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Dielectric Jackets as Lenses and Application to Generalized Coaxes and Bends in Coaxial Cables

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Abstract

This paper considers the general properties of a jacket consisting of two closely spaced, but curved, conducting plates separated by a medium of uniform isotropic permeability and nonuniform isotropic permittivity for propagating dispersionless TEM waves. A simple form of this is a body of revolution with generalized axial propagation (generalized coax) which can be synthesized with a uniform permittivity. One can also use a nonuniform permittivity for a bend in a coaxial cable.

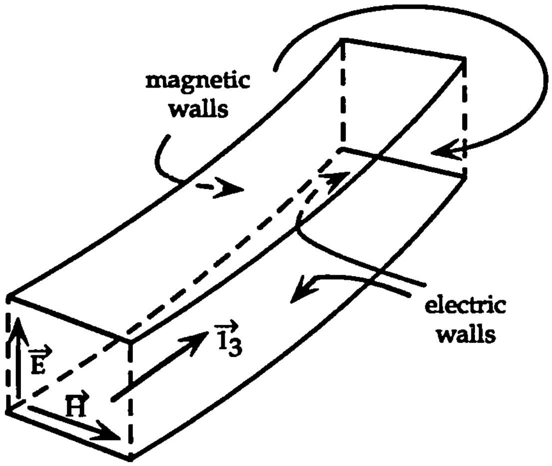
1. Introduction

In the design of lenses for transporting transverse electromagnetic (TEM) waves in desirable ways, there are techniques from differential geometry and the concepts of transit-time and differential-impedance matching [5]. In its most general form such lenses are three-dimensional structures in which wavelengths are allowed to be small compared to characteristic dimensions in all three coordinate directions. In such cases, one can consider the full Maxwell equations for appropriate solutions.

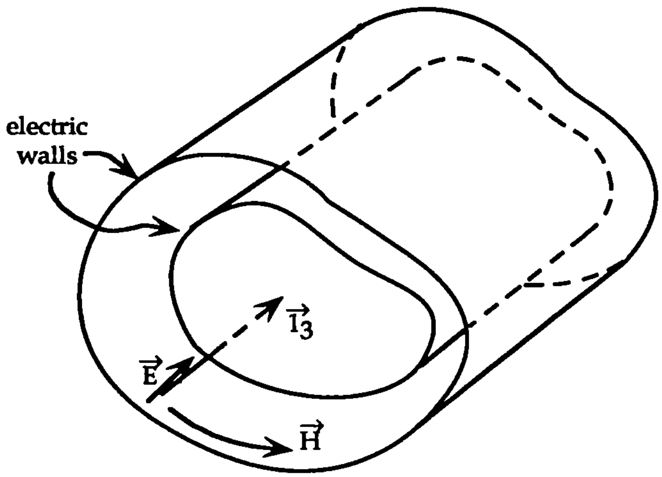
Under appropriate conditions, the propagation of electromagnetic waves as governed by the Maxwell equations can be reduced to propagation in less than three dimensions. A common case is that of a transmission line in which the two cross-section dimensions are typically assumed to be small compared to wavelength. If the transmission line is constructed of straight perfect conductors in a uniform isotropic medium the dominant mode of propagation is a TEM mode, and one can even go to frequencies high enough that the cross section is not electrically small if one is careful not to introduce higher order (E and H) modes.

In [5] the concept of a duct is used to describe the propagation of an elementary part of a TEM wave through a lens. As illustrated in fig. 1.1A a duct consists of a waveguide bounded by two each electric and magnetic walls (boundaries) in the shape of a curvilinear rectangle (a rectangle in the limit of small cross-section dimensions). In a general (u_1, u_2, u_3) orthogonal curvilinear coordinate system the electric field $\vec{E}$ is in the $\vec{1}_1$ direction, the magnetic field $\vec{H}$ is in the $\vec{1}_2$ direction, and propagation is in the $\vec{1}_3$ direction. All of the directions can change (smoothly) as one moves along the u_3 coordinate (in general curved). The small changes in the cross-section coordinates can be labelled as Δu_1 and Δu_2, these being taken as constant along a uniform duct. Multiplying the electric field by the spacing in the $\vec{1}_1$ direction between electric conductors gives a voltage; multiplying the magnetic field by the spacing in the $\vec{1}_2$ direction gives a current. Together with the u_3 coordinate the voltage and current satisfy the usual telegrapher (transmission-line) equations (one-dimensional).

