Sensor and Simulation Notes

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Two-Dimensional Inhomogeneous Dielectric Lenses for E-Plane Bends of TEM Waves Guided Between Perfectly Conducting Sheets

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Abstract

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1. Introduction

In guiding fast-rising TEM waves between perfectly conducting parallel plates (say of separation d, but sufficiently wide to be approximated as infinitely wide), one has the problem of how to bend such a waveguide while preserving the TEM character of a wave with arbitrarily small risetime t_r, with

$$
\begin{aligned}
& t_r << \frac{d}{v} \\
& v = [\mu\ \varepsilon]^{-\frac{1}{2}} \equiv \text{speed of propagation} \\
& \quad \mu \equiv \text{guide permeability} \\
& \quad \varepsilon \equiv \text{guide permittivity}
\end{aligned}
\tag{1.1}
$$

where μ and ε are assumed real and frequency independent with

$$
\begin{aligned}
& \mu \geq \mu_0 \\
& \varepsilon \geq \varepsilon_0
\end{aligned}
\tag{1.2}
$$

In this paper, E-plane bends in such a waveguide are considered with appropriate dielectric lenses in the bend region. Previous papers have considered a dielectric wedge [1] and arrays of parallel conducting sheets [2] for accomplishing this purpose.

The present paper extends the differential-geometry scaling method for such two-dimensional lenses [5 (Section 2.5 and Appendix F)] to the present problem of such E-plane bends. A general form related to conformal transformations is developed and a simple example is given.

2. Two-dimensional lenses with constant permeability for E waves

Summarizing from [5] we have the (u_1, u_2, u_3) orthogonal curvilinear coordinate system related to the usual Cartesian (x, y, z) system via the scale factors

$$h_n^2 = \left(\frac{2x}{2u_n}\right)^2 = \left(\frac{2y}{2u_n}\right)^2 = \left(\frac{2z}{2u_n}\right)^2 \quad , \; n = 1,2,3 \tag{2.1}$$

giving the line element

$$(d\ell) = \sum_{n=1}^{3} h_n^2 \; (du_n)^2 \tag{2.2}$$

For our present purposes, let us take

$$u_3 \equiv z \;, \quad h_3 = 1 \tag{2.3}$$

reducing the problem to a two-dimensional one in the (x, y) and (u_1, u_2) planes. Furthermore let us take u_1 as the propagation direction with formal fields with the only components as

$$\begin{gathered} E_2' = E_{2_0}' \; f\left(t - \frac{u_1}{c'}\right) , \quad H_3' = H_{3_0}' \; f\left(t - \frac{u_1}{c'}\right) \\ \frac{E_2'}{H_3'} = \left[\frac{\mu_3'}{\varepsilon_2'}\right]^{\frac{1}{2}} \quad , \quad c' = \left[\mu_3' \varepsilon_2'\right]^{-\frac{1}{2}} \\ \mu_3' \, , \, \varepsilon_2' \equiv \text{real and positive constant formal permeability and permittivity} \end{gathered} \tag{2.4}$$

This describes a simple uniform plane wave with the magnetic field in the z direction. Note that it is a uniform plane TEM wave in general only with respect to the u_n coordinates. In the Cartesian coordinates the wave is still TEM but in general neither plane nor uniform. The propagation direction varies as the wave progresses along the u_1 coordinate. Since the magnetic field is fixed along the z direction, but the electric field can turn in the x, y plane the wave as referenced to the Cartesian coordinates, thereby describing in general an E (or TM) wave. It is this type of wave we can use for an E-plane bend, surfaces of constant u_2 being chosen for the perfectly conducting guiding cylindrical sheets.

