Sensor and Simulation Notes

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Radiation from Self-Reciprocal Apertures

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Abstract

We define here a new symmetry operator called reciprocation. Apertures that remain unchanged after reciprocation are called self-reciprocal, and have a number of unique properties. When excited suddenly by a voltage, a self-reciprocal aperture becomes an antenna. This antenna is an approximation to a TEM horn, to a lens IRA (TEM feed + lens), or to a reflector IRA (TEM feed + reflector). Because of the unique properties of self-reciprocal apertures, these antenna configurations have radiation characteristics that are of a particularly simple form. The simple form for the radiation from self-reciprocal apertures greatly simplifies the design and analysis of this class of antenna.

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I. Introduction

Recent interest in the radiation of fast transients has led to investigations of the fields radiated from apertures [1]. Using the concepts of [1], we can describe the prompt (early-time) radiation of most of the common antennas used for radiating fast transients, including long TEM horns [2], lens IRAs (TEM feed with a lens) [1-3], and reflector IRAs [1, 4-8]. Although the techniques of [1] are simple in principal, involving a contour integration around the aperture, often the contour integral can only be carried out numerically. It would be useful if the integral could be simplified. We define here a new class of geometries, called self-reciprocal geometries, for which these contour integrations simplify considerably. This simplification leads to a closed-form result, and is proportional to the field at the center of the aperture [13].

We begin by describing the reciprocation operator. A geometry that remains unchanged after reciprocation is called self-reciprocal. We calculate here the field radiated from a self-reciprocal aperture in a number of ways; using contour integration, using the line dipole moment of the aperture, and using the field at the center of the aperture. The technique of using the field at the center is particularly simple to implement. Finally, we describe how this new symmetry operator fits into the larger picture of symmetry operators in electromagnetics.

II. The Reciprocation Operator and Self-Reciprocal Apertures

An entire field of endeavor has grown up around the concept that by understanding symmetry in physical systems, one can more simply describe these systems [9]. One often takes advantage of left-right symmetry (reflection) to simplify a problem, but the applications of symmetry go far beyond this. One can use a wide variety of rotations, inversions, and other operators, as described in [9] to describe such devices and phenomena as self-complementary antennas, dihedral capacitors, and wave propagation through chiral media. The symmetry we describe here can be considered an additional method of simplifying the description of the physical world with a new type of symmetry.

Let us consider now the new symmetry operator, reciprocation, in two dimensions. This operator replaces each point in the original geometry by a constant times its reciprocal conjugate. The original geometry is described in terms of the usual x and y coordinates as

$$\zeta = x + j\,y = \Psi\, e^{j\phi} \tag{2.1}$$

After invoking reciprocation, all points in the plane are mapped to new locations, indicated by a subscript 2. This mapping is

$$\begin{aligned} \zeta_2 &= x_2 + j\,y_2 = a^2 / \zeta^*, \quad x_2 = a^2 / x, \quad y_2 = a^2 / y \\ \zeta_2 &= \Psi_2\, e^{j\phi_2}, \quad \Psi_2 = a^2 / \Psi, \quad \phi_2 = \phi \end{aligned} \tag{2.2}$$

This is in some ways similar to reflecting an object through a circle of radius a centered at the origin. A single point maps to a new point which appears at the same angle from the x-axis, but its new distance from the origin is a^2 / Ψ. Thus, points outside the circle are mapped inside, and points inside the circle are mapped outside. Points on the circle remain unchanged. Note that one could extend this idea to three dimensions trivially.

The ideas behind this transformation are not particularly new, as they have appeared earlier in [10, 11]. This transformation, for example, has been used to simplify the problem of a line charges near a conducting cylinder, parallel to the line charge [10, pp. 69-70]. Toward the end of this paper we develop this operation in the context of symmetry groups and symmetry operators; a concept that is new. However, there is some precedent for using this transformation to simplify problems in electromagnetics.

The operator we have defined here is not analytic in ζ. A related operator which maps ζ to a^2 / ζ, (without the conjugate), is both analytic and conformal. The reciprocation operator we defined above is not analytic, nor is it conformal. It is actually a combination of the analytic operator a^2 / ζ, and a reflection through the x axis. Thus, it is "almost" a conformal mapping, the angles being preserved, but reversed.

It is pointed out in [11] that under this transformation we call reciprocation, every straight line is transformed into a circle passing through the origin, and every circle passing through the origin is transformed into a straight line. Furthermore, any circle that does not pass through the origin is transformed into another circle. Finally, any straight line passing through the origin is transformed onto itself. We can consider a number of pairs of structures that are complementary under reciprocation (Figure 2.1). It is clear these structures are consistent with the above rules.