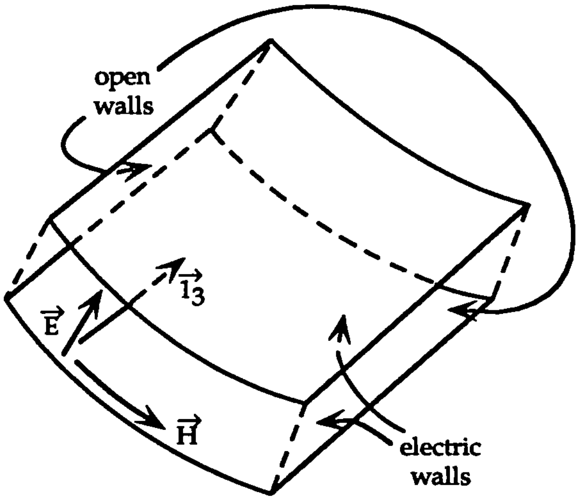
A jacket is a structure as illustrated in fig. 1.1B. Now the u_2 coordinate extends over many wavelengths, at least at the higher frequencies of interest, but Δu_1 (the extent of the u_1 coordinate) is still electrically small. Constraining the wave to propagate in the u_3 direction, one can think of a jacket as a set of ducts combined together such that two adjacent ducts have a common magnetic wall. Provided that the u_3 coordinate is continuous in going between adjacent ducts, then waves with identical voltages on all ducts having the same properties with respect to u_3 can be considered as a single wave on the entire jacket and the magnetic walls can be removed with no effect. As illustrated in fig. 1.1B the u_2



A. Duct



B. Jacket



C. Open jacket

Fig. 1.1. Ducts and Jackets

coordinate closes on itself such that (as in a coaxial cable) lines of $\vec{H}$ close on themselves with no magnetic walls (divergenceless $\vec{B}$). As such, one can speak of inner and outer electric walls. A jacket is fundamentally a two-dimensional electromagnetic structure with propagation possible in u_2 and u_3 directions, except that by design, excitation, and termination propagation is made to occur in the u_3 direction with appropriate parameters independent of u_2. Thus a jacket, as a two-dimensional kind of electromagnetic structure, is intermediate between a one-dimensional transmission line (or duct) and a three-dimensional electromagnetic structure (e.g., lens). (More generally, one can also consider propagation in both u_2 and u_3 coordinates as in [5 (Appendix B)].)

One can also consider an open jacket as in fig. 1.1C, in which the u_2 coordinate does not close on itself. (Perhaps one could call such a structure a "sandwich".) With no magnetic walls at the two truncations of the u_2 coordinate, there is leakage of the fields through these open walls. As discussed in [1], as long as the spacing between the electric walls is small compared to their "width" in the u_2 direction, one can consider this as an approximation in which the error due to the fringe fields is sufficiently small for some applications. Another way to think of such an open jacket is as (in general) deformed "parallel plates" with perhaps an inhomogeneous medium between the plates.

The dual of a jacket is a slice [5] in which the ducts are combined so as to remove common electric boundaries. However, this has less application due to the important magnetic boundaries. For present purposes, our emphasis is also on media of constant permeability μ (typically μ_0), but of variable (but isotropic) permittivity ε ($\geq \varepsilon_0$).

2. Two-Dimensional TEM Waves and Waves in Jackets

One way to look at a jacket is as a two-dimensional space [5 (Appendix B)]. In terms of a general (u_1, u_2, u_3) coordinate system the two dimensions of concern are u_2 and u_3 where u_1 is the coordinate normal to the two-dimensional surface characterizing the jacket, the extent of the u_1 coordinate variation (i.e., Δu_1) being small by hypothesis. The assumed form of our TEM wave is uniform in terms of the formal fields (primed quantities in terms of which the u_n coordinates are like Cartesian coordinates). The assumed wave is then like

$$\begin{aligned} \vec{E} &= E_1 \vec{1}_1 = \frac{E_0'}{h_1} f\left(t - \frac{u_3}{c'}\right) \\ \vec{H} &= H_2 \vec{1}_2 = \frac{E_0'}{h_1 Z_0'} f\left(t - \frac{u_3}{c'}\right) \\ c' &= [\mu' \varepsilon']^{-\frac{1}{2}} \quad , \quad Z_0' = \left[\frac{\mu'}{\varepsilon'}\right]^{\frac{1}{2}} \end{aligned} \tag{2.1}$$

where μ' (formal permeability) and ε' (formal permittivity) are positive real constants (frequency independent).

For diagonal $\overleftrightarrow{\mu}$ (permeability) and $\overleftrightarrow{\varepsilon}$ (permitivity), these are related to the formal parameters via

$$\begin{aligned} \overleftrightarrow{\mu}' &= (\gamma_{n,m}) \cdot \overleftrightarrow{\mu} \quad , \quad \overleftrightarrow{\varepsilon}' = (\gamma_{n,m}) \cdot \overleftrightarrow{\varepsilon} \\ (\gamma_{n,m}) &= \begin{pmatrix} \frac{h_2 h_3}{h_1} & 0 & 0 \\ 0 & \frac{h_3 h_1}{h_2} & 0 \\ 0 & 0 & \frac{h_1 h_2}{h_3} \end{pmatrix} \end{aligned} \tag{2.2}$$

Noting that the electric field has only a u_1 component and the magnetic field has only a u_2 component we have the scalar relations

$$\varepsilon' = \frac{h_2 h_3}{h_1} \varepsilon \quad , \quad \mu' = \frac{h_3 h_1}{h_2} \mu \tag{2.3}$$

