Sensor and Simulation Notes

Note 405

5 December 1996

Use of Generalized Inhomogeneous TEM Plane Waves in Differential Geometric Lens Synthesis

Carl E. Baum
Phillips Laboratory

Abstract

This paper develops a generalized form of inhomogeneous TEM plane wave involving inhomogeneous (but isotropic) constitutive parameters. This is then used in conjunction with differential-geometric scaling to synthesize inhomogeneous dielectric lenses for transient TEM waves. A special class of these has a uniform permeability allowing a purely dielectric lens. A special example involving propagation in the azimuthal direction in cylindrical coordinates is shown to agree with previous results based on a different method.

1. Introduction

A recent paper [5] showed that guided TEM waves (modes) of a very general form could be propagated in an inhomogeneous isotropic dielectric medium with permittivity ε proportional to Ψ^{-2} where Ψ is the cylindrical radius in the usual (Ψ, ϕ, z) cylindrical coordinate system. This has the property of practical importance that the permeability μ is a constant (homogeneous), such as μ_o, the permeability of free space. Propagation is in the ϕ direction with electric and magnetic fields both perpendicular to $\vec{1}_\phi$ and to each other, with ratio given by the local wave impedance. The TEM wave is guided by (ideally) perfect conductors with cross sections in each (z, Ψ) plane independent of ϕ over some range $\phi_1 \leq \phi \leq \phi_2$ for which the special lens is desired.

Special cases of TEM waves (simple forms with field components in only single coordinate directions in cylindrical and spherical coordinates) in this kind of medium have been discussed in [2, 3]. One application of this type of lens is a dispersionless bend in a coaxial cable, a special but common form of TEM transmission line. This is considered in an approximate form in [4] and in an exact form in [5].

The exact form of inhomogeneous ϕ-directed TEM waves in [5] allows for very general conductor cross sections in (z, Ψ) coordinates. It is developed from a magnetostatic solution for ϕ-independent, ϕ-directed currents and the associated vector potential with only a ϕ-independent ϕ component A_{0_ϕ}. The field components are shown to be derivable from $\nabla(\Psi A_{0_\phi})$ times the waveform function describing the propagation in the ϕ direction. The quantity ΨA_{0_ϕ} serves the role of an electric scalar potential.

In [7] lens synthesis (design) is considered for various types of inhomogeneous media supporting TEM modes. This synthesis is based on a differential-geometric scaling of coordinates, fields, and media. One form of wave, say a TEM plane wave, is thereby transformed into a TEM wave propagating in some curved manner. The question then arises as to whether the new types of solutions can also be cast into this differential-geometric format so as to extend this method to new classes of solutions. This is the subject of this paper.

2. Inhomogeneous TEM Plane Waves in Uniform Isotropic Media

As discussed in [7 (Section 2.4)] we have inhomogeneous TEM plane waves of the form

$$
\begin{aligned}
\vec{E}' &= \vec{E}_0'(u_1,u_2)\, f(t-\frac{u_3}{c'}) \\
\vec{H}' &= \vec{H}_0'(u_1,u_2)\, f(t-\frac{u_3}{c'}) \\
\vec{1}_3 \times \vec{E}_0'(u_1,u_2) &= Z'\vec{H}_0'(u_1,u_2) \;,\; Z'\vec{H}_0'(u_1,u_2) = -\vec{1}_3 \times \vec{E}_0'(u_1,u_2) \\
Z' &= \left[\frac{\mu'}{\varepsilon'}\right] \equiv \text{wave impedance} \\
c' &= [\mu' \varepsilon']^{-1/2} \equiv \text{wave speed} \\
\vec{E}_0'(u_1,u_2) &= -\nabla_t' \Phi_e(u_1,u_2) \;,\; \nabla_t'^2 \Phi_e(u_1,u_2) = 0 \\
\vec{H}_0'(u_1,u_2) &= -\nabla_t' \Phi_h(u_1,u_2) \;,\; \nabla_t'^2 \Phi_h(u_1,u_2) = 0 \\
f(t-\frac{u_3}{c}) &\equiv \text{waveform function} \\
\nabla_t' &= \vec{1}_1 \frac{\partial}{\partial u_1} + \vec{1}_2 \frac{\partial}{\partial u_2} \\
\mu' &\equiv \text{permeability} \;,\; \varepsilon' = \text{permittivity}
\end{aligned}
\tag{2.1}
$$