The formal fields are related to usual fields via

$$\vec{E}' = (\alpha_{n,m}) \cdot \vec{E} \ , \quad \vec{H}' = (\alpha_{n,m}) \cdot \vec{H}$$
$$\vec{D}' = (\beta_{n,m}) \cdot \vec{D} \ , \quad \vec{B}' = (\beta_{n,m}) \cdot \vec{B}$$
$$(\alpha_{n,m}) = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.5}$$
$$(\beta_{n,m}) = \begin{pmatrix} h_2 h_3 & 0 & 0 \\ 0 & h_3 h_1 & 0 \\ 0 & 0 & h_1 h_2 \end{pmatrix} = \begin{pmatrix} h_2 & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 0 & h_1 h_2 \end{pmatrix}$$

The formal constitutive parameter are related to the usual ones via

$$\overleftrightarrow{\varepsilon}' = (\beta_{n,m}) \cdot \overleftrightarrow{\varepsilon} \cdot (\alpha_{n,m})^{-1} \ , \quad \overleftrightarrow{\mu}' = (\beta_{n,m}) \cdot \overleftrightarrow{\mu} \cdot (\alpha_{n,m})^{-1} \tag{2.6}$$

and if the $\overleftrightarrow{\varepsilon}$ and $\overleftrightarrow{\mu}$ are diagonal we have

$$\overleftrightarrow{\varepsilon}' = (\gamma_{n,m}) \cdot \overleftrightarrow{\varepsilon} \ , \quad \overleftrightarrow{\mu}' = (\gamma_{n,m}) \cdot \overleftrightarrow{\mu}$$
$$(\gamma_{n,m}) = (\beta_{n,m}) \cdot (\alpha_{n,m})^{-1} = (\alpha_{n,m})^{-1} \cdot (\beta_{n,m})$$
$$= \begin{pmatrix} \frac{h_2 h_3}{h_1} & 0 & 0 \\ 0 & \frac{h_3 h_1}{h_2} & 0 \\ 0 & 0 & \frac{h_1 h_2}{h_3} \end{pmatrix} = \begin{pmatrix} \frac{h_2}{h_1} & 0 & 0 \\ 0 & \frac{h_1}{h_2} & 0 \\ 0 & 0 & h_1 h_2 \end{pmatrix} \tag{2.7}$$

Note the simplifications that occur in the two-dimensional problem due to the removal of h_3 from (2.3). Furthermore, with E_2' and H_3' as the only non-zero formal-field components, then ε_2' and μ_3' are the only relevant components of the formal permittivity and permeability dyadics (assumed diagonal in the (u_1, u_2, u_3) coordinate system). This allows us to take

$$\overleftrightarrow{\varepsilon} = \varepsilon \overleftrightarrow{1} \ , \quad \overleftrightarrow{\mu} = \mu \overleftrightarrow{1} \ , \quad \overleftrightarrow{1} \equiv \vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y + \vec{1}_z \vec{1}_z \tag{2.8}$$

so that the medium is isotropic. The electric and magnetic fields then take the form

$$\vec{E} = \frac{1}{h_2} \vec{E}' = \frac{1}{h_2} E_2' \vec{1}_2$$
$$\vec{H} = \vec{H}' = H_3' \vec{1}_3 = H_3' \vec{1}_z \tag{2.9}$$

where the $\vec{1}_n$ are unit vectors pointing in the direction of increasing u_n. In component form this is

$$E_2 = \frac{1}{h_2} E_2' \ , \ H_z = H_3' \tag{2.10}$$

where E_2 is an actual electric-field component (V/m), but oriented in the $\vec{1}_2$ direction.

In [5] such an E-wave is considered for the case of constant ε but variable μ. Here we consider the converse, i.e.

$$\mu = \text{constant} \geq \mu_o \tag{2.11}$$

and μ may be set to μ_o for many practical applications. For convenience we set

$$\mu \equiv \mu_3' \ . \tag{2.12}$$

implying from (2.7)

$$h_1 h_2 = 1 \tag{2.13}$$

leaving only one scale factor to determine. For the permittivity this implies

$$\varepsilon = \frac{h_2}{h_1} \varepsilon_2' = h_1^{-2} \varepsilon_2' = h_2^2 \varepsilon_2' \tag{2.14}$$

Constraining for convenience

$$\varepsilon_{\text{min}} = \varepsilon_2' \tag{2.15}$$

where ε_{min} is the minimum value of ε in the domain of interest, giving

$$\begin{gathered} \varepsilon = \frac{h_1}{h_2} \varepsilon_{\text{min}} = h_1 \ \varepsilon_{\text{min}} = h_2^2 \ \varepsilon_{\text{min}} \\ h_1 \leq 1 \ , \quad h_2 \geq 1 \end{gathered} \tag{2.16}$$

in the lens domain.