Constraining for convenience that μ be uniform let us choose giving

$$\frac{h_3 h_1}{h_2} = 1$$
$$\varepsilon = \frac{h_1}{h_2 h_3}\,\varepsilon' = h_3^{-2}\,\varepsilon' \quad , \quad \varepsilon' \geq \varepsilon_o \tag{2.5}$$

The local wave speed and wave impedance are

$$c_w = [\mu\ \varepsilon]^{-\frac{1}{2}} = c'\left[\frac{\varepsilon'}{\varepsilon}\right]^{\frac{1}{2}} = c'\,h_3$$
$$Z_w = \left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}} = Z_c'\left[\frac{\varepsilon'}{\varepsilon}\right]^{\frac{1}{2}} = Z_o'\,h_3 \tag{2.6}$$

If we restrict (for causality)

$$\varepsilon \geq \varepsilon_{\text{min}} \geq \varepsilon_0 \tag{2.7}$$

we can set

$$\varepsilon_{\text{min}} \equiv \varepsilon' \tag{2.8}$$

which in turn testricts

$$h_3 = \left[\frac{\varepsilon_{\text{min}}}{\varepsilon}\right]^{\frac{1}{2}} \leq 1 \tag{2.9}$$

The line element is

$$(d\ell)^2 = \sum_{n=1}^{3} h_n^2\,(du_n)^2$$
$$(h_n)^2 = \left(\frac{\partial x}{\partial u_n}\right)^2 + \left(\frac{\partial y}{\partial u_n}\right)^2 + \left(\frac{\partial z}{\partial u_n}\right) \tag{2.10}$$
$$h_n \equiv \text{scale factors}$$

In the $\vec{1}_1$ direction, we have by typothesis a small separation between the electric walls, say designated by $u_1^{(1)}$ and $u_1^{(2)}$, given by

$$h_\Delta \equiv h_\Delta(u_2, u_3) \simeq h_1 \; \Delta u_1 = h_1\left[u_1^{(2)} - u_1^{(1)}\right] \tag{2.11}$$

so h_Δ is simply a constant times h_1 (exact in the limit of small Δu_1) which we can regard as something to be found in the solution instead of being specified a priori. Note that

$$h_1 = \frac{h_2}{h_3} \tag{2.12}$$

so the u_2 and u_3 coordinates give h_2 and h_3 which , in turn give h_1 and h_Δ. This seems to offer lots of flexibility.

3. Jacket as a Body of Revolution with Generalized Axial Propagation

A simple kind of jacket is a body of revolution (BOR) in the form of a generalized coax. The electric walls are independent of rotation about the axis of rotation symmetry (z axis). Propagation can be considered as in a generalized axial direction (combination of axial and radial).

A special case of this has constant

$$\begin{aligned} \mu &= \mu' \quad , \quad \varepsilon = \varepsilon' \\ h_3 &= 1 \quad , \quad h_1 = h_2 \end{aligned} \tag{3.1}$$

implying a constant propagation speed in the u_3 direction. One example of this is a coax of slowly varying radius as illustrated in fig. 3.1. In this case, we have

$$\begin{aligned} u_3 &\simeq z \\ Z_c &= \left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}} \frac{1}{2\pi} \ell n\left(\frac{\Psi_2(z)}{\Psi_1(z)}\right) \equiv \text{characteristic impedance} \\ \frac{\Psi_2(z)}{\Psi_1(z)} &\neq \text{function of } z \text{ (i.e., constant)} \end{aligned} \tag{3.2}$$

Another obvious example is the conical transmission line with closely spaced circular conical electric walls as illustrated in fig. 3.2. In the usual spherical (r, θ, ϕ) and cylindrical (Ψ, ϕ, z) coordinate systems we have

$$\begin{aligned} x &= \Psi\cos(\phi) \quad , \quad y = \Psi\sin(\phi) \\ z &= r\cos(\theta) \quad , \quad \Psi = \Psi\sin(\phi) \end{aligned} \tag{3.3}$$

With rotational (ϕ) symmetry we have

$$\begin{aligned} u_3 &= r \quad , \quad h_3 = 1 \\ u_2 &= \Psi_j\,\phi \quad , \quad h_2 = \frac{\Psi}{\Psi_j} = \frac{r\sin(\theta)}{r_j\sin(\theta_o)} \\ \left.\begin{aligned} du_3 &= dr \\ du_r &= \frac{\Psi}{\Psi_j}du_2 = \Psi d\phi \end{aligned}\right\} &\equiv \text{line elements} \end{aligned} \tag{3.4}$$