Here the right-handed orthogonal curvilinear coordinate system (u_1, u_2, u_3) describes the formal fields, i.e., fields that satisfy the Maxwell equations with the coordinates treated as Cartesian. All the primed parameters and operators have this interpretation. As a special case, these coordinates can be taken as (x, y, z), in which case (2.1) describes fields appropriate to a TEM transmission line derivable from a scalar electric potential Φ_e or scalar magnetic potential Φ_h, including the complex form $\Phi_e' + j\,\Phi_h'$ and $E_0' = j\,Z'\,H_0'$ with complex coordinate $u_1 + j\,u_2$ as in [1, 8].

In [7] this type of wave is then transformed by a coordinate transformation where u_3 is in general curved to form what are called the real fields, etc. In this process ε' and μ' are changed to ε and μ which may, in general, be dyadics. It is important to note that these constitutive parameters have the forms

$$
\begin{aligned}
(\varepsilon_{n,m}') &= \begin{pmatrix} \varepsilon' & 0 & 0 \\ 0 & \varepsilon' & 0 \\ 0 & 0 & \varepsilon_3' \end{pmatrix} \;,\; (\mu_{n,m}') = \begin{pmatrix} \mu' & 0 & 0 \\ 0 & \mu' & 0 \\ 0 & 0 & \mu_3' \end{pmatrix} \\
(\varepsilon_{n,m}) &= \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \;,\; (\mu_{n,m}) = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}
\end{aligned}
\tag{2.2}
$$

where the components operating on the third components of the fields can be anything since these field components are zero by hypothesis. This gives a certain degree of freedom on selection of suitable orthogonal curvilinear coordinate systems for (u_1, u_2, u_3).

3. Generalized Inhomogeneous TEM Plane Waves in Inhomogeneous but Isotropic Media

Still thinking of our (u_1, u_2, u_3) coordinates as though they were Cartesian, let us allow ε' and μ' to be inhomogeneous and derive a more general form of TEM plane wave. Let us look for solutions, again in the (u_1, u_2, u_3) coordinates as formal fields, etc., of the form

$$\begin{aligned} \vec{E}' &= \vec{E}_0'(u_1,u_2)\ f(t-\tau(u_3)) \\ \vec{H}' &= \vec{H}_0'(u_1,u_2)\ f(t-\tau(u_3)) \\ \vec{E}_0'(u_1,u_2)\cdot\vec{1}_3 &= 0\ ,\quad \vec{H}_0'(u_1,u_2)\cdot\vec{1}_3 = 0 \end{aligned} \tag{3.1}$$

Again, restricting the fields to have no third components makes ε_3' and μ_3' in (2.2) play no role. However, now we let ε' and μ' be functions of the coordinates (i.e., inhomogeneous). Note also that u_3/c' in (2.1) is replaced by the more general $\tau(u_3)$ in (3.1). Thus the wave is propagating in the u_3 direction, but perhaps with variable speed.

For convenience we have

$$\begin{aligned} v' &= [\mu'\varepsilon']^{-1/2} \equiv \text{wave speed} \\ Z' &= \left[\frac{\mu'}{\varepsilon'}\right]^{1/2} \equiv \text{wave impedance} \\ \mu' &> 0\ ,\quad \varepsilon' > 0 \end{aligned} \tag{3.2}$$

where now these may be coordinate dependent. Note also that

$$\frac{\partial f}{\partial u_3} = \frac{\partial f}{\partial(t-\tau)}\left[-\frac{d\tau}{du_3}\right] = -\frac{\partial f}{\partial t}\frac{d\tau}{du_3} \tag{3.3}$$