In the domain of concern (the lens domain) there is some minimum acceptable value of ε as

$$\varepsilon \geq \varepsilon_{\text{min}} \geq \varepsilon_o \tag{2.17}$$

which is related to physical realizability. The lens domain is restricted to values of u_2 as

$$u_{2_{\text{min}}} < u_2 < u_{2_{\text{max}}} \tag{2.18}$$

where the minimum and maximum values correspond to the perfectly conducting sheets guiding the E-wave. Further restriction may be based on whether some (u_1, u_2) values have negligible fields or the lens is truncated for various practical reasons. Note also that

$$c' = \left[\mu_3' \varepsilon_2'\right] = \left[\mu\, \varepsilon_{\text{min}}\right]^{-\frac{1}{2}} \leq c = \left[\mu_o \varepsilon_o\right]^{-\frac{1}{2}} \tag{2.19}$$

as required. The local wave speed is

$$c_w = \left[\mu\, \varepsilon\right]^{-\frac{1}{2}} = h_1 \left[\mu\, \varepsilon_{\text{min}}\right]^{-\frac{1}{2}} \tag{2.20}$$

Defining

$$c_{w_{\text{max}}} = \left[\mu\, \varepsilon_{\text{min}}\right]^{-\frac{1}{2}} \tag{2.21}$$

then we have

$$c_w \leq c_{w_{\text{max}}} \leq c \tag{2.22}$$

in the lens domain. Similarly for the local wave impedance we have

$$Z_w = \frac{E_2}{H_2} = h_2^{-1} \frac{E_2'}{H_3'} = h_2^{-1}\left[\frac{\mu}{\varepsilon_{\text{min}}}\right]^{\frac{1}{2}} = h_1\left[\frac{\mu}{\varepsilon_{\text{min}}}\right]^{\frac{1}{2}} \leq Z_{w_{\text{max}}} \equiv \left[\frac{\mu}{\varepsilon_{\text{min}}}\right]^{\frac{1}{2}} \tag{2.23}$$

3. Relation to conformal transformation

Now consider the question of how to construct orthogonal curvilinear coordinates (u_1, u_2) subject to the constraints in the previous section, specifically from (2.13) as

$$h_1 \, h_2 = 1 \tag{3.1}$$

In the two dimensional cases in [5], the constraint is that h_1 and h_2 are the same which is a conformal transformation, i.e., u_1 and u_2 are stretched/contracted the same at each location giving what is sometime described as curvilinear squares. Here our constraint is different but we can still use conformal transformation as an intermediate step to construct our (u_1, u_2) system, thereby giving a plethora of possible solutions.

Consider the complex coordinates

$$\begin{aligned} \zeta &\equiv x + jy \\ v &\equiv v_1 + jv_2 \qquad (v_1 \text{ and } v_2 \text{ real}) \end{aligned} \tag{3.2}$$

where $v(\zeta)$ is the analytic function describing a conformal transformation between the (x, y) plane and the (v_1, v_2) plane. The line element is

$$\begin{aligned} |d\zeta|^2 &= (dx)^2 + (dy)^2 = h_{v_1}^2 (dv_1)^2 + h_{v_2}^2 (dv_2)^2 \\ &= h_v^2\left[(dv_1)^2 + (dv_2)^2\right] = h_v^2 |dv|^2 \\ h_v &= \left|\frac{d\zeta}{dv}\right| = \left|\frac{dv}{d\zeta}\right|^{-1} \\ h_v^2 &= h_{v_n}^2 = \left(\frac{\partial x}{\partial v_n}\right)^2 + \left(\frac{\partial y}{\partial v_n}\right)^2 \end{aligned} \tag{3.3}$$

Functions $v(\zeta)$ are the convenient way to construct (v_1, v_2) coordinates and many examples exist in the literature. (See, e.g., [3, 4]).