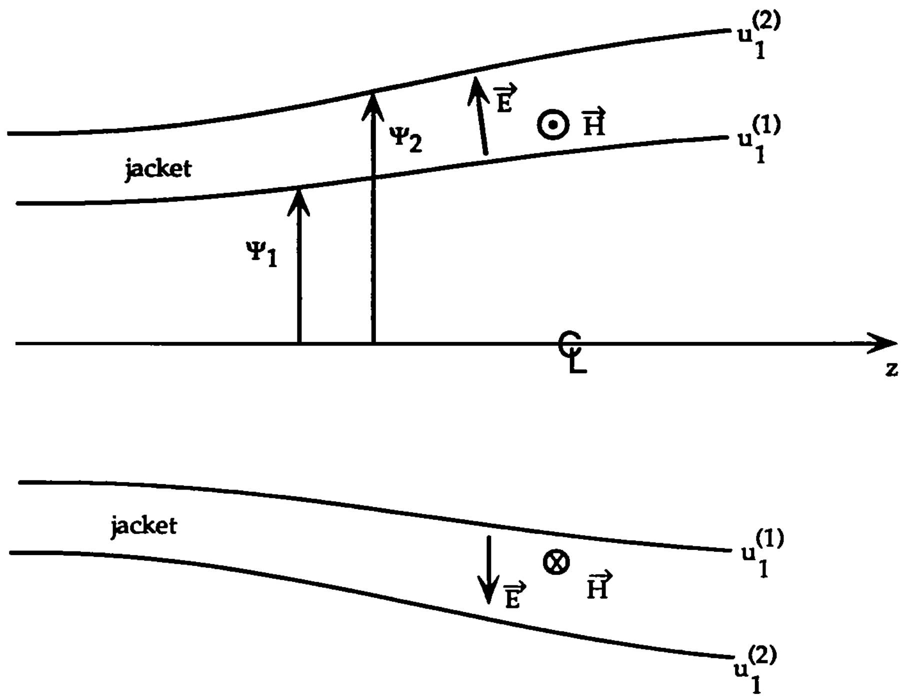


Fig. 3.1. Coax with Slowly Varying Radius

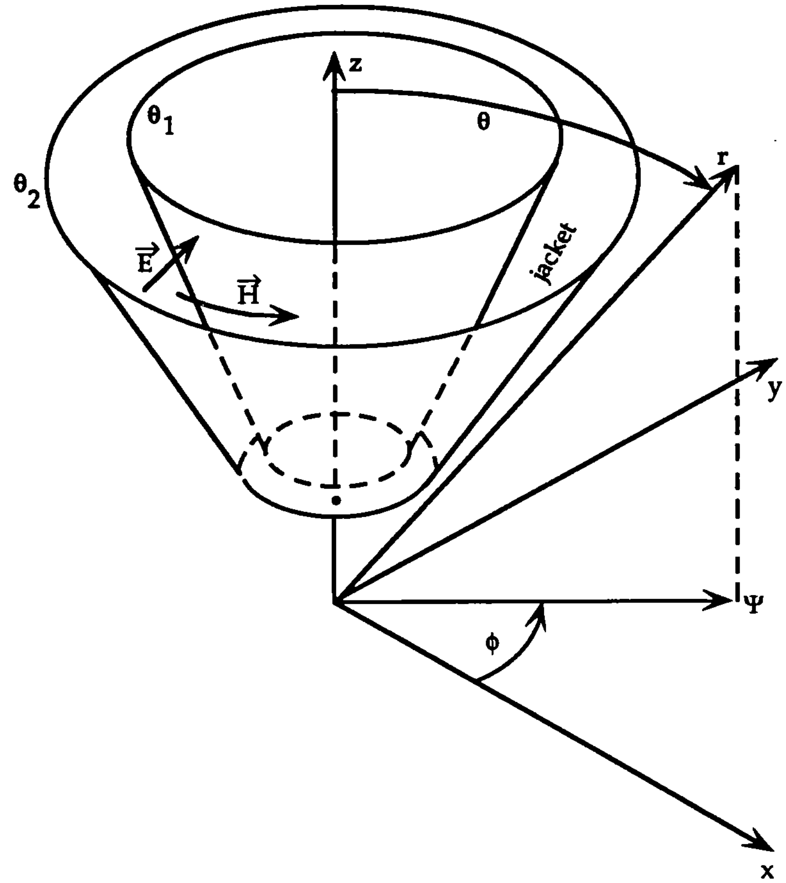


Fig. 3.2. Low-Impedance Circular Conical Transmission Line

where Ψ_j is some convenient normalizing cylindrical radius and θ_j is some mean value of θ in the jacket, i.e.