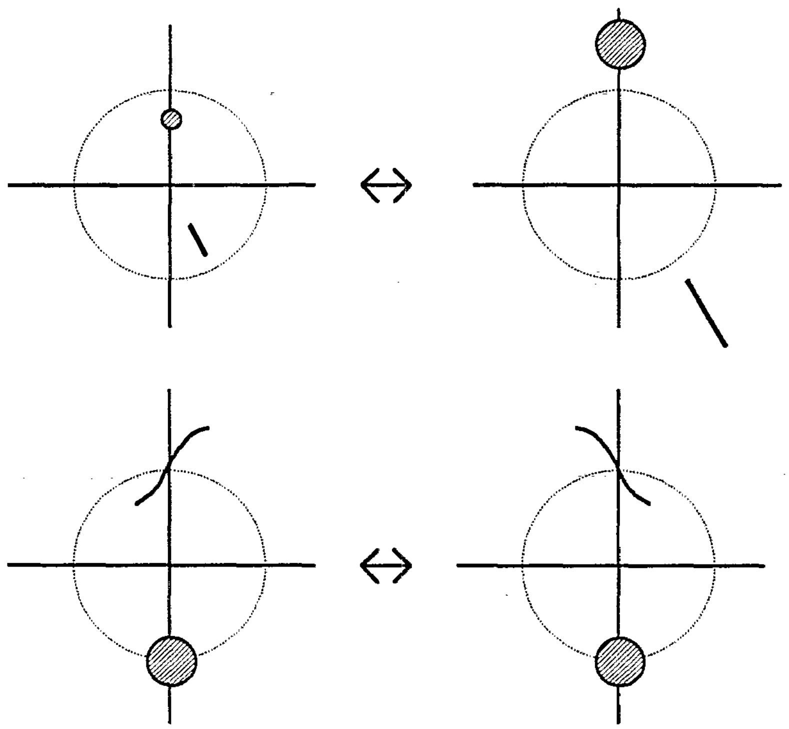

Figure 2.1. Some examples of structures and their reciprocals.

If we now assume that the shapes in Figure 2.1 are made of perfect electric conductors, and we assume a potential difference V between the two conductors, then we have in two dimensions the cross section of a transmission line. We also assume that there is no net charge on the combination of the two conductors. This two dimensional problem must satisfy Laplace's equation in two dimensions, so one can use a complex potential function to describe the static fields that result [10]. Thus, the complex potential is

$$w(\zeta) = u(\zeta) + j\,v(\zeta) \tag{2.3}$$

where the function $u(\zeta)$ is usually taken to be proportional to the electric potential. The contours of the two conductors that form the transmission line are contours of constant u. The fields can be described in a number of ways as

$$
\begin{aligned}
E(\zeta) &= E_x(\zeta) - j\,E_y(\zeta) = -\frac{V}{\Delta u}\frac{dw(\zeta)}{d\zeta} \\
&= -\frac{V}{\Delta u}\left[\frac{du(\zeta)}{d\zeta} + j\frac{dv(\zeta)}{d\zeta}\right] = -\frac{V}{\Delta u}\left[\frac{\partial u(\zeta)}{\partial x} + j\frac{\partial v(\zeta)}{\partial x}\right] \\
&= -\frac{V}{\Delta u}\left[\frac{\partial u(\zeta)}{\partial x} - j\frac{\partial u(\zeta)}{\partial y}\right] = -\frac{V}{\Delta u}\left[\frac{\partial v(\zeta)}{\partial y} + j\frac{\partial v(\zeta)}{\partial x}\right]
\end{aligned}
\tag{2.4}
$$

$$
H(\zeta) = H_x(\zeta) - j\,H_y(\zeta) = E(\zeta)/(j Z_o)
$$

where Δu is the change in u from the first conductor to the second, and V is the potential difference between them. In these equations we have used the Cauchy-Riemann expressions in rectangular coordinates [12],

$$
\frac{\partial u(\zeta)}{\partial x} = \frac{\partial v(\zeta)}{\partial y}, \qquad \frac{\partial u(\zeta)}{\partial y} = -\frac{\partial v(\zeta)}{\partial x}
\tag{2.5}
$$

These are valid for all complex functions $w(\zeta)$, provided its derivative $w'(\zeta)$ exists.

Continuing with the development of the transmission-line analogy, we can also identify a geometric factor f_g, which is the ratio of the characteristic impedance of such an arrangement to the impedance of free space. Thus, we have

$$
f_g = \frac{Z_c}{Z_o} = \frac{\Delta u}{\Delta v}
\tag{2.6}
$$

where Δu is the change in u from one conductor to the other, Δv is the change in v as one encircles one of the conductors, and Z_o is the impedance of free space.

We can also write analogous expressions for a second problem, which is the reciprocated geometry of the first. The second problem has a complex potential that is described by

$$
\begin{aligned}
w_2(\zeta) &= w^*(\zeta_2) + j v_o = w^*(a^2/\zeta^*) + j v_o \\
&= u(a^2/\zeta^*) - j v(a^2/\zeta^*) + j v_o
\end{aligned}
\tag{2.7}
$$

where v_o is some real constant. Another way of expressing this is

$$
\begin{aligned}
w(\zeta) &= w_2^*(\zeta_2) - j v_o = w_2^*(a^2/\zeta^*) - j v_o \\
&= w_2(a^2/\zeta) - j v_o
\end{aligned}
\tag{2.8}
$$

We can now derive some interesting analytic properties from (2.7) and (2.8). Of course, the function w is analytic in ζ. Because of (2.8), the function w_2 must also be an analytic function of ζ, except at $\zeta = 0$. Finally, w_2 is not an analytic function of ζ_2, but it is an analytic function of ζ_2^*. Thus, the function $w_2^*(\zeta_2)$ is analytic in ζ_2. We will find these properties useful later, when we calculate contour integrals.