Consider one of the Maxwell equations

$$\nabla' \times \vec{E}' = -\frac{\partial \vec{B}'}{\partial t} = -\mu'\frac{\partial \vec{H}'}{\partial t} \tag{3.4}$$

where we think of the (u_1, u_2, u_3) formally as Cartesian (x, y, z) coordinates. This leads to the requirements (given the constraints of (3.1))

$$\frac{\partial}{\partial u_1} E'_{0_2} - \frac{\partial}{\partial u_2} E'_{0_1} = 0$$

$$\vec{1}_3 \times \vec{E}'_0 \frac{d\tau}{du_3} = \mu' \vec{H}'_0 \ , \ \vec{E}'_0 \frac{d\tau}{du_3} = -\mu' \vec{1}_3 \times \vec{H}'_0 \tag{3.5}$$

The second Maxwell equation

$$\nabla' \times \vec{H}' = \frac{\partial \vec{D}'}{\partial t} = \varepsilon' \frac{\partial \vec{E}'}{\partial t} \tag{3.6}$$

similarly leads to

$$\frac{\partial}{\partial u_1} H'_{0_2} - \frac{\partial}{\partial u_2} H'_{0_1} = 0$$

$$\vec{1}_3 \times \vec{H}'_0 \frac{d\tau}{du_3} = -\varepsilon' \vec{E}'_0 \ , \ \vec{H}'_0 \frac{d\tau}{du_3} = \varepsilon' \vec{1}_3 \times \vec{E}'_0 \tag{3.7}$$

With our constraints, satisfaction of (3.5) and (3.7) is equivalent to satisfying the Maxwell equations.

From the two forms of the cross products in (3.5) and (3.7) we have the constraint

$$\frac{1}{\varepsilon'} \frac{d\tau}{du_3} = \mu' \frac{du_3}{d\tau}$$

$$\frac{d\tau}{du_3} = \pm [\mu' \varepsilon']^{1/2} = \pm \, v'^{-1} \tag{3.8}$$

Choosing the + sign for convenience (propagation in the $+u_3$ direction) we have

$$\frac{d\tau}{du_3} = v'^{-1}(u_3)$$

$$\tau = \int_0^{u_3} \frac{du'_3}{v'(u'_3)} = \tau(u_3) \tag{3.9}$$

where we have taken the lower limit of integration as zero, but any convenient value of u_3 can be used. Note that we have also shown that

$$v' = [\mu' \varepsilon']^{-1/2} = v'(u_3) \text{ (a function of only } u_3) \tag{3.10}$$

so that the product $\mu'\varepsilon'$ is a function of only u_3.

The requirements of (3.5) and (3.7) may now be summarized as

$$\frac{\partial}{\partial u_1} E'_{0_2} - \frac{\partial}{\partial u_2} E'_{0_1} = 0 \ , \quad \frac{\partial}{\partial u_1} H'_{0_2} - \frac{\partial}{\partial u_2} H'_{0_1} = 0$$
$$\vec{1}_3 \times \vec{E}'_0 = Z'\vec{H}'_0 \ , \quad \vec{1}_3 \times \vec{H}'_0 = -Z'\vec{E}'_0 \tag{3.11}$$

Note the local TEM wave condition with the local field ratio given by the local wave impedance (Z' in general being coordinate dependent). There are three conditions in (3.9) now to be satisfied (the last two equations being equivalent). Note that since $\vec{E}'_0$ and $\vec{H}'_0$ are not functions of u_3 (by hypothesis in (3.1)) we have

$$Z' = \left[\frac{\mu'}{\varepsilon'}\right]^{1/2} = Z'(u_1,u_2) \ \ \text{(not a function of } u_3) \tag{3.12}$$

so that the ratio μ'/ε' is independent of u_3.

The forms of these equations suggest that we try solutions of the form

$$\vec{E}'_0(u_1,u_2) = -\nabla_t\Phi'_e(u_1,u_2) \ , \quad \vec{H}'_0(u_1,u_2) = -\nabla_t\Phi'_h(u_1,u_2) \tag{3.13}$$

where the transverse gradients in the formal coordinates make this a two-dimensional potential problem. As usual, the electric potential is solved subject to specified potentials on the conductors passing through the lens, and the magnetic potential has to allow for a discontinuity in passing around a conductor (the discontinuity being equal to the current in the conductor).

