\qquad +\sin^2(\phi')\sin(n\phi')+\cos(\phi')\sin(\phi')\cos(n\phi')\Big]d\phi' \\
&= b_n\int_0^{2\pi}\sin(n\phi')d\phi' = 0
\end{aligned} \tag{4.7}
$$

Hence there are only E_x and E_y components on the z axis, these coming from the n = 1 term in the expansion.

For the magnetic field in (1.3) first consider the similar integrals for H_x and H_y. Each is written as the sum of two integrals. Looking at the second integrals note first that some of the terms have no x' or y' dependence shown (other than in E_y' and E_x' respectively). These lead to integrals over ϕ' just like those for E_x and E_y in (4.5) and result in contributions only from $n=1$. This leaves terms with coefficients $x'y', y'^2$, and x'^2 (noting $(x,y)=(0,0)$) which fortunately all multiply a term with R^{-4} times the three-term sum involving the same time derivative, constant, and integral. Then for H_x group the coefficients of these terms according to the a_n and b_n coefficients given, for $n\neq 1$

$$
\begin{aligned}
a_n\int_0^{2\pi}&\Big[-\cos^2(\phi')\sin(\phi')\cos(n\phi')+\cos(\phi')\sin^2(\phi')\sin(n\phi') \\
&\qquad -\sin^3(\phi')\cos(n\phi')-\cos(\phi')\sin^2(\phi')\sin(n\phi)\Big]d\phi' \\
&= -a_n\int_0^{2\pi}\sin(\phi')\cos(n\phi')d\phi' = 0 \\
b_n\int_0^{2\pi}&\Big[-\cos^2(\phi')\sin(\phi')\sin(n\phi')+\cos(\phi')\sin^2(\phi')\cos(n\phi') \\
&\qquad -\sin^3(\phi')\sin(n\phi')-\cos(\phi')\sin^2(\phi')\cos(n\phi)\Big]d\phi' \\
&= -b_n\int_0^{2\pi}\sin(\phi')\sin(n\phi') = 0
\end{aligned} \tag{4.8}
$$

Similarly for H_y group the coefficients of these terms the same way giving, for $n\neq 1$

$$
\begin{aligned}
a_n\int_0^{2\pi}&\Big[\cos(\phi')\sin^2(\phi')\cos(n\phi')-\cos^2(\phi')\sin(\phi')\sin(n\phi') \\
&\qquad +\cos^3(\phi')\cos(n\phi')+\cos^2(\phi')\sin(\phi')\sin(n\phi')\Big]d\phi' \\
&= a_n\int_0^{2\pi}\cos(\phi')\cos(n\phi')d\phi' = 0
\end{aligned}
$$

$$
\begin{aligned}
b_n \int_0^{2\pi} \Big[&\cos(\phi')\sin^2(\phi')\sin(n\phi') + \cos^2(\phi')\sin(\phi')\cos(n\phi') \\
&+ \cos^3(\phi')\sin(n\phi') - \cos^2(\phi')\sin(\phi')\cos(n\phi)\Big] d\phi' \\
&= b_n \int_0^{2\pi} \cos(\phi')\sin(n\phi')d\phi' = 0
\end{aligned} \tag{4.9}
$$

For H_z the situation is somewhat simpler and we have the coefficients for all n as

$$
\begin{aligned}
a_n \int_0^{2\pi} \Big[&\cos(\phi')\sin(\phi')\cos(n\phi') + \sin^2(\phi')\sin(n\phi') \\
&- \cos(\phi')\sin(\phi')\cos(n\phi') + \cos^2(\phi')\sin(n\phi')\Big] d\phi' \\
&= a_n \int_0^{2\pi} \sin(n\phi')d\phi' = 0 \\
b_n \int_0^{2\pi} \Big[&\cos(\phi')\sin(\phi')\sin(n\phi') - \sin^2(\phi')\cos(n\phi') \\
&- \cos(\phi')\sin(\phi')\sin(n\phi') - \cos^2(\phi')\cos(n\phi)\Big] d\phi' \\
&= -b_n \int_0^{2\pi} \cos(n\phi')d\phi' = 0
\end{aligned} \tag{4.10}
$$

similar to the results for E_z in (4.7). Hence there are only H_z and H_y components on the z axis, these coming from the $n = 1$ term in the expansion.