The fields in the reciprocal problem can be related to those in the original problem as

$$\begin{aligned} E_2(\zeta_2) &= -E^*(\zeta)\frac{\zeta^{*2}}{a^2} \\ H_2(\zeta_2) &= E_2(\zeta_2)/(j Z_o) \end{aligned} \tag{2.9}$$

In order to prove this, we note that the reciprocal transform consists of a combination of an analytic transform and a reflection through the x axis (i.e. conjugate) (Figure 2.2). First we consider the relationship between the original problem and the analytic transform, which maps ζ into $\zeta_1 = a^2/\zeta$. We have

$$\begin{aligned} w(\zeta) &= w_1(\zeta_1) + j v_o \\ \frac{dw(\zeta)}{d\zeta} &= \frac{dw_1(\zeta_1)}{d\zeta_1}\frac{d\zeta_1}{d\zeta} = \frac{dw_1(\zeta_1)}{d\zeta_1}\left(-\frac{a^2}{\zeta^2}\right) \end{aligned} \tag{2.10}$$

so the relationship between the fields is

$$\begin{aligned} -\frac{V}{\Delta u}\frac{dw(\zeta)}{d\zeta} &= \frac{V}{\Delta u}\frac{dw_1(\zeta_1)}{d\zeta_1}\frac{a^2}{\zeta^2} \\ E(\zeta) &= -E_1(\zeta_1)\frac{a^2}{\zeta^2} \\ E_1(\zeta_1) &= -E(\zeta)\frac{\zeta^2}{a^2} \end{aligned} \tag{2.11}$$

We now find the relationship between the original problem and the reciprocal problem. The mapping between the analytic problem and the reciprocal problem is $\zeta_2 = \zeta_1^*$. This is just a reflection through the x axis. Thus, we have

$$E_2(\zeta_2) = E_1^*(\zeta_1) = -E^*(\zeta)\frac{\zeta^{*2}}{a^2} \tag{2.12}$$

which proves the relationship in (2.10).

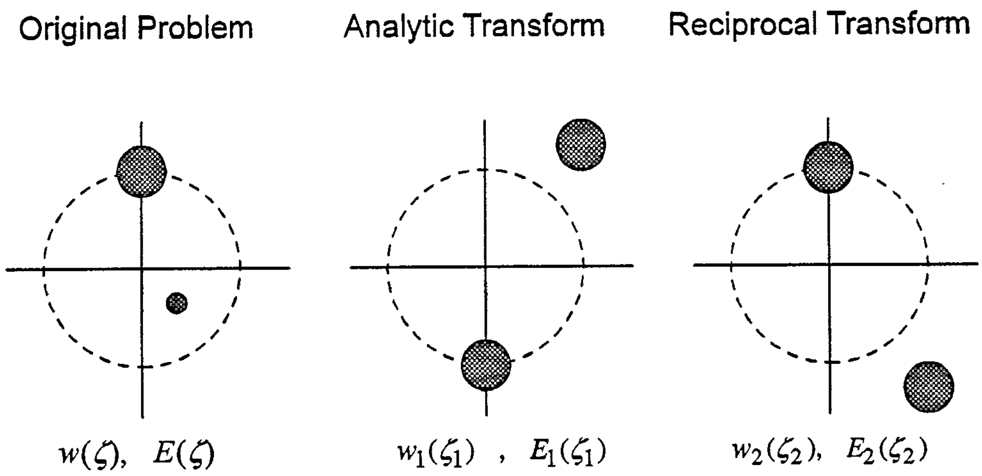

Figure 2.2. Three configurations for calculating the potentials and fields.

Let us now consider the special case of self-reciprocal structures. These are structures that remain unchanged after reciprocation. A sampling of self-reciprocal structures is shown in Figure 2.3. In this case, the structure is unchanged after reciprocation, so the potential function maps onto itself (to within a constant) as

$$\begin{aligned} w(\zeta) &= w^*(\zeta_2) + jv_o = w^*(a^2/\zeta^*) + jv_o \\ u(\zeta) &= u(\zeta_2) \\ v(\zeta) &= -v(\zeta_2) + jv_o \end{aligned} \tag{2.13}$$

where v_o is a real constant, which determines the location of the branch cut. We will see later that a convenient location for the branch cut is on the circle through $\zeta = -a$, which makes the value of v_o equal to zero. We can also establish a relationship between the fields in self-reciprocal structures. Using (2.12) we have

$$E(\zeta_2) = -E^*(\zeta)\frac{\zeta^{*2}}{a^2} \tag{2.14}$$

Having established some of the basic properties of self-reciprocal apertures, let us now consider the radiation from such apertures.

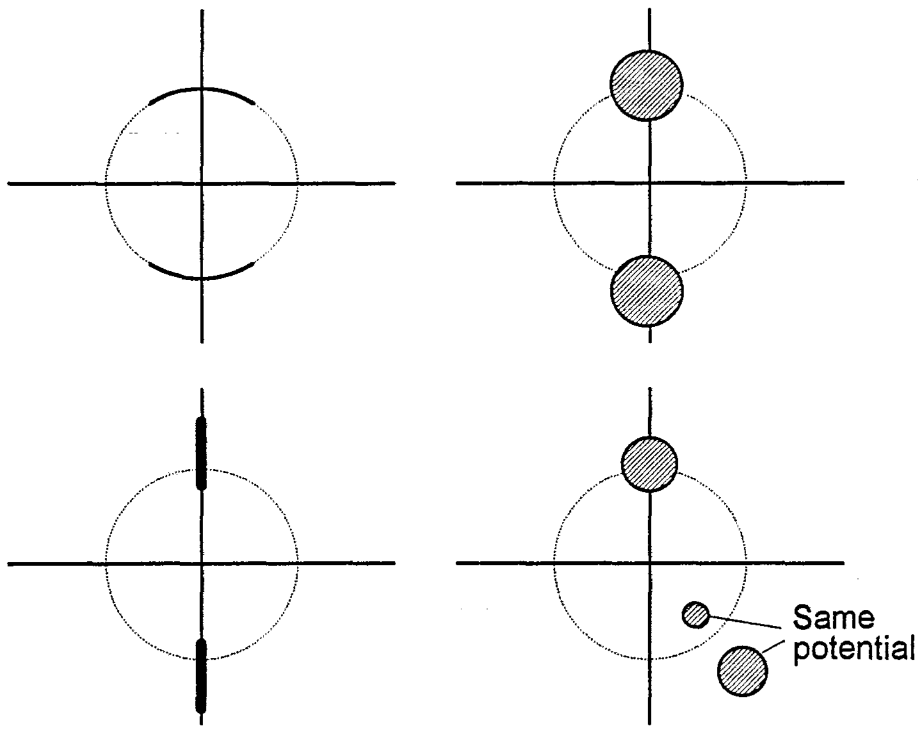

Figure 2.3. Some examples of self-reciprocal apertures.

