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Azimuthal TEM Waveguides in Dielectric Media

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Abstract

This paper develops exact solutions of the Maxwell equations for TEM waves with azimuthal (ϕ) propagation in a dielectric lens medium with permittivity proportional to Ψ^{-2} ($\Psi \equiv$ cylindrical radius). This class of solutions is based on magnetostatic fields from ϕ-independent ϕ-directed currents to give the transverse parts (magnetic and electric) of the TEM wave. These can be derived from the magnetostatic vector potential. This leads to various geometries of TEM waveguides in the lens medium, including bends in coaxial cable.

I. Introduction

Dielectric lenses for bending the direction of propagation of transverse electromagnetic (TEM) waves have been investigated in three recent papers [2-4]. A common feature of these lenses, and those to be discussed here, is that they are portions of a dielectric body of revolution (BOR) with permittivity ε proportional to Ψ^{-2} where Ψ is the cylindrical radius from the axis of revolution (the z axis). In [2] an E-plane bend with translation symmetry in the z direction is discussed. In [3] this is generalized to several cases involving planar, circular cylindrical, circular conical and spherical conducting boundaries. This includes the H-plane bend (planar conducting boundaries) based on the class of lenses in [7 (appendix F)]. In [4] this problem is approached from the point of view of a jacket in which the spacing of the two conductors perpendicular to the electric field is electrically small, and small compared to the dimensions in the direction of the magnetic field. Such a low-impedance structure is basically two-dimensional in terms of dimensions that can be large compared to wavelength and thereby admit propagation. This jacket technique gives an approximate solution for a bend in a coaxial cable.

The present paper continues the discussion of such dielectric bending lenses by exploring the general properties of the TEM waves with azimuthal (ϕ - directed) propagation. Recalling the properties of the solutions [3], it was noted there how the transverse behavior of the magnetic field was like a static magnetic field multiplied by a waveform function propagating in the ϕ direction. As we shall see, this is a characteristic of the general solution. An interesting canonical magnetostatic solution is that of a filamentary circular current loop. The closed magnetic field lines encircling the current can be used to define closed conducting surfaces which look like a circularly bent coaxial cable. (This insight came to me in a dream the night of 25-26 March 1996.) By superposition of such loops any ϕ - independent static ϕ - directed current distribution and the associated magnetic field can be produced for defining "coaxial" bends with various cross-section shapes.

2. TEM Waves with Azimuthal Propagation

The cylindrical (Ψ, ϕ, z) coordinate system is related to the Cartesian (x,y,z) coordinate system by:

$$x = \Psi \cos(\phi) \quad , \quad y = \Psi \sin(\phi) \tag{2.1}$$

For our (u_1, u_2, u_3) othoganal curvilinear coordinate system as in [3], let us choose the right-handed form

$$\begin{aligned} &u_1 \equiv z \quad , \quad u_2 \equiv \Psi \quad , \quad u_3 \equiv \Psi_{max}\,\phi \\ &\vec{1}_1 = \vec{1}_z \;,\; \vec{1}_2 = \vec{1}_\Psi \quad , \quad \vec{1}_3 = \vec{1}_\phi \;\text{(unit vectors)} \end{aligned} \tag{2.2}$$

with scale factors

$$h_1 = h_2 = 1 \quad , \quad h_3 = \frac{\Psi}{\Psi_{max}} \tag{2.3}$$

The medium is specified by

$$\begin{aligned} &\mu = \mu' \geq \mu_o \qquad \text{(uniform permeability)} \\ &\frac{\varepsilon}{\varepsilon_{min}} = h_3^{-2} = \left(\frac{\Psi_{max}}{\Psi}\right)^2 \\ &\varepsilon_{min} = \varepsilon \geq \varepsilon_0 \\ &\Psi_{max} \equiv \text{ maximum allowable } \Psi \text{ in lens region} \end{aligned} \tag{2.4}$$

This gives wave parameters

$$\begin{aligned} &c_w = [\mu\varepsilon]^{-1/2} = h_3 c' = \frac{\Psi}{\Psi_{max}} c' \leq c' \text{ in lens region} \\ &c' = [\mu\varepsilon']^{1/2} \leq c \\ &Z_w = \left[\frac{\mu}{\varepsilon}\right]^{1/2} = h_3 Z_0' = \frac{\Psi}{\Psi_{max}} Z_0' \leq Z_0' \text{ in lens region} \\ &Z_0' = \left[\frac{\mu'}{\varepsilon'}\right]^{1/2} \end{aligned} \tag{2.5}$$