$$\theta_1 < \theta_j < \theta_2 \quad , \quad \theta_2 - \theta_1 \equiv \Delta\theta \tag{3.5}$$

with θ_1 and θ_2 as the θ values of the two circular conical electric walls. It is well known [4] that this structure admits an exact three-dimensional solution, which corresponds in this case to

$$h_1 = \frac{\Psi}{\Psi_j} = \frac{r\sin(\theta)}{r_j \sin(\theta_0)} \tag{3.6}$$

$$h_1 du_1 = r d\theta = \text{line element}$$

$$u_1 = r_j \sin(\theta_j)\, \ell n\left[\frac{\tan\left(\frac{\theta}{2}\right)}{\tan\left(\frac{\theta_j}{2}\right)}\right] = \Psi_j\, \ell n\left[\frac{\tan\left(\frac{\theta}{2}\right)}{\tan\left(\frac{\theta_j}{2}\right)}\right] \tag{3.7}$$

with the integration constant chosen for convenience. The characteristic impedance is just

$$Z_c = \left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}} \frac{1}{2\pi}\, \ell n\left[\frac{\tan\left(\frac{\theta_2}{2}\right)}{\tan\left(\frac{\theta_1}{2}\right)}\right] \tag{3.2}$$

For small Δu_1, this is also well approximated using the formulae in Section 2 as

$$h_\Delta = h_1\,\Delta u_1 \simeq \frac{\Psi}{\Psi_j}\Delta u_1 \simeq r\Delta\theta \simeq \frac{\Psi}{\sin(\theta_j)}\,\Delta\theta$$

$$Z_c = \left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}} \frac{\Delta u_1}{2\pi\Psi_j} \simeq \left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}} \frac{h_\Delta}{2\pi\Psi} \simeq \left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}} \frac{\Delta\theta}{2\pi\sin(\theta_j)} \tag{3.9}$$

The foregoing can be generalized by considering the jacket to have some mean cylindrical radius $\Psi(z)$. (One can also use a spherical form $r(\theta)$.) Then we have

$$u_3 = \int d\ell \equiv \text{arc length along curve } \Psi(z)$$

$$h_3 = 1 \tag{3.10}$$

$$u_2 = \Psi_j \phi \quad , \quad h_2 = \frac{\Psi(z)}{\Psi_j}$$

Note also that we can now also parameterize Ψ or r as a function of u_3. Regarding now u_1 as a generalized radial direction (orthogonal to u_3 as well as u_2), and noting that Δu_1 is small (by hypothesis), we have the spacing of the electric walls as $h_\Delta(z)$ in the $\vec{1}_1$ direction (orthogonal to $\vec{1}_3$) as

$$h_\Delta = h_1 \, \Delta u_1 \simeq \frac{\Psi(z)}{\Psi_j} \Delta u_1 \tag{3.11}$$

So the spacing in the generalized radial direction is proportional to the local cylindrical radius. The characteristic impedance is

$$Z_c \simeq \left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}} \frac{h_\Delta}{2\pi\Psi} = \left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}} \frac{\Delta u_1}{2\pi\Psi_j} \tag{3.12}$$

which is independent of u_3 as has been required.

This kind of BOR jacket is especially simple because of its invariance with respect to ϕ. If we think of each elementary $\Delta\phi$ as defining a duct, then all ducts are identical. This is not the most general case where there can be variation (say in ε) from duct to duct.

4. Waves in Ducts Combined to Give Waves in Jackets

Consider a *uniform* duct (analogous to a uniform transmission line) with characteristic impedance (with small Δu_2 as well as small Δu_1)

$$Z_d = \frac{h_1}{h_2} \frac{\Delta u_1}{\Delta u_2} \left[\frac{\mu}{\varepsilon} \right]^{\frac{1}{2}} \neq \text{function of } u_3 \tag{4.1}$$

Following the previous constant μ assumption (Section 2) we have

$$Z_d = \frac{\Delta u_1}{\Delta u_2} \left[\frac{\mu}{\varepsilon_{\min}} \right]^{\frac{1}{2}} \tag{4.2}$$

This constant impedance assures that there are no reflections along the duct.

Note now that ε is allowed in general to be a function of u_2 and u_3. In going from one duct to an adjacent one (a small change in u_2) we require that u_3 remain the same, which is merely another way of saying that (u_2, u_3) is an orthogonal coordinate system for the jacket. Constant u_3 contours represent wavefronts on the jacket. Combining the various ducts together and removing the magnetic walls then gives a jacket as discussed in Section 1. Its characteristic impedance is the same as in (4.2) with Δu_2 now reinterpreted as the total change in u_2 around the jacket. (All ducts are effectively connected in parallel.)

5. Nonuniform Ducts and Jackets

As an aside now let each duct be *nonuniform*, i.e., modify (4.1) and (4.2) as

$$Z_d(u_3) = \frac{h_1}{h_2}\,\frac{\Delta(u_3)}{\Delta u_2}\left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}} = \frac{\Delta(u_3)}{\Delta u_2}\left[\frac{\mu}{\varepsilon_{\min}}\right]^{\frac{1}{2}} \tag{5.1}$$

where Δu_1 is now replaced by $\Delta(u_3)$, a function of u_3, but $h_1\ \Delta(u_3)$ is still considered small. The spacing of the electric walls is then

$$h_\Delta(u_2, u_3) = h_1(u_2, u_3)\ \Delta(u_3) \tag{5.2}$$

Requiring $\Delta(u_3)$ not to be a function of u_2 makes all the ducts again the same, but now nonuniform. Each duct can be treated as a nonuniform transmission line which is dispersive, but can be treated analytically (even for pulses) for various tapers (functional forms of $\Delta(u_3)$ in this case).