Now the scale factors for the v_n coordinates do not satisfy (3.1) so let us construct a special form of the u_n as $u_n(v_n)$, i.e., u_1 is a function of only v_1 and u_2 is only a function of v_2. With lines of constant v_1 and v_2 forming curvilinear squares (equal decrements for both v_1 and v_2), then lines of constant u_1 and u_2 form curvilinear rectangles. The scale factors for the u_n coordinates are now

$$h_n^2 = \left(\frac{\partial x}{\partial u_n}\right)^2 + \left(\frac{2y}{2u_n}\right)^2 = \left(\frac{d\mathrm{v}_n}{du_n}\right)^2 \left[\left(\frac{\partial x}{\partial \mathrm{v}_n}\right) + \left(\frac{\partial y}{\partial \mathrm{v}_n}\right)\right] = \left(\frac{d\mathrm{v}_n}{du_n}\right)^2 h_\mathrm{v}^2 \tag{3.4}$$

which implies

$$h_1 h_2 = \left|\frac{d\mathrm{v}_1}{du_1}\right| \left|\frac{d\mathrm{v}_2}{du_2}\right| h_\mathrm{v}^2 = 1 \tag{3.5}$$

Given $\mathrm{v}(\zeta)$ and one of the u_n coordinates, say $u_1(\mathrm{v}_1)$ (the propagation direction), then (3.5) can be used to solve for $u_2(\mathrm{v}_2)$ to within an integration constant.

4. Concentric perfectly conducting circular cylinders guiding azimuthally propagating wave

Consider a simple example of an E-plane bend as indicated in fig. 4.1. For this we have the usual complex coordinates

$$\zeta = x + jy = \Psi e^{j\phi} \tag{4.1}$$

The lens domain is defined by the two perfectly conducting sheets on Ψ_1 and Ψ_2 with

$$\Psi_1 \leq \Psi \leq \Psi_2 \quad , \quad \Psi_1 < \Psi_2 \tag{4.2}$$

and some extent of consideration in terms of ϕ as

$$\phi_1 \leq \phi \leq \phi_2 \quad , \quad \phi_2 < \phi_1 \tag{4.3}$$

the difference $\phi_2 - \phi_1$ depending on how much of a bend angle one wishes.

Define the v_n coordinates by

$$\begin{aligned} v &= -j\ell n(\zeta) = \phi - j\ell n(\Psi) = v_1 + jv_2 \\ v_1 &= \phi \quad , \quad v_2 = -\ell n(\Psi) = \ell n\left(\Psi^{-1}\right) \\ \frac{dv}{d\zeta} &= -j\zeta^{-1} \\ h_v &= \left|\frac{dv}{d\zeta}\right|^{-1} = |\zeta| = \Psi\ e^{-v_2} \end{aligned} \tag{4.4}$$

This makes our wave propagate in the direction of increasing ϕ.

Converting to u_n coordinates, choose first

$$u_1(v_1) = \text{constant times } v_1 = \text{constant times } \phi \tag{4.5}$$

Letting this be specified by the longest path length for constant Ψ in the lens region we have