Let us look for waves of the form

$$\begin{aligned}
\vec{E} &= \vec{E}_t(z,\Psi)\ f\left(t - \frac{u_3}{c'}\right) \\
\vec{H} &= \vec{H}_t(z,\Psi)\ f\left(t - \frac{u_3}{c'}\right) \\
&\left.\begin{aligned} \vec{E}_t \cdot \vec{1}_\phi &= 0 \\ \vec{H} \cdot \vec{1}_\phi &= 0 \end{aligned}\right\} \text{transverse wave} \\
&f(t) \equiv \text{temporal waveform}
\end{aligned} \tag{2.6}$$

where the argument $t - \frac{u_3}{c'}$ gives the propagation in the $+\phi$ direction. Let us show that this kind of wave can satisfy the Maxwell equations.

From

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{2.7}$$

we have

$$\begin{aligned}
&-\frac{1}{h_3}\vec{1}_z E_{t\Psi}\frac{\partial}{\partial u_3} f\left(t - \frac{u_3}{c'}\right) + \frac{1}{h_3}\vec{1}_\Psi E_{t_z}\frac{\partial}{\partial u_3} f\left(t - \frac{u_3}{c'}\right) \\
&+\vec{1}_\phi\left[\frac{\partial}{\partial z}E_{0\Psi} - \frac{\partial}{\partial \Psi}E_{0_z}\right] f\left(t - \frac{u_3}{c'}\right) = -\mu \vec{H}_t \frac{\partial}{\partial t} f\left(t - \frac{u_3}{c'}\right)
\end{aligned} \tag{2.8}$$

This requires that

$$\frac{\partial}{\partial z}E_{t\Psi} - \frac{\partial}{\partial z}E_{t_z} = 0 \tag{2.9}$$

Furthermore, from

$$\frac{\partial}{\partial t} f\left(t - \frac{u_3}{c'}\right) = \frac{1}{c'}\frac{\partial}{\partial u_3} f\left(t - \frac{u_3}{\partial u_3}\right) \tag{2.10}$$

we have

$$\frac{1}{h_3 c'}E_{t_\Psi} = -\mu H_{t_z} \quad , \quad \frac{1}{h_3 c'}E_{t_z} = \mu H_{t\Psi} \tag{2.11}$$

which implies

$$\vec{1}_\phi \times \vec{E}_t = Z_w \vec{H}_t \quad , \quad \vec{E}_t = -Z_w \vec{1}_\phi \times \vec{H}_t \tag{2.12}$$

so that the wave must be TEM, i.e. have the fields mutually orthoganal and related by the local wave impedance.

From

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \tag{2.13}$$

we have

$$\frac{1}{h_3} \vec{1}_z H_{t\Psi} \frac{\partial}{\partial u_3} f\left(t - \frac{u_3}{c'}\right) + \frac{1}{h_3} \vec{1}_\Psi H_{t_z} \frac{\partial}{\partial u_3} f\left(t - \frac{u_3}{c'}\right)$$

$$+ \vec{1}_\phi \left[\frac{\partial}{\partial z} H_{t\Psi} - \frac{\partial}{\partial \Psi} H_{t_z}\right] f\left(t - \frac{u_3}{c'}\right) = \varepsilon \vec{E}_t \frac{\partial}{\partial t} f\left(t - \frac{u_3}{c'}\right) \tag{2.14}$$

This requires that

$$\frac{\partial}{\partial z} H_{t_\Psi} - \frac{\partial}{\partial \Psi} H_{t_z} = 0 \tag{2.15}$$

Again using (2.10) we have

$$\frac{1}{h_3 c'} H_{t_\Psi} = \varepsilon E_{t_z} \quad , \quad -\frac{1}{h_3 c'} H_{t_z} = \varepsilon E_{t_z} \tag{2.16}$$

which gives exactly the same TEM condition as in (2.12).

Thus for TEM solutions of the form in (2.6) we need only to satisfy (2.9). Equations (2.12), and (2.15) with non-zero $\vec{E}_t$ and $\vec{H}_t$.