All ducts having the same form as in (5.1), they can be combined to form a *nonuniform jacket* with $\Delta(u_3)$ as the *jacket taper*. Then such a jacket can be also solved as a nonuniform transmission line [2, 3]. Such jackets then can be used as pulse transmission-line transformers.

6. Bend in Coax

Now let us consider an example of a jacket in which ε is required to vary with spatial position. As illustrated in fig. 6.1, let us consider a bend in a coax. The bend region (jacket of interest) is contained within the region $\phi_1 \le \phi \le \phi_2$ and the coax is bent in a circular arc with a reference arc ("axis") on $(\Psi, z) = (\Psi_0, 0)$ in our first cylindrical (Ψ, ϕ, z) coordinate system. As seen in cross section (plane of constant ϕ) there is a second cylindrical (Ψ', ϕ') coordinate system used to describe the conductors and permittivity on the local cross section.

The straight portions of the coaxial cable are connected to the bend on the ϕ_1 and ϕ_2 planes with the axes on $(\Psi, z) = (\Psi_0, 0)$ on these two planes and with the axes perpendicular to these two planes. With Ψ_1' and Ψ_2' as the inner and outer radii, respectively, and ε as the permittivity in the jacket, the characteristic impedance is

$$\begin{aligned} Z_c &= \left[\frac{\mu}{\varepsilon_1}\right]^{\frac{1}{2}} \frac{1}{2\pi} \ell n\left(\frac{\Psi_2'}{\Psi_1'}\right) \\ &\simeq \left[\frac{\mu}{\varepsilon_1}\right]^{\frac{1}{2}} \frac{1}{2\pi} \frac{\Psi_2' - \Psi_1'}{\Psi_0'} \quad \text{for} \quad \frac{\Psi_2'}{\Psi_1'} \quad \text{near } 1 \end{aligned} \tag{6.1}$$

where Ψ_0' is a mean coax radius as

$$\Psi_0' \equiv [\Psi_2' \; \Psi_1']^{\frac{1}{2}} \simeq \frac{\Psi_2' + \Psi_1'}{2} \tag{6.2}$$

For later use we have

$$g_o \equiv \ell n\left(\frac{\Psi_2'}{\Psi_1'}\right) = 2\,\ell n\left(\frac{\Psi_2'}{\Psi_0'}\right) = 2\,\ell n\left(\frac{\Psi_0'}{\Psi_1'}\right) \tag{6.3}$$

Considering such a coax as a jacket strictly requires Ψ_2' near Ψ_1' so that

$$g_o \simeq \frac{\Psi_2' - \Psi_1'}{\Psi_0'} << 1 \tag{6.4}$$

and propagation in the Ψ' direction (local radial direction) can be neglected. Considering a duct in the straight coax as characterized by some small $\Delta\phi'$, we have