$$u_1 = \Psi_2\,\phi = \Psi_2\, v_1 \tag{4.6}$$

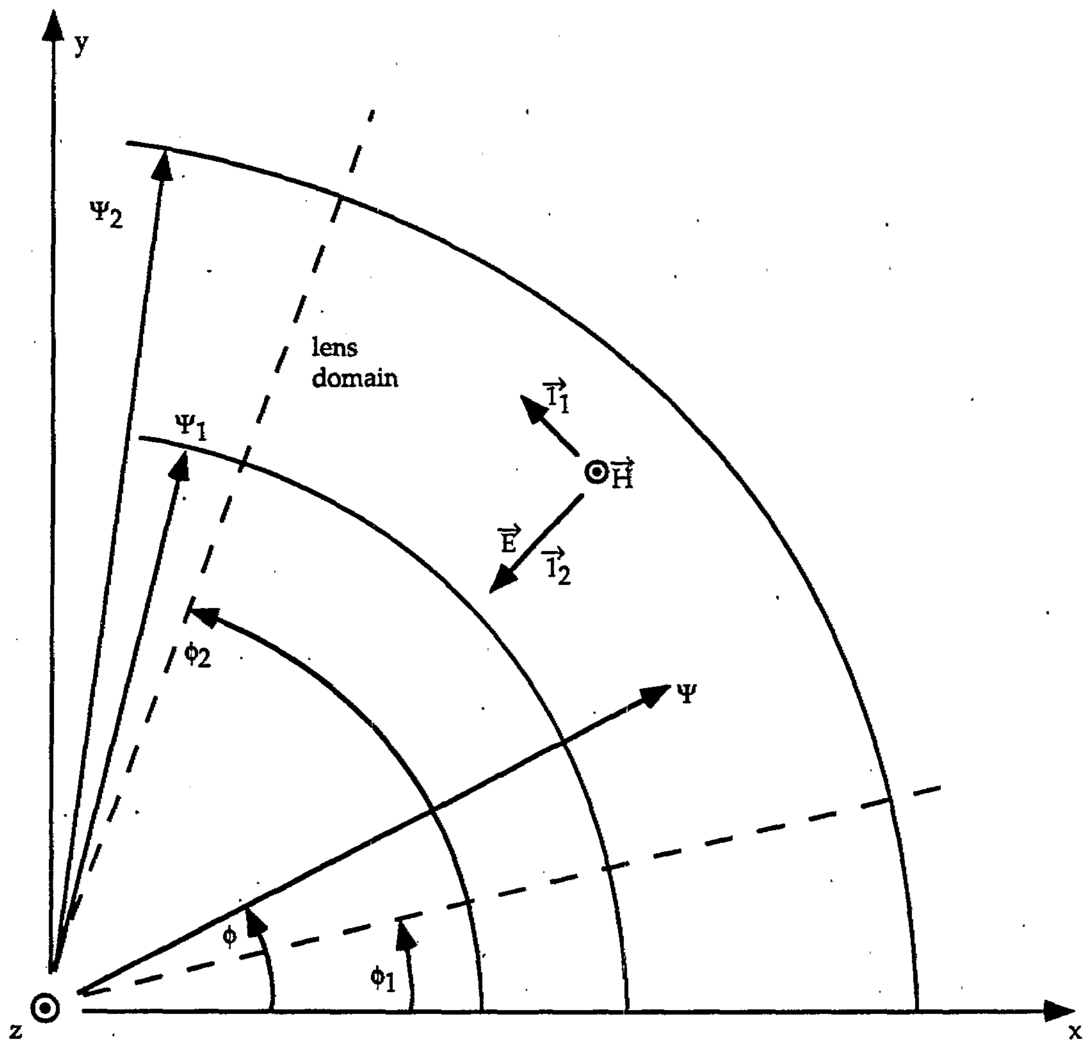

Fig. 4.1. Circular Cylindrical Lens for E-Plane Bend

On $\Psi = \Psi_2$ this is the actual path length in meters. This is where ε is its smallest, i.e., ε_{min}. Other paths between two different values of ϕ are shorter, and the speed of propagation is correspondingly less on such paths. Note now that the simple form in (4.6) gives

$$\begin{aligned} \frac{du_1}{dv_1} &= \frac{du_1}{d\phi} = \Psi_2 \\ h_1 &= \left|\frac{dv_1}{du_1}\right| h_v = \frac{\Psi}{\Psi_2} = \frac{e^{-v_2}}{\Psi_2} \\ &= h_2^{-1} \end{aligned} \tag{4.7}$$

In the lens domain $\Psi \leq \Psi_2$ making (2.16) satisfied.