Consider first constructing the solution via the electric potential. Then

$$\vec{E}'_0 = -\nabla'_t\Phi'_e \Rightarrow \frac{\partial}{\partial u_1} E'_{0_2} - \frac{\partial}{\partial u_2} E'_{0_2} = 0 \tag{3.14}$$

satisfying the first condition in (3.11). The magnetic field has the form

$$\vec{H}'_0 = Z'^{-1}\vec{1}_3 \times \vec{E}'_0 = Z'^{-1}\left[E'_{0_1}\vec{1}_2 - E'_{0_2}\vec{1}_1\right] \ . \tag{3.15}$$

Then require

$$0 = \frac{\partial}{\partial u_1} H'_{0_2} - \frac{\partial}{\partial u_2} H'_{0_1} = \frac{\partial}{\partial u_1}\left[\frac{E'_{0_1}}{Z'}\right] + \frac{\partial}{\partial u_2}\left[\frac{E'_{0_2}}{Z'}\right]$$

$$= \nabla'_t \cdot \left[\frac{\vec{E'_0}}{Z'}\right] = -\nabla'_t \cdot \left[Z'^{-1} \nabla'_t \Phi_e\right] \tag{3.16}$$

Consider the fact that

$$\nabla' \cdot \vec{D'} = 0 \Rightarrow$$

$$0 = \nabla'_t \cdot \left[\varepsilon' \vec{E'}_0\right] = \nabla'_t \cdot \left[v'^{-1} Z'^{-1} \vec{E'_0}\right] = v'^{-1} \nabla'_t \cdot \left[Z'^{-1} \vec{E'}_0\right] \tag{3.17}$$

$$0 = \nabla'_t \cdot \left[Z'^{-1} \vec{E'}_0\right] = -\nabla'_t \left[Z'^{-1} \nabla'_t \; \Phi'_e\right]$$

where the fact that v' is a function of only u_3 has been used. Hence, requiring that Φ'_e satisfy a Laplace-type equation for an inhomogeneous medium (variable ε') gives a two-dimensional potential which satisfies all three conditions ((3.14) – (3.17)).

The alternate approach to the solution is to use the magnetic potential. Then

$$\vec{H_0} = -\nabla'_t \Phi'_h \Rightarrow \frac{\partial}{\partial u_1} H'_{0_2} - \frac{\partial}{\partial u_2} H'_{0_1} = 0 \tag{3.18}$$

satisfying the second condition in (3.11). The electric field has the form

$$\vec{E'_0} = -Z' \vec{1}_3 \times \vec{H'_0} \tag{3.19}$$

Then require

$$0 = \frac{\partial}{\partial u_1} E'_{0_2} - \frac{\partial}{\partial u_2} E'_{0_1} = -\frac{\partial}{\partial u_1}\left[Z' H'_{0_1}\right] - \frac{\partial}{\partial u_2}\left[Z' H'_{0_2}\right]$$

$$= -\nabla'_t \cdot \left[Z' \vec{H'_{0_1}}\right] = \nabla'_t \cdot \left[Z' \nabla'_t \Phi_h\right] \tag{3.20}$$

Consider the fact that

$$\nabla' \cdot \vec{B}' = 0 \Rightarrow$$

$$0 = \nabla_t' \cdot \left[\mu' \vec{H}_0'\right] = \nabla_t' \cdot \left[v'^{-1} Z' \vec{H}_0'\right] = v'^{-1} \nabla_t' \cdot \left[Z' \vec{H}_0'\right] \tag{3.21}$$

$$0 = \nabla_t' \cdot \left[Z' \vec{H}_0'\right] = -\nabla_t' \cdot \left[Z' \nabla_t' \Phi_h'\right]$$

where again v' is factored through the transverse divergence, being a function of only u_3. Hence, requiring that Φ_h' satisfy a Laplace-type equation for an inhomogeneous medium (variable u') gives a two-dimensional potential which satisfies all three conditions ((3.18) – (3.20)).

4. Examples of Discrete Changes in Permeability and Permittivity

As an aid to understanding the properties of such generalized inhomogeneous TEM plane waves, let us consider some simple special cases. While μ' and ε' can have continuous variation with the coordinates, let us consider two simple cases of discrete variation. As illustrated in fig. 4.1, we can think of dividing a parallel-plate waveguide (infinitely wide) into two regions. Here we take two examples designated parallel and series. Both cases are approached from the electric-potential point of view.