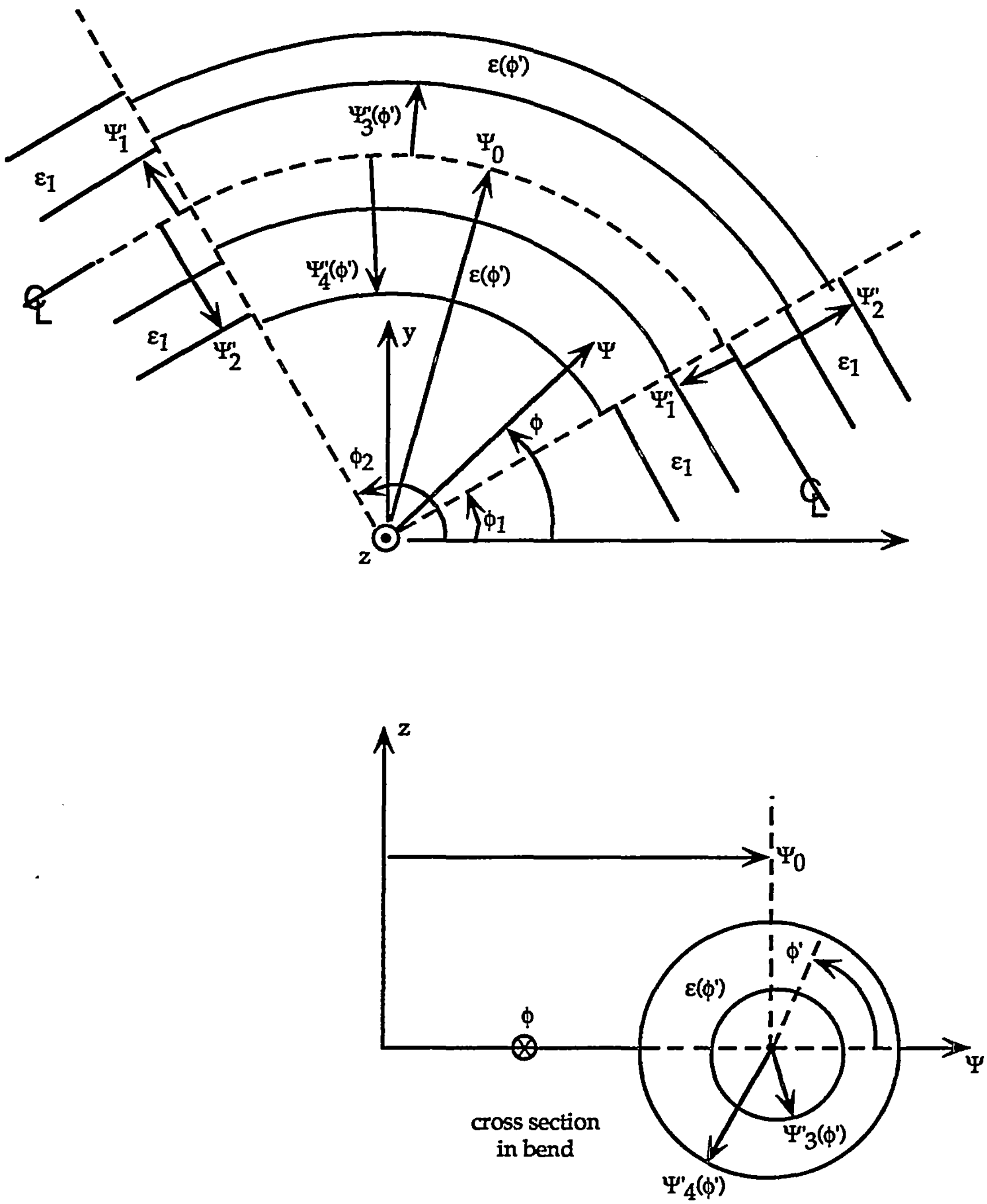


Fig. 6.1. Bend in Coax

$$Z_d = \left[\frac{\mu}{\varepsilon_1}\right]^{\frac{1}{2}} \frac{g_o}{\Delta\phi'} \tag{6.5}$$

as the characteristic impedance of the duct. This will be useful later for matching to the ducts in the bend.

Considering the bend region we have a distorted coax cross section with inner radius $\Psi_3'(\phi')$, outer radius $\Psi_4'(\phi')$, and permittivity $\varepsilon(\phi')$ in the jacket. The u_3 coordinate for the jacket is given by [1]

$$u_3 = \Psi_{\text{max}}\phi \quad , \quad h_3 = \frac{\Psi}{\Psi_{\text{max}}} \leq 1 \tag{6.6}$$

which implies

$$\begin{aligned} \varepsilon &= \left(\frac{\Psi_{\text{max}}}{\Psi}\right)^2 \varepsilon_{\text{min}} \\ c_w &= c'\left[\frac{\varepsilon_{\text{min}}}{\varepsilon}\right]^{\frac{1}{2}} = \frac{\Psi}{\Psi_{\text{max}}}\, c' \\ Z_w &= Z_o'\left[\frac{\varepsilon_{\text{min}}}{\varepsilon}\right]^{\frac{1}{2}} = \frac{\Psi}{\Psi_{\text{max}}}\, Z_o' \end{aligned} \tag{6.7}$$

This makes constant ϕ surfaces the wavefront surfaces (simultaneous arrival time).

Thinking of the bend region as a jacket implies that

$$h_\Delta(\phi') = \Psi_4'(\phi') - \Psi_3'(\phi') << \Psi_0' \tag{6.8}$$

and we neglect variations in ε with respect to Ψ' for each ϕ'. For each ϕ' we assign Ψ_0' as the appropriate value of Ψ' to use in each duct where we constrain

$$\Psi_0' = \left[\Psi_4'(\phi')\,\Psi_3'(\phi')\right]^{\frac{1}{2}} \simeq \frac{\Psi_4'(\phi') + \Psi_3'(\phi')}{2} \tag{6.9}$$

as an appropriate mean value of Ψ' in the jacket. Note that Ψ_0' in the jacket is chosen as the same as in (6.2) for the straight coaxes for matching between the straight coaxes and the bend. Thus converting from Ψ in (6.7) to ϕ' we have for the jacket

$$\Psi = \Psi_0 + \Psi_0' \cos(\phi') \quad , \quad \Psi_{max} = \Psi_0 + \Psi_0'$$

$$\frac{\varepsilon(\phi')}{\varepsilon_{min}} = \left[\frac{\Psi_0 + \Psi_0'}{\Psi_0 + \Psi_0' \cos(\phi')}\right]^2 \qquad (6.10)$$

$$\varepsilon_{min} = \varepsilon(0)$$

Alternate forms for this include

$$\begin{aligned} \varepsilon(\phi') &= \varepsilon(0)\left[\frac{\Psi_0 + \Psi_0'}{\Psi_0 + \Psi_0' \cos(\phi')}\right]^2 \\ &= \varepsilon\left(\frac{\pi}{2}\right)\left[\frac{\Psi_0}{\Psi_0 + \Psi_0' \cos(\phi')}\right]^2 \\ &= \varepsilon(\pi)\left[\frac{\Psi_0 - \Psi_0'}{\Psi_0 + \Psi_0' \cos(\phi')}\right]^2 \end{aligned} \qquad (6.11)$$