3. Relation to Magnetostatics

From Appendix A we have the general properties of magnetostatic fields (subscript 0) associated with ϕ-directed currents independent of ϕ. Let us try this type of field which from (A.8) is

$$\vec{H} \equiv \vec{H}_0 = -\frac{1}{\mu\Psi}\vec{1}_\phi \times \nabla\left(\Psi A_{0_\phi}\right) \tag{3.1}$$

where A_{0_ϕ} is the only non-zero component (ϕ -independent) of the static vector potential. In component form this is

$$\begin{aligned} H_{t_z} &= \frac{1}{\mu\Psi}\frac{\partial}{\partial\Psi}\left(\Psi A_{0_\phi}\right) \\ H_{t_\Psi} &= -\frac{1}{\mu\Psi}\frac{\partial}{\partial z}\left(\Psi A_{0_\phi}\right) \end{aligned} \tag{3.2}$$

Note that with this identification (A.10) is exactly the same as (2.15), thereby satisfying this requirement, provided we restrict ourselves to regions where there are no currents (i.e., away from the perfectly conducting boundaries).

Having an acceptable magnetic field, let us turn now to the electric field. Define $\vec{E}_t$ from the $\vec{H}_t$ (which we now have) by the TEM condition (2.12). With (3.1) this gives

$$\vec{E}_t = -\frac{Z_w}{\mu\Psi}\nabla\left(\Psi A_{0_\phi}\right) = -\frac{Z_0'}{\mu\Psi_{\max}}\nabla\left(\Psi A_{0_\phi}\right) = -\frac{c'}{\Psi_{\max}}\nabla\left(\Psi A_{0_\phi}\right) \tag{3.3}$$

In component form this is

$$\begin{aligned} E_{t_z} &= -\frac{c'}{\Psi_{\max}}\frac{\partial}{\partial_z}\left(\Psi A_{0_\phi}\right) \\ E_{t_\Psi} &= -\frac{c'}{\Psi_{\max}}\frac{\partial}{\partial_z}\left(\Psi A_{0_\phi}\right) \end{aligned} \tag{3.4}$$

Since $\vec{E}_t$ is proportional to a gradient in a plane of constant ϕ, it is easy to verify that (2.9) is satisfied.

All the requirements for satisfying the Maxwell equations having been met, then TEM waves of the form in (2.6) with (2.12) relating the two fields are admissible solutions of the Maxwell equations in regions of no currents.

4. Azimuthal TEM Waveguide

With the TEM wave satisfying the Maxwell equations, attention now turns to the boundary conditions. Consider a bend in the coaxial cable as illustrated in fig. 4.1. Note the similarity to the same problem in [4]. Besides the previously introduced cylindrical (Ψ,ϕ,z) coordinate system, one can introduce coordinates (Ψ',ϕ') in a plane of constant ϕ, based on a reference point (center) at $(z,\Psi) = (0,\Psi_0)$. The coax is then described by perfectly conducting surfaces as

$$\begin{aligned} \Psi_{3'}(\phi') &\equiv \text{inner conductor} \\ \Psi_{4'}(\phi') &\equiv \text{outer conductor} \end{aligned} \tag{4.1}$$

where the notation corresponds to [4]. However, ε is now taken as a function of Ψ as in (2.4) instead of the approximate form as a function of ϕ'. Besides the boundaries in (4.1) there is a region $\phi_1 \le \phi \le \phi_2$ defining the lens region (the bend).

Considering the boundary conditions on $\Psi_3'(\phi')$and $\Psi_4'(\phi')$ we need $\vec{E}_t$ perpendicular and $\vec{H}_t$ parallel. With these two mutually perpendicular as in (2.12), it suffices to consider either of the fields at these boundaries. As we have seen, these fields can be derived from magnetostatic considerations. For this purpose extend the bend to a full circle by setting $(\phi_1, \phi_2) = (0, 2\pi)$ so that the conductors form closed surfaces with complete azimuthal symmetry. Then as a gedankenexperiment place a current source in the inner conductor to force a static current I_0 (in the ϕ direction) around the inner-conductor loop with a surface current density $J_{s_\phi}^{(3)}(\phi')$ on it (independent of ϕ). Then with no magnetic field allowed to penetrate the outer perfect conductor (and the inner one for that matter) there will be an opposite current - I_0 on the outer conductor with some distribution $J_{s_\phi}^{(4)}(\phi')$. The magnetic field between the two surfaces (encircling the inner conductor) is an example of the magnetostatic field discussed in Appendix A. Identify this (whether calculated or measured) with $\vec{H}_t$ which now matches the boundary condition of zero normal component on the inner conductors. The TEM mode then satisfies the boundary conditions on these two surfaces. (The same reasoning applies to multiple inner conductors surrounded by the outer conductor, giving multiple TEM modes).