Now we find the u_2 coordinate from (3.5) as

$$\begin{aligned} \left|\frac{du_2}{dv_2}\right| &= \left|\frac{dv_1}{du_1}\right| h_v^2 = \frac{\Psi^2}{\Psi_2} = \frac{e^{-2v_2}}{\Psi_2} \\ u_2 &= \pm\frac{e^{-2v_2}}{2\Psi_2} + \text{constant} \\ &= \pm\frac{\Psi^2}{2\Psi_2^2} + \text{constant} \end{aligned} \tag{4.8}$$

Choose

$$u_2 = -\frac{\Psi^2}{2\Psi_2} = -\frac{e^{-2v_2}}{2\Psi^2} \tag{4.9}$$

so that increasing u_2 corresponds to decreasing Ψ, making (u_1, u_2, z) a right-handed coordinate system. The integration constant is taken as zero for convenience.

The required permittivity is now

$$\varepsilon = h_2^2\,\varepsilon_{min} = \left(\frac{\Psi_2}{\Psi}\right)^2 \varepsilon_{min} \tag{4.10}$$

so that $\varepsilon \geq \varepsilon_{min}$ in the lens domain. The local wave speed is

$$c_w = [\mu\,\varepsilon]^{-\frac{1}{2}} = h_1[\mu\,\varepsilon_{\min}]^{-\frac{1}{2}} = \frac{\Psi}{\Psi_2}\,[\mu\,\varepsilon_{\min}]^{-\frac{1}{2}} \\ = \frac{\Psi}{\Psi_2}\,c_{w_{\max}} \tag{4.11}$$

The local wave impedance is

$$Z_w = \frac{E_2}{H_2} = -\frac{E_\Psi}{H_z} = h_1\left[\frac{\mu}{\varepsilon_{\min}}\right]^{\frac{1}{2}} = \frac{\Psi}{\Psi_2}\,Z_{w_{\max}} \\ Z_{w_{\max}} = \left[\frac{\mu}{\varepsilon_{\min}}\right]^{\frac{1}{2}} \tag{4.12}$$

Since H_z is independent of Ψ this also shows that the electric field is maximum at $\Psi = \Psi_2$, the outer bend radius.

5. Concluding Remarks

With this extension of the differential-geometry scaling method for lens design, one can now consider many possible designs for dielectric E-plane bends in parallel-plate waveguides (for large ratios of plate width to spacing). The example in Section 4 is a simple example to illustrate this method of lens design. There is still the problem of how to join such a lens to parallel plate waveguides without significant discontinuities (e.g., at ϕ_1, and ϕ_2 in fig. 4.1). One may accept some scattering at such places if it is not too large. In the example, this would require that Ψ_1/Ψ_2 be not too much less than 1. Other conformal transformations that blend the bend region more smoothly into the parallel-plate regions (by a continuous change of the plate curvatures to zero curvature on each end) may also be developed. Noting that conformal transformations give solutions to the two-dimensional Laplace equation, one can construct the v_2 function with boundary conditions $v_2^{(1)}$ and $v_2^{(2)}$ on the perfectly conducting boundaries and numerically solve the Laplace equation to find both v_2 and v_1 for given boundary shapes and then u_1, u_2, and the corresponding ε distribution. One still needs to restrict ε_{min} in the lens domain for physically realizability.

There is, of course, the dual problem with the electric field in the z direction and the magnetic field in the $-u_2$ direction. The procedure in Section 2 then applies to the case of constant ε, but variable μ. In this case, perfectly conducting boundaries are planes of constant z.

References

1. C. E. Baum, Wedge Dielectric Lenses for TEM Waves Between Parallel Plates, Sensor and Simulation Note 332, September 1991.

2. C. E. Baum, Arrays of Parallel Conducting Sheets for Two-Dimensional E-Plane Bending Lenses, Sensor and Simulation Note 341, April 1992.

3. H. Kober, *Dictionary of Conformal Representations*, Dover, 1957.

4. R. V. Churchill, *Complex Variables and Applications*, McGraw Hill, 1960.

5. C. E. Baum and A. P. Stone, *Transient Lens Synthesis: Differential Geometry in Electromagnetic Theory*, Hemisphere Publishing Corp. (Taylor & Francis), 1991.