An individual duct in the jacket has a characteristic impedance

$$Z_d = \left[\frac{\mu}{\varepsilon(\phi')}\right]^{\frac{1}{2}} \frac{g(\phi')}{\Delta\phi'}$$

$$g(\phi') \equiv \ell n\left(\frac{\Psi_4'(\phi')}{\Psi_3'(\phi')}\right) \equiv 2\,\ell n\left(\frac{\Psi_4'(\phi')}{\Psi_0'(\phi')}\right) \equiv 2\,\ell n\left(\frac{\Psi_0'}{\Psi_3'(\phi')}\right) \qquad (6.12)$$

Matching the ducts in the bend to those in the coaxes for common values of ϕ' and equating the duct characteristic impedances gives

$$\frac{g(\phi')}{g_o} = \left[\frac{\varepsilon(\phi')}{\varepsilon_1}\right]^{\frac{1}{2}} \qquad (6.13)$$

Using these formulae one can determine $\Psi_3'(\phi')$ and $\Psi_4'(\phi')$ from

$$\begin{aligned} g(\phi') &= 2\,\ell n\left(\frac{\Psi_4'(\phi')}{\Psi_0'}\right) = 2\,\ell n\left(\frac{\Psi_0'}{\Psi_3'(\phi')}\right) \\ &= g_0\left[\frac{\varepsilon(\phi')}{\varepsilon_1}\right]^{\frac{1}{2}} \\ g_0 &= 2\,\ell n\left(\frac{\Psi_2'}{\Psi_0'}\right) = 2\,\ell n\left(\frac{\Psi_0'}{\Psi_2'}\right) \end{aligned}$$

$$\left[\frac{\varepsilon(\phi')}{\varepsilon_1}\right]^{\frac{1}{2}} = \left[\frac{\varepsilon(\phi'')}{\varepsilon_1}\right]^{\frac{1}{2}} \frac{\Psi_0 + \Psi_0' \cos(\phi'')}{\Psi_0 + \Psi_0' \cos(\phi')} \tag{6.14}$$

where ϕ'' is any reference angle for the permittivity, such as those in (6.11). Comparing $\varepsilon(\phi')$ to ε_1 one may wish to make some choice which minimizes the change on transitioning between the straight coaxes and the bend. One choice would have

$$\varepsilon\left(\frac{\pi}{2}\right) = \varepsilon_1 \tag{6.15}$$

so that the mean ε matches ε_1 and Ψ_4' and Ψ_3' match Ψ_2' and Ψ_1' respectively at $\phi' = \pm\pi/2$. This minimizes the maximum deviation (approximately) of the permittivities and conductor radii at $\phi' = 0$ and $\phi' = \pi$. Of course this assumes that $\varepsilon_1 > \varepsilon_0$ enough that $\varepsilon_{min} = \varepsilon(0) \geq \varepsilon_0$. This presumes that ε_1 is associated with some dielectric such as polyethylene, foam polyethylene, etc. In the bend, the permittivity needs to be graded, but in a limiting case $\varepsilon(0)$ can be air. If the straight coax sections have air dielectric, then one can choose $\varepsilon_{min} = \varepsilon_0$ and have a larger deviation of permittivity and conductors at $\phi' = \pi$.

7. Concluding Remarks

The concept of a jacket as a two-dimensional space for TEM-wave propagation can lead to some practical dielectric lens designs. From a theoretical point of view this is of interest as a two-dimensional wave concept, midway between a one-dimensional duct or transmission line, and a full three-dimensional lens. Constraining the permeability to be uniform and isotropic (e.g., μ_0) and the permittivity to be isotropic, this still leaves the spatial variation of the permittivity to consider. Examples of both uniform (generalized coax) and nonuniform (bend in coax) permittivity have been considered.

The approximation of a lens as a jacket is basically a low-frequency approximation (as electrically small in the direction of the electric field). As such the lens takes the form of a low-impedance transmission line. In the example of a bent coax, one would like to use this result for higher impedances of interest (50 Ω, 100 Ω). One can search for three-dimensional solutions in the spirit of [1]. Another approach is to extend the present approximations by making the transition more smoothly between the straight coaxes and the bend, and using the more exact dependence of ε on Ψ (as in (6.7)) instead of an average for each ϕ'. The bend radius Ψ_0 need not be a constant over the bend. Near the straight coaxes the curvature Ψ_0^{-1} can be smoothly decreased (being 0 in the limit of connecting to the straight coaxes). Adjusting ε at each cross section to correspond with the local Ψ_0 will then remove abrupt changes in ε at both ends of the bend.

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