III. Radiation from Self-Reciprocal Apertures Using Contour Integration

The theory of radiation from apertures using contour integration has already been developed well in [1]. Our purpose here is to show some special results that relate to self-reciprocal apertures.

First, we review the theory of radiation from apertures from [1]. If some potential is turned on suddenly, then we have a radiated field on boresight of

$$E(r,t) = -\frac{V}{r}\frac{h_a}{2\pi c f_g}\delta_a(t) \tag{3.1}$$

where c is the speed of light, f_g is the normalized impedance of the transmission line formed by the two wires, and $\delta_a(t)$ is an approximate Dirac delta function, which becomes exact in the limit as r approaches infinity. Furthermore, the aperture height, h_a can be calculated as

$$\vec{h}_a = h_{a_x}\vec{1}_x + h_{a_y}\vec{1}_y = -\frac{f_g}{V}\iint_{S_a}\vec{E}(x,y)\,dx\,dy \tag{3.2}$$

where the surface integral is taken over the entire cross section, excluding the conductors. Alternatively, this can be expressed in complex field notation as

$$h_a = h_{a_x} - j\,h_{a_y} = -\frac{f_g}{V}\iint_{S_a}E(\zeta)\,dx\,dy = \frac{1}{\Delta v}\iint_{S_a}\frac{dw(\zeta)}{d\zeta}dxdy = \frac{1}{\Delta v}W \tag{3.3}$$

Note that the real and imaginary parts of h_a correspond to the x and y components of the vector aperture height. The integral can be written in a variety of ways as

$$\begin{aligned} W &= \iint_{S_a}\frac{dw(\zeta)}{d\zeta}\,dx\,dy = \iint_{S_a}\frac{\partial w}{\partial x}\,dx\,dy = \oint_{C_a} w\,dy \\ &= -j\iint_{S_a}\frac{\partial w}{\partial y}\,dx\,dy = j\oint_{C_a} w\,dx \end{aligned} \tag{3.4}$$

or by combining the above two forms of the contour integral,

$$W = \frac{1}{2}\oint_{S_a} w(\zeta)\left[j\,dx + dy\right] = \frac{j}{2}\oint_{C_a} w(\zeta)\,d\zeta^* \tag{3.5}$$

where the contour integral is taken around the entire cross section, as shown in Figure 3.1. Recall that this final result can be compared to the Cauchy integral theorem,

$$\oint_{C_a} w(\zeta)\,d\zeta = 0 \tag{3.6}$$

which is valid for all $w(\zeta)$ as long as there is no singularity within C_a. The difference between (3.5) and (3.6) is the conjugation.

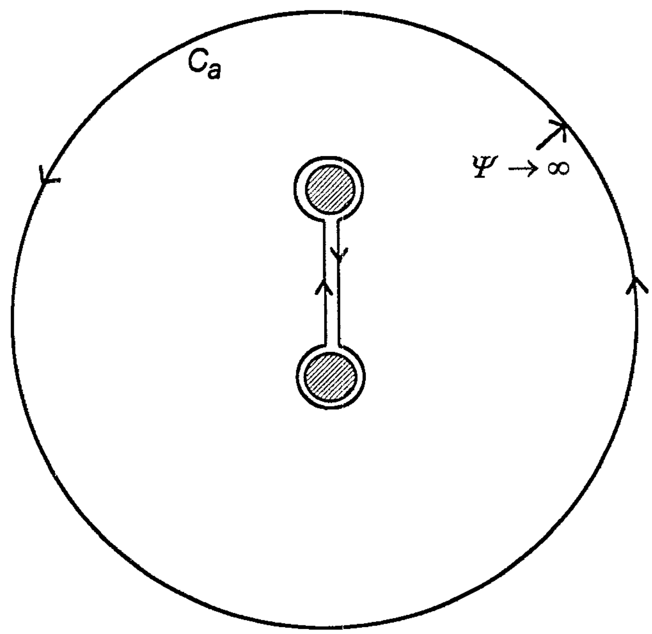

Figure 3.1. The contour C_a, where surface S_a is the area contained within C_a. The net charge on the conductors is zero.

Another way of expressing the integral W involves breaking up the potential function into its real and imaginary parts. Thus, [1]

$$\begin{aligned} W &= \iint_{S_a} \frac{\partial u}{\partial x}\,dx\,dy + j\iint_{S_a} \frac{\partial v}{\partial x}\,dx\,dy \\ &= \iint_{S_a} \frac{\partial u}{\partial x}\,dx\,dy - j\iint_{S_a} \frac{\partial u}{\partial y}\,dx\,dy = \oint_{C_a} u\,dy + j\oint_{C_a} u\,dx = j\oint_{C_a} u\,d\zeta^* \qquad (3.7) \\ &= \iint_{S_a} \frac{\partial v}{\partial y}\,dx\,dy + j\iint_{S_a} \frac{\partial v}{\partial x}\,dx\,dy = -\oint_{C_a} v\,dx + j\oint_{C_a} v\,dy = -\oint_{C_a} v\,d\zeta^* \end{aligned}$$

This allows one to separate out the x and y components of h_a in (3.3).