We are still regarding the (u_1, u_2, u_3) coordinates as Cartesian as illustrated in fig. 4.1. In this case the formal fields and the real fields are the same. Note that the two regions (subscripts 1 and 2) have

$$\mu_1' \varepsilon_1' = \mu_2' \varepsilon_2' = v'^{-2} \tag{4.1}$$

as required by (3.10). Here we have a cross section as a plane of constant u_3 on which to solve our potential problem. Variation with u_3 of the above products is allowed, but does not affect the results.

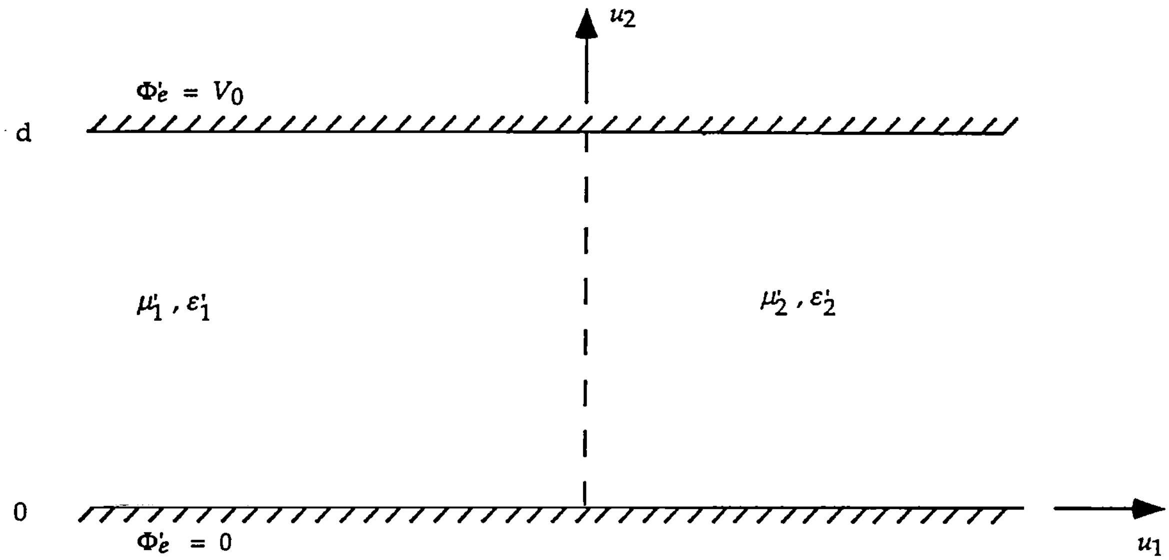
4.1 Two media in parallel

As our first example, fig. 4.1A shows a region bounded by two equipotential surfaces (perfectly conducting) as planes on $u_2 = 0, d$. The two media meet on the $u_1 = 0$ plane. The symmetry of the problem allows us to choose a solution of the form

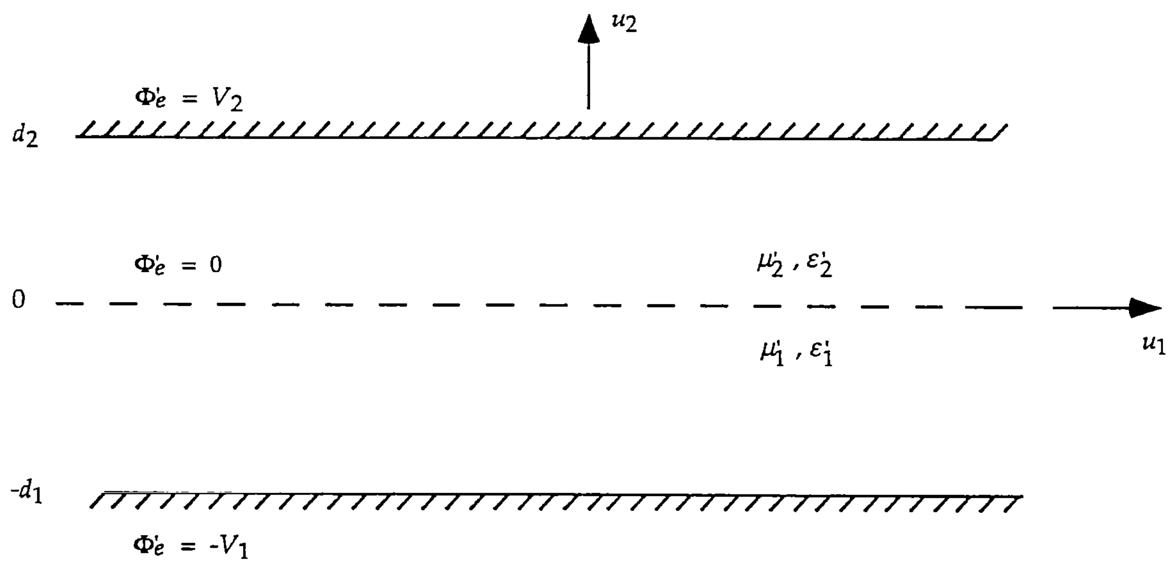
$$\begin{aligned} \Phi_e' &= \frac{u_2}{d} V_0 \\ \vec{E}_0' &= -\frac{V_0}{d} \vec{1}_2 \end{aligned} \tag{4.1}$$