From an analytic or computational point of view, one can solve the second order differential equation for ΨA_{0_ϕ} in Appendix A, subject to the boundary condition that this be two different constant values on the inner and outer boundaries (like an electric potential function). Alternatively, one can formulate an integral equation with ΨA_{0_ϕ} for a filamentary wire loop (as

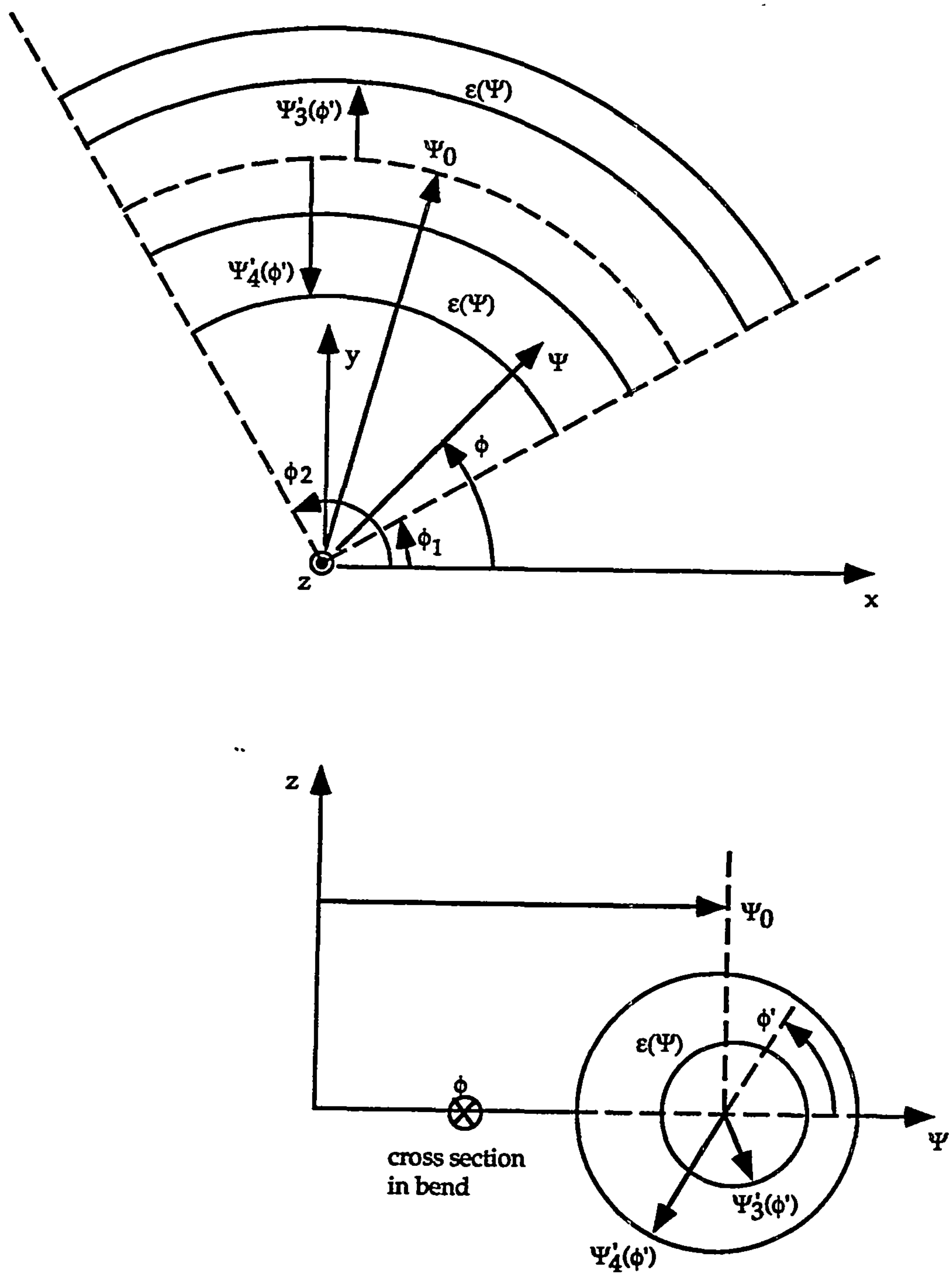

Fig. 4.1. Bend in Coax

in Appendix B) taken as a Green's function. Integrating this times an unknown ϕ-directed static current density on the two boundaries, and imposing the two constant values for ΨA_{0_ϕ} on the two boundaries gives an appropriate integral equation on the two boundaries as curves on a plane of constant ϕ. This can be solved for the surface current density by techniques such as the moment method. Of course, one can choose some number of current loops located on the $(z,\Psi) = (z_n, \Psi_n)$ with currents I_n, compute ΨA_{0_ϕ} and choose paths in the (z,Ψ) plane on which this is constant to define the inner and outer conductors.