Let us now consider the effect of using a self-reciprocal structure. The shape of the contour can now be deformed to be the sum of four contours (Figure 3.2). The two contours near the circle of reciprocation are the "diverted" contours, C_+ and C_-, and they map onto each other under reciprocation. The two other contours, at $\Psi \to \infty$ and $\Psi \to 0$, also map onto each other under reciprocation.

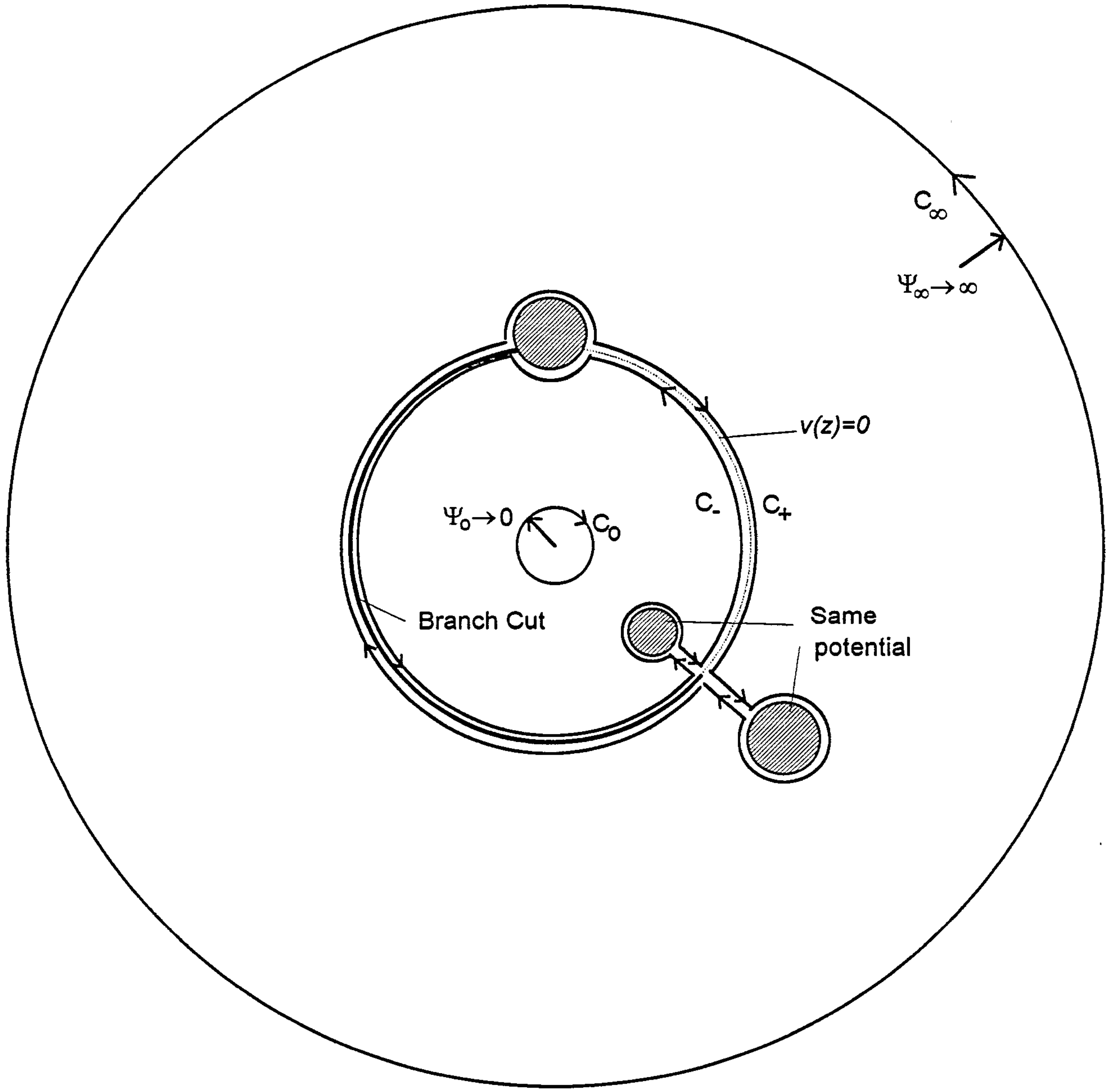

Figure 3.2. The location of C_+ and C_-. Note that $C_a = C_+ \cup C_\infty \cup C_- \cup C_o$. Also, the net charge on the conductors is zero.

Recall that a self-reciprocal structure has the property that

$$\begin{aligned} w(\zeta) &= w^*(\zeta_2) + j v_o \\ u(\zeta) &= u(\zeta_2) \\ v(\zeta) &= -v(\zeta_2) + j v_o \end{aligned} \tag{3.8}$$

with v_o being an arbitrary real constant. It is convenient, for self-reciprocal structures, to impose the condition that $w(\zeta{=}a) = 0$. This leads to a branch cut on the reciprocation circle through $\zeta = -a$, as shown in Figure 3.2. This is a convenient place to have a branch cut, since the branch cut does not move when the structure is reciprocated. Thus, if $w(\zeta)$ is chosen conveniently, the value of v_o in the above relation is zero. Other possible choices for the branch cut might be on the y axis at either $|y|<a$ or $|y|>a$. These two choices transform onto each other under reciprocation.