in both media. Finishing out the solution we have

$$\begin{aligned} Z_1' &= \left[\frac{\mu_1'}{\varepsilon_1'}\right]^{1/2} \quad , \quad Z_2' = \left[\frac{\mu_2'}{\varepsilon_2'}\right]^{1/2} \\ \vec{H}_0 &= \begin{cases} Z_1'^{-1} \vec{1}_3 \times \vec{E}_0' = \dfrac{V_0}{dZ_1'} \vec{1}_1 & \text{for } u_1 < 0 \\ Z_2'^{-1} \vec{1}_3 \times \vec{E}_0' = \dfrac{V_0}{dZ_2'} \vec{1}_1 & \text{for } u_1 > 0 \end{cases} \end{aligned} \tag{4.2}$$



A. Two media in parallel



B. Two media in series

Fig. 4.1. Examples of Discrete Changes in μ' and ε'.

On the boundary ($u_1 = 0$) we have continuity of the tangential electric field as in (4.1) and continuity of the normal magnetic flux density as

$$\vec{B}_0' = \mu \vec{H}_0' = \begin{cases} \dfrac{\mu_1'}{Z_1'} \dfrac{V_0}{d} \vec{1}_1 = [\mu_1' \varepsilon_1']^{1/2} \dfrac{V_0}{d} \vec{1}_1 & \text{for } u_1 < 0 \\ \dfrac{\mu_2'}{Z_2'} \dfrac{V_0}{d} \vec{1}_1 = [\mu_2' \varepsilon_2']^{1/2} \dfrac{V_0}{d} \vec{1}_1 & \text{for } u_1 > 0 \end{cases} \tag{4.3}$$

So the Maxwell equations and boundary conditions are satisfied.

4.2 Two media in series

As our second example, fig. 4.1B shows a region bounded by two equipotential surfaces (perfectly conducting) as planes on $u_2 = -d_1, d_2$. The two media meet on the $u_2 = 0$ plane. Now we have a solution of the form

$$\Phi_e' = \begin{cases} \dfrac{u_2}{d_1} V_1 & \text{for } u_2 < 0 \\ \dfrac{u_2}{d_2} V_2 & \text{for } u_2 > 0 \end{cases}$$

$$\vec{E}_0 = \begin{cases} -\dfrac{V_1}{d_1} \vec{1}_2 & \text{for } u_2 < 0 \\ -\dfrac{V_2}{d_2} \vec{1}_2 & \text{for } u_2 > 0 \end{cases} \tag{4.4}$$

Making normal electric displacement continuous through the boundary ($u_2 = 0$) gives

$$\vec{D}_0' = \varepsilon' \vec{E}_0' = \begin{cases} \varepsilon_1' \dfrac{V_1}{d_1} & \text{for } u_2 < 0 \\ \varepsilon_2' \dfrac{V_2}{d_2} & \text{for } u_2 > 0 \end{cases}$$

$$\frac{V_1}{V_2} = \frac{\varepsilon_2'}{\varepsilon_1'} \frac{d_1}{d_2} \quad \text{(constraint on potentials)} \tag{4.5}$$