As far as the TEM mode (or modes in the case of multiple inner conductors) is concerned, this is sufficient to satisfy the existence and properties. However, other modes can exist in the geometry and medium. The boundary conditions at ϕ_1 and ϕ_2 need to be satisfied. One can impose the boundary conditions required to match the TEM mode with electric and magnetic sources on the ϕ_1 and ϕ_2 planes as specified by the field-equivalence principle [8]. As a practical matter one would like (at least approximately) to match this TEM mode to the well-known TEM mode of a straight coaxial cable. An approximate way of doing this based on ducts of equal $\Delta\phi'$ is discussed in [4]. In this case, the duct impedances are matched through the boundaries, but this still leaves some perturbations due to the shifts of the conductor boundaries and jump in ε in crossing the ϕ_1 and ϕ_2 boundaries. Another approach is to construct the bend with sections of various bend radius Ψ_0 with Ψ_0^{-1} tending toward zero near the two straight coaxes.

5. Concluding Remarks

We now have a theory of TEM modes in a BOR section with ε proportional to Ψ^{-2}. This gives the desired dispersionless character of TEM modes for propagating pulses through bends in coaxial cables, thereby extending the bandwidth of such bends to higher frequencies, or equivalently preserving shorter rising times. This still leaves questions of how best to transition into and out of such a bend, and how to best manufacture such a bend while closely approximating the required conductors and permittivity.

The method of constructing the solution relies on the properties of static magnetic fields and the associated currents on boundaries. For the coaxial bend the topology is one of two (or more) perfectly conducting surfaces, the closed inner loop(s) being contained within the outer loop. Another such topology for comparison is that discussed in [1] for containing static magnetic fields. One might ask whether other kinds of magnetostatic field geometries can be used to construct TEM fields in transient lens design.

Appendix A. Magnetostatics for Azimuthally Directed Currents with Rotation Symmetry.

For magnetostatics (subscript zero) we have the appropriate Maxwell equation

$$\nabla\times\vec{H}_0=\vec{J}_0 \quad , \quad \nabla\cdot\vec{J}_0 \quad \text{(static current density)} \tag{A.1}$$

Assuming the permeability μ is uniform and isotropic, we have

$$\nabla\cdot\vec{B}_0=\vec{0} \quad , \quad \vec{B}_0=\mu\vec{H}_0 \quad , \quad \nabla\cdot\vec{H}_0=0 \tag{A.2}$$

It is well known that the magnetic field can be derived from a vector potential [6] where

$$\vec{B}_0=\mu\vec{H}_0=\nabla\times\vec{A}_0 \quad , \quad \nabla\cdot\vec{A}_0=0$$

$$\nabla^2\vec{A}_0=-\mu\vec{J}_0 \quad , \quad \vec{A}_0(\vec{r})=\mu\int_V\frac{\vec{J}_0(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}\,dV' \tag{A.3}$$

Note also that we have

$$\nabla\times\vec{H}_0=\vec{0} \quad , \quad \nabla^2\vec{A}_0=\vec{0} \tag{A.4}$$

away from currents.

Specialize this to a current density of the form

$$\vec{J}_0=J_{0_\phi}(z,\Psi)\vec{1}_\phi \tag{A.5}$$

which, in cylindrical coordinates (Ψ,ϕ,z), has only a ϕ component independent of ϕ. This rotation symmetry, $C_{\infty a}$ in [8], carries over to the other parameters as well. In particular, the vector potential has the form

$$\vec{A}_0=A_{0_\phi}(z,\Psi)\vec{1}_\phi \tag{A.6}$$

from which we have the magnetic field

$$\begin{aligned}\vec{H}&=H_{0_z}(z,\Psi)\vec{1}_z+H_{0_\Psi}(z,\Psi)\vec{1}_\Psi\\ H_{0_z}&=\frac{1}{\mu\Psi}\frac{\partial}{\partial\Psi}\left(\Psi A_{0_\phi}\right)\\ H_{0_\Psi}&=-\frac{1}{\mu}\frac{\partial}{\partial z}A_{0_\phi}=-\frac{1}{\mu\Psi}\frac{\partial}{\partial z}\left(\Psi A_{0_\phi}\right)\end{aligned} \tag{A.7}$$