Let us now simplify the contour integral. Starting from (3.5), we can express W as

$$W = W_+ + W_- = \frac{j}{2}\oint_{C_+\cup C_\infty} w(\zeta)\, d\zeta^* + \frac{j}{2}\oint_{C_-\cup C_o} w(\zeta)\, d\zeta^* \tag{3.9}$$

since $C_a = C_+ \cup C_\infty \cup C_- \cup C_o$. Consider first the integral around the combination of $C_+ \cup C_\infty$. We may invoke reciprocation to simplify the integral W_+, and we find that C_∞ maps to $-C_o$, and C_+ maps to $-C_-$. These mappings are true because the aperture is self-reciprocal. Thus, we find

$$W_+ = \frac{j}{2}\oint_{C_+\cup C_\infty} w(\zeta)\, d\zeta^* = \frac{j}{2}\oint_{C_-\cup C_o} w_2^*(\zeta_2)\frac{a^2}{\zeta_2^2}\, d\zeta_2 \tag{3.10}$$

where $w_2(\zeta_2)$ is the potential function for the reciprocal problem. In the above equation we used the fact that $\zeta^* = a^2/\zeta_2$, so $d\zeta^*/d\zeta = -a^2/\zeta_2^2$. We now need to test the analyticity of the function $w_2^*(\zeta_2)\,(a^2/\zeta_2^2)$, which appears in the integral in (3.10). We have already shown that $w^*(\zeta_2)$ is an analytic function of ζ_2 in Section II, equation (2.8). This implies that the function $w_2^*(\zeta_2)(a^2/\zeta_2)$ is an analytic function of ζ_2 except at $\zeta_2 = 0$. So the integral in (3.10) is around a closed contour that contains no singularities. Thus, by Cauchy's integral theorem (equation (3.6)),

$$W_+ = 0 \tag{3.11}$$

Substituting into (3.9), noting that the integral around C_o is trivially zero in the limit as $\Psi_o \to 0$, we have

$$W = \frac{j}{2}\oint_{C_-} w(\zeta)\, d\zeta^* \tag{3.12}$$

We therefore find that the portion of the aperture integral outside the circle of reciprocation provides no net contribution to the total integral W. This implies that the aperture fields outside the circle of reciprocation in a self-reciprocal structure make no net contribution to the radiated field on boresight. Using (3.7), the integral W can be expressed in various forms as

$$W = W_- = \frac{j}{2}\oint_{C_-} w(\zeta)\, d\zeta^* = j\oint_{C_-} u\, d\zeta^* = -\oint_{C_-} v\, d\zeta^* \tag{3.13}$$

Therefore, the final result for h_a for a self-reciprocal aperture is

$$h_a = \frac{W}{\Delta v} = \frac{j}{2\,\Delta v}\oint_{C_-} w\, d\zeta^* = \frac{j}{\Delta v}\oint_{C_-} u\, d\zeta^* = \frac{-1}{\Delta v}\oint_{C_-} v\, d\zeta^* \tag{3.14}$$

A considerable simplification has been achieved by only needing the contour integral around C_-, rather than around all of C_a.

As a consequence of this complex field theory, we may infer some practical results. In particular, this provides insight concerning the size of lens one might use in a lens IRA. Consider the lens IRA shown in Figure 3.3. The horn portion of the TEM horn is constructed out of a portion of a circular cone. It had previously been thought that one would like to have a lens that included the entire aperture, extending out to infinity. Clearly this is not practical, but we have just shown that the fields outside the horn aperture make no net contrubution to the radiated field on boresight. In other words, if the lens extends only out to the edge of the horn, then one can achieve the same effect as if the lens extended out to infinity.

Similarly, one might wonder if the truncation of the aperture in a reflector IRA reduces the total output of the antenna. We assume here a circular reflector and also the projection of the conical feed onto the aperture plane is self-reciprocal. Under this assumption, the fields outside the aperture make no net contribution to the radiated field on boresight. This had been known previously for thin wires [1], but we have just proven it for arbitrary self-reciprocal apertures.

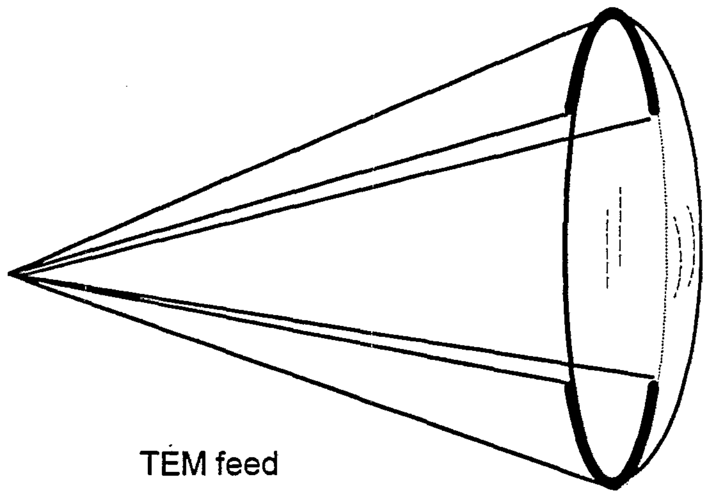

Figure 3.3. A lens IRA.