The magnetic field is then

$$\vec{H}_0' = \begin{cases} Z_1'^{-1}\, \vec{1}_3 \times \vec{E}_0' = \dfrac{V_1}{d_1 Z_1'}\, \vec{1}_1 = \dfrac{\varepsilon_1'}{[\mu_1'\, \varepsilon_1']^{1/2}} \dfrac{V_1}{d_1} \\ Z_2'^{-1}\, \vec{1}_3 \times \vec{E}_0' = \dfrac{V_2}{d_2 Z_2'}\, \vec{1}_1 = \dfrac{\varepsilon_2'}{[\mu_2'\, \varepsilon_2']^{1/2}} \dfrac{V_2}{d_2} \end{cases} \tag{4.6}$$

showing with (4.1) and (4.5) that tangential magnetic field is continuous through the boundary ($u_2 = 0$). Again the Maxwell equations and boundary conditions are satisfied.

5. Scaling Generalized Inhomogeneous TEM Plane Waves

Summarizing our generalized form of TEM wave we have

$$
\begin{aligned}
&\vec{E}' = \vec{E}_0'(u_1,u_2)\, f(t-(u_3)) \quad , \quad \vec{E}_0' \cdot \vec{1}_3 = 0 \\
&\vec{H}' = \vec{H}_0'(u_1,u_2)\, f(t-(u_3)) \quad , \quad \vec{H}_0' \cdot \vec{1}_3 = 0 \\
&\vec{1}_3 \times \vec{E}_0' = Z' \vec{H}_0' \quad , \quad \vec{H}_0 = -Z'^{-1}\, \vec{1}_3 \times \vec{E}_0' \\
&Z' = \left[\frac{\mu'}{\varepsilon'}\right]^{1/2} = Z'(u_1,u_2) \text{ (a function of only } u_1 \text{ and } u_2) \\
&v' = \left[\mu' \varepsilon'\right]^{1/2} = v'(u_3) \text{ (a function of only } u_3) \\
&\tau(u_3) = \int_0^{u_3} \frac{du_3'}{v'(u_3)} \\
&\vec{E}_0'(u_1,u_2) = -\nabla_t' \Phi_h(u_1,u_2) \quad , \quad \nabla_t' \cdot \left[\varepsilon' \nabla_t' \Phi_e'\right] = 0 \\
&\vec{H}_0'(u_1,u_2) = -\nabla_t' \Phi_h(u_1,u_2) \quad , \quad \nabla_t' \cdot \left[u' \nabla_t' \Phi_h'\right] = 0
\end{aligned}
\tag{5.1}
$$

Here Φ_h' may be multiple valued (as in a conformal transformation) as one goes around a conductor.

Now regard the fields as formal in the sense of [7] and let the (u_1, u_2, u_3) coordinates be some orthogonal curvilinear coordinate system. Then we have a distinction between ε' and ε and between μ' and μ in (2.2). The general transformation equations have the forms

$$
\begin{aligned}
&\vec{E}' = (\alpha_{n,m}) \cdot \vec{E} \quad , \quad \vec{H}' = (\alpha_{n,m}) \cdot \vec{H} \\
&(\varepsilon_{n,m}') = (\gamma_{n,m}) \cdot (\varepsilon_{n,m}) \quad , \quad (\mu_{n,m}') = (\gamma_{n,m}) \cdot (\mu_{n,m}) \\
&(\alpha_{n,m}) = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \quad , \quad (\gamma_{n,m}) = \begin{pmatrix} \frac{h_2 h_3}{h_1} & 0 & 0 \\ 0 & \frac{h_3 h_1}{h_2} & 0 \\ 0 & 0 & \frac{h_1 h_2}{h_3} \end{pmatrix}
\end{aligned}
\tag{5.2}
$$