An alternate way to show that only the $n = 1$ term contributes to the z-axis fields which are transverse to the z axis avoids looking at the details of the integrals. Instead symmetry considerations can be applied [7]. The circular aperture itself has C_∞ symmetry with respect to the z axis. However, the particular terms in (4.3) indexed by n and a_n or b_n have discrete rotation symmetry C_n in which rotation by $2\pi / n$ brings the fields (magnitude and orientation) back to their origin configurations. Each of these field terms has $2n$ symmetry planes (containing the z axis), n with respect to which the fields are symmetric and n antisymmetric. These symmetries with respect to such a plane are defined by a reflection dyad which relates electric fields at mirror positions by a plus sign (plus for components parallel to plane) for symmetric fields and minus sign for antisymmetric fields, and conversely for the magnetic field [1,6].

For $n \geq 2$ with n planes giving antisymmetric fields, the electric field must be perpendicular to each symmetry plane. Hence on the z axis, not only does this mean that here is no E_z, but also no transverse field (E_x and E_y) on the z axis due to the fact that a field perpendicular to one of the n planes is not perpendicular to any of the other n-1 planes (on the z axis). For the magnetic field use the n planes giving symmetric fields and repeat the above argument. Hence, for $n \geq 2$, there are no fields on the z axis.

For $n=1$ there is only one plane for antisymmetric fields allowing a transverse, but no longitudinal, electric field on the z axis. Similarly, the one plane for symmetric fields allows a transverse, but no longitudinal magnetic field on the z axis. This agrees with the foregoing detailed calculations.

Since the $n=1$ term in the aperture field is the important term, let us look at this term in more detail. Using a subscript "1" to distinguish this term (4.2) gives

$$\begin{aligned}\Phi_1(x',y') &= \Psi'\left[a_1\cos(\phi')+b_1\sin(\phi')\right] \\ &= a_1x'+b_1y'\end{aligned} \tag{4.11}$$

and (4.3) gives

$$\begin{aligned}\vec{g}_1(x',y') &= -\vec{1}_{\Psi'}\left[a_1\cos(\phi')+b_1\sin(\phi')\right]-\vec{1}_{\phi'}\left[-a_1\sin(\phi')+b_1\cos(\phi')\right] \\ &= -a_1\vec{1}_x-b_1\vec{1}_y\end{aligned} \tag{4.12}$$

This is merely a uniform electric field with a constant direction (polarization) on the aperture. Furthermore, this represents the tangential electric field at the center of the aperture where only the $n=1$ term in (4.3) is non zero. So (4.1) becomes at the origin

$$\vec{E}_t(0,0;t) = E_o\vec{g}(0,0)f(t) = -E_of(t)\left[a_1\vec{1}_x+b_1\vec{1}_y\right] \tag{4.13}$$

Since only the $n=1$ term gives the fields on the z axis then it is a field as in (4.13), extended to the entire circular aperture, which contributes to the integrals for these fields. To evaluate such fields note that (4.13) can be rotated about the z axis to give any polarization as desired. In particular it can be rotated to be parallel to the y direction, and thereby assume exactly the form in (2.1). The conclusion is then that the planar TEM field throughout a circular aperture of radius a (no singularities for $0\le\Psi'<a$) with zero tangential electric field outside the circle (as in (4.1)) gives the same result as a uniform field in (2.1) and the results of sections II and III apply here. One replaces the distribution in (4.1) by the distribution at the origin to define the equivalent uniform field for the entire aperture. This gives an alternate way to view the approximate impulse δ_a in [3,4] in terms of the exact results in the previous sections.

V. Concluding Remarks

So now we have exact expressions for both $\bar{E}$ and $\bar{H}$ for the fields on the z axis for a circular aperture antenna. This applies not only to a uniform tangential electric field on the aperture, but also the electric field due to a TEM plane wave (in general inhomogeneous) on the aperture by using the field at the center of the aperture as an equivalent uniform field. By the duality principle these results apply to an electric current sheet with the same spatial distributions on interchanging the roles of electric and magnetic fields.

Of course, this is an ideal type of aperture distribution, applying approximately to real antennas, such as IRAs (with reflectors or lenses). For its intended use, however, to the impulsive part of the waveform it is a rather useful approximation. Other parts of the time-domain waveform are associated with feeds and how the portions of the aperture plane outside of the circular disk of interest here are treated.

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