IV. Radiation from Self-Reciprocal Apertures Using the Field at the Center of the Aperture

Although we have already succeeded in simplifying the contour integral, it turns out that we can simplify it even more by calculating the field at the center of the aperture. This is an extension of [13], which calculated the radiated field for any circular aperture with no conductors or singularities within the aperture. We demonstrate now that a similar result applies to self-reciprocal apertures.

The result of [13] was that the radiated far field could be expressed in terms of the field at the center of a circular aperture of radius a as

$$E(r,t) = \frac{E_o}{r}\frac{a^2}{2c}\delta_a(t) \tag{4.1}$$

where E_o is the field at the center of the aperture. Although E_o was a scalar quantity in [13], one can simply extend this formula to a vector quantity by superposition, so

$$\begin{aligned} \vec{E}(r,t) &= \left[E_{o_x}\vec{1}_x + E_{o_y}\vec{1}_y\right]\frac{a^2}{2cr}\delta_a(t) \\ E(r,t) &= \left[\,E_{o_x} - jE_{o_y}\,\right]\frac{a^2}{2cr}\delta_a(t) \end{aligned} \tag{4.2}$$

Let us now make a conjecture that the field at the center of self-reciprocal apertures describes the radiated far field in a similar manner. If this is true, then by comparing (4.1) to (3.1) it must also be true that

$$-\frac{V}{r}\frac{h_a}{2\pi c f_g} = \frac{E_o}{r}\frac{a^2}{2c} \tag{4.3}$$

so the field at the center must be

$$E_o = E_{o_x} - jE_{o_y} = -\frac{V}{f_g}\frac{1}{\pi a^2}\left(h_{a_x} - jh_{a_y}\right) \tag{4.4}$$

Furthermore, it was shown in [1] that

$$h_a = \frac{p}{2Q'} = \frac{p}{2C'V} = \frac{p f_g}{2\varepsilon_o V} \tag{4.5}$$

where p is the complex electric line dipole moment of the aperture, $p = p_x - j\,p_y$, and Q' is the charge per unit length on one of the conductors. Thus, in order for the conjecture to be true, it must also be true that

$$\begin{aligned} E_o &= -\frac{p}{2\pi\varepsilon_o a^2} \\ E_{o_x} - j\,E_{o_y} &= -\frac{1}{2\pi\varepsilon_o a^2}\left(p_x - j\,p_y\right) \end{aligned} \tag{4.6}$$

If the above relationship holds true for arbitrary self-reciprocal apertures, then we can use (4.1) to describe the radiated field in terms of the field at the center. This is simpler than carrying out one of the contour integrations of (3.14).

In order to express the field at $\zeta = 0$ in terms of the line dipole moment, we must first find the field at infinity in terms of the line dipole moment. Consider two wires which create a line dipole moment p_y (Figure 4.1). For this arrangement it is simple to show that E_y on the x-axis is

$$E_y(x \to \infty, y = 0) = -\frac{2Q'a}{2\pi\varepsilon_o x^2} = -\frac{p_y}{2\pi\varepsilon_o x^2} \tag{4.7}$$

By symmetry, we can write a similar expression relating p_x to E_x on the y axis. Note that this field at infinity is dependent only upon the line dipole moment, so this field at infinity applies equally well to an arbitrary self-reciprocal structure.

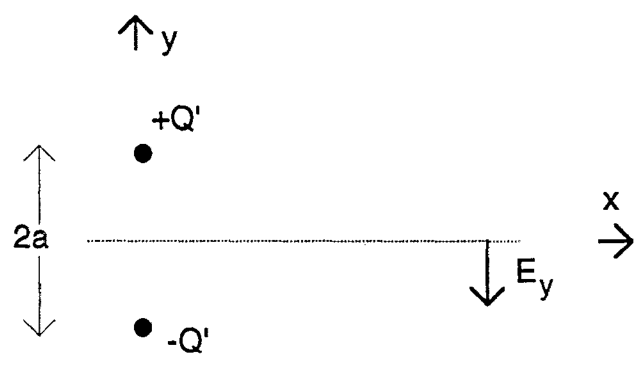

Figure 4.1. The field at infinity due to a simple line dipole, $p_y = 2aQ'$.

Now that we have the field at infinity for a self-reciprocal structure, all that is left is to find a relationship between the field at infinity and the field at $\zeta = 0$. Using (2.14), at $\zeta = x + j0$ where $x \to \infty$, we find

$$E_y(x=0, y=0) = \lim_{x \to \infty} E_y(x, y=0) \frac{x^2}{a^2} \tag{4.8}$$

Combining this with (4.7), we find the field in the center of the aperture of a self-reciprocal structure in terms of its line dipole moment as

$$E_y(x=0, y=0) = -\frac{p_y}{2\pi\varepsilon_o a^2} \tag{4.9}$$

By symmetry, the field at $\zeta = 0$ due to a line dipole in the x direction in a self-reciprocal structure is

$$E_x(x=0, y=0) = -\frac{p_x}{2\pi\varepsilon_o a^2} \tag{4.10}$$

so the total field at the center is

$$E(\zeta = 0) = -\frac{p}{2\pi\varepsilon_o a^2} \tag{4.11}$$

This proves (4.6), which was the condition for being able to express the radiated field in terms of the field at the center. of the aperture. Thus, the radiated field on boresight can be described by (4.1). This is much simpler than using any of the contour integrals or surface integrals we provided earlier.

Note also that we now can describe the early-time step response in the near field on boresight in somewhat more detail. According to [13], the early-time radiated field on boresight must be simply the field at the center of the aperture, up to a certain clear time. This clear time is just the time difference between the ray from the center of the aperture, and the ray from the edge of the conductor that is closest to the center (Figure 4.2). Thus, this clear time is t_2 - t_1.