An alternate form of this is

$$\vec{H}_0 = -\frac{1}{\mu\Psi}\vec{1}_\phi \times \nabla\left(\Psi A_{0_\phi}\right) \tag{A.8}$$

Away from currents this rotationally symmetric magnetic field has

$$\nabla \times \vec{H}_0 = \vec{0} = \left[\frac{\partial}{\partial z} H_{0_\Psi} - \frac{\partial}{\partial \Psi} H_{0_z}\right]\vec{1}_\phi \tag{A.9}$$

showing that all such magnetic fields have

$$\frac{\partial}{\partial z} H_{0_\Psi} - \frac{\partial}{\partial \Psi} H_{0_z} = 0 \tag{A.10}$$

in currentless regions. For the vector potential away from currents we have

$$\nabla^2 \vec{A}_0 = \vec{0} = \left[\nabla^2 A_{0_\phi} - \Psi^{-2} A_{0_\phi}\right]\vec{1}_\phi \tag{A.11}$$

This gives a solar differential equation for the one non-zero component of the vector potential with various forms as

$$\begin{aligned} &\Psi^{-1}\frac{\partial}{\partial \Psi}\left(\Psi \frac{\partial A_{0_\phi}}{\partial \Psi}\right) + \frac{\partial^2 A_{0_\phi}}{\partial z^2} - \Psi^{-2} A_{0_\phi} = 0 \\ &\frac{\partial^2}{\partial \Psi^2}\left(\Psi A_{0_\phi}\right) - \Psi^{-1}\frac{\partial}{\partial \Psi}\left(\Psi A_{0_\phi}\right) + \frac{\partial^2}{\partial z^2}\left(\Psi A_{0_\phi}\right) = 0 \\ &\Psi \frac{\partial}{\partial \Psi}\left[\Psi^{-1}\frac{\partial}{\partial \Psi}\left(\Psi A_{0_\phi}\right)\right] + \frac{\partial}{\partial z^2}\left(\Psi A_{0_\phi}\right) = 0 \end{aligned} \tag{A.12}$$

So this gives a partial differential equation (similar to a Laplace equation) for A_{0_ϕ} or ΨA_{0_ϕ} on a domain in a (z, Ψ) plane. If we are given perfectly conducting boundaries, (say an inner and outer conductor) this can be solved subject to the boundary condition that $\vec{H}_0$ be parallel to these boundaries. From (A.7) this is a condition on the derivatives of ΨA_{0_ϕ}, in particular that the tangential derivative of the parameter be zero on these boundaries. Equivalently, this means that ΨA_{0_ϕ} must achieve a constant value on each boundary (like an electric potential function).

Appendix B. Magnetostatic Filametary Circular Loop

An important case of the kind of current distribution treated in Appendix A is that of a filamentary current I_0 on a circular loop, located on $(z,\Psi)=(0,a)$. By varying a (say as Ψ') and shifting z as $z-z'$ this can be used to give a current-density distribution of the form $J_{0_\phi}(z,\Psi)$ discussed previously. This canonical problem in effect gives a Green's function over which one can integrate to give the vector potential and magnetic field for the more general case.

Summarizing from [6], we have the vector potential

$$\begin{aligned} A_{0_\phi} &= \frac{\mu I_0}{\pi}\left[\frac{a}{m\Psi}\right]^{1/2}\left[\left(1-\frac{m}{2}\right)K(m)-E(m)\right] \\ m &= 4a\Psi\left[(a+\Psi)^2+z^2\right]^{-1} \end{aligned} \qquad \text{(B.1)}$$

and magnetic field

$$\begin{aligned} H_{0_z} &= \frac{I_0}{2\pi}\left[(a+\Psi)^2+z^2\right]^{1/2}\left[K(m)+\frac{a^2-\Psi^2-z^2}{(a-\Psi)^2+z^2}E(m)\right] \\ H_{0_\Psi} &= \frac{I_0}{2\pi}\frac{z}{\Psi}\left[(a+\Psi)^2+z^2\right]^{-1/2}\left[K(m)+\frac{a^2+\Psi^2+z^2}{(a-\Psi)^2+z^2}E(m)\right] \end{aligned} \qquad \text{(B.2)}$$

Here m is called the parameter in modern notation for elliptic integrals [5], and is equivalent to k^2, where k is called the modulus. On the z axis that magnetic field reduces to

$$H_{0_z} = \frac{I}{2}a^2\left[a^2+z^2\right]^{-3/2} \quad , \quad H_{0_\Psi}=0 \qquad \text{(B.3)}$$

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