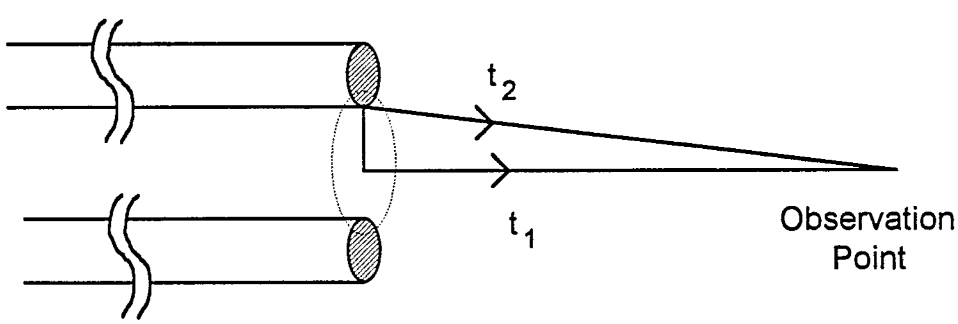

Figure 4.2. The clear time for which the radiated field on boresight is the field at the center.

V. Definition of the Reciprocation Group P

We now define the reciprocation group P, and show how it relates to other symmetry groups. Much of the notation and concepts in this section are extensions of [9], so it is assumed the reader has some familiarity with this reference.

Recall that a Finite Group is a set of a finite number of elements, one of which is the identity. Furthermore, each element of the group has an inverse, and all "products" are also elements. The reciprocation group P (where P is understood to be upper-case ρ (rho)) is defined as

$$P \equiv \{(1),(P)\} \tag{5.1}$$

where (P) is a means of implementing the transformation of equations (2.1) and (2.2). The operator (P) is meant to be in two dimensions, but its extension to three dimensions is straightforward [3]. Note that $(P) = (P)^{-1}$, so $(P)\,(P)^{-1} = 1$. A geometry which remains invariant under reciprocation is called "self-reciprocal".

The reciprocation operator is an example of an involution group [14]. An involution group is any group consisting of a single operator (in this case (P)) and the identity operator. There is only one such abstract group, but there are many matrix representations of such a group. Other examples of involution are the reflection groups through the y and x axes,

$$R_x = \{(1),(R_x)\}, \qquad R_y = \{(1),(R_y)\} \tag{5.2}$$

whose matrix representations (for dot multiplication) are

$$(1) \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (R_x) = (R_x)^{-1} \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (R_y) = (R_y)^{-1} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.3}$$

These are often thought of in terms of simple mirror images.

Let us consider now the practical case where all three involutions can be applied. By that, we mean that a geometry is invariant under either reciprocation (P), or reflection through the x or y axes (R_x) and (R_y). The group representing all the combinations of these three operators is

$$G \equiv \{(1),\ (P),\ (R_x),\ (R_y),\ (P)(R_x),\ (P)(R_y),\ (R_x)(R_y),\ (P)(R_x)(R_y)\} \tag{5.4}$$

Since all of the above combinations of operators commute, this is a commutative (or Abelian) group. Certain of the above combinations can be expressed in other ways. For example a reflection through the x and y axes, $(R_x)\,(R_y)$, is equivalent to either an inversion through the origin, (I), or a rotation though 180°, (C_2) [9]. Note also that the combination of $(P)\,(R_y)$ is the analytic version of the reciprocation operator, i.e., the operation that maps ζ to a^2/ζ, without the conjugation.

A structure that is invariant under the three transformations described above has a potential function with certain special properties. In particular

$$\begin{aligned} w(\zeta^*) &= \pm w^*(\zeta) + w' && \text{Reflection through } x \text{ axis} \\ w(-\zeta^*) &= \pm w^*(\zeta) + w'' && \text{Reflection through } y \text{ axis} \\ w(a^2/\zeta^*) &= w^*(\zeta) + w''' && \text{Reciprocation} \end{aligned} \tag{5.5}$$

where w', w'', and w''' are arbitrary complex constants. The sign ambiguity in the two reflection expressions must be resolved by whether or not the transformation is symmetric or antisymmetric [9] (electric field parallel or perpendicular to the plane of symmetry). One can add a constant to the potentials because it is only the fields that must be preserved, and the fields depend only on the derivatives of the potentials.

If a structure has a potential function which satisfies the above two relations for reflection symmetry, then the fields radiating from that aperture can be evaluated from the potential in just a single quadrant, multiplying the result by four. If the structure's potential functions satisfies the above condition for reciprocation symmetry, then the radiated fields can be evaluated from just the potential inside the circle $|\zeta| \leq a$. In this case, the potential function outside the circle makes no net contribution, so an evaluation of the potential function contained within the circle is the total solution. If all three of the above symmetry conditions are satisfied, then the radiated fields can be evaluated with just the portion of the potential contained within a single quadrant, inside the circle $|\zeta| \leq a$, again multiplying the result by four.

VI. Conclusions

We have described here a technique for calculating the field radiated from a self-reciprocal aperture. Such apertures are good approximations to a variety of antennas used to radiated transients, including Impulse Radiating Antennas and long TEM horns, or TEM horns with lenses. The technique developed here demonstrates that if the aperture has reciprocal symmetry, then one only needs to know the field at the center of the aperture in order to calculate the far field on boresight. This is a considerable simplification over previous methods, which required either a surface integral or a contour integral.

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