This lets both the formal and real media be inhomogeneous but isotropic, noting that only 1, 1 and 2, 2 components are relevant since the 3-components of the fields are zero. The scale factors and line element are

$$
\begin{aligned}
h_n^2 &= \left[\frac{\partial x}{\partial u_n}\right]^2 + \left[\frac{\partial y}{\partial u_n}\right]^2 + \left[\frac{\partial z}{\partial u_n}\right]^2 \\
(d\ell)^2 &= \sum_{n=1}^{3} h_n^2 (du_n)^2
\end{aligned}
\tag{5.2}
$$

Since the first two diagonal elements of the constitutive-parameter dyadics are equal we have

$$
\mu' = \frac{h_2 h_3}{h_1}\mu = \frac{h_3 h_1}{h_2}\mu \quad , \quad \varepsilon' = \frac{h_2 h_3}{h_1}\varepsilon = \frac{h_3 h_1}{h_2}\varepsilon
\tag{5.3}
$$

which implies (taking the positive square root)

$$
h_1 = h_2
\tag{5.4}
$$

This, in turn implies [7 (Section 2.4), 6] that *surfaces of constant u_3 can only be spheres or planes.* Furthermore we have

$$
\begin{aligned}
\mu' &= h_3\mu \quad , \quad \varepsilon' = h_3\,\varepsilon \\
\frac{\mu'}{\varepsilon'} &= \frac{\mu}{\varepsilon} \qquad \text{(a function of } u_1 \text{ and } u_2 \text{ only)} \\
Z \equiv \left[\frac{\mu}{\varepsilon}\right]^{1/2} &= Z' \qquad \text{(a function of } u_1 \text{ and } u_2 \text{ only)} \\
v \equiv [\mu\,\varepsilon]^{-1/2} &= h_3[\mu'\,\varepsilon']^{-1/2} = h_3\,v'
\end{aligned}
\tag{5.5}
$$

While v' is a function of only u_3, v has the extra factor h_3 which can in general be a function of the various u_n.

6. Uniform Permeability Lenses

Now add the requirement that

$$\mu = \mu_0 \tag{6.1}$$

i.e., the permeability is uniform, typically taken as the permeability of free space for some practical lens applications. The permittivity is allowed to be more generally variable. For physical realizability we also require in the lens region that

$$\varepsilon \geq \varepsilon_{\min} \geq \varepsilon_0 \tag{6.2}$$

i.e., a permittivity bounded below by that of free space. Note that μ' and ε', being formal, do not have the same restrictions. For the wave speed we now have

$$v = [\mu_0 \varepsilon]^{-1/2} \leq [\mu_0 \varepsilon_0]^{-1/2} \equiv Z_0 \tag{6.3}$$

which is the basic causality restriction in the lens region. For the wave impedance we have

$$Z = \left[\frac{\mu_0}{\varepsilon}\right]^{1/2} \leq \left[\frac{\mu_0}{\varepsilon_0}\right]^{1/2} \equiv Z_0 \tag{6.4}$$

From (5.5) we have

$$Z = \left[\frac{\mu_0}{\varepsilon}\right]^{1/2} \Rightarrow$$
$$\varepsilon = \varepsilon(u_1, u_2) \quad \text{(a function of } u_1 \text{ and } u_2 \text{ only)} \tag{6.5}$$

Then we also have

$$v = [\mu_0 \varepsilon]^{-1/2} = v(u_1, u_2) \quad \text{(a function of } u_1 \text{ and } u_2 \text{ only)} \tag{6.6}$$

This restricts h_3 to the form

$$h_3 = \frac{v(u_1, u_2)}{v'(u_3)} \tag{6.7}$$

7. Example of Dielectric Bending Lens Based on Modified Cylindrical Coordinates

Consider the usual cylindrical coordinates (Ψ, ϕ, z) with

$$x = \Psi\cos(\phi) \quad , \quad y = \Psi\sin(\phi) \tag{7.1}$$

Then identify the (u_1, u_2, u_3) coordinates as

$$\begin{aligned} &u_1 = z \; , \; u_2 = \Psi \; , \; u_3 = \Psi_{max}\phi \\ &\vec{1}_1 = \vec{1}_2 , \; \vec{1}_2 = \vec{1}_\Psi , \; \vec{1}_3 = \vec{1}_\phi \end{aligned} \tag{7.2}$$

Note that ϕ has been scaled to $\Psi_{max}\phi$ to give it distance units (meters). The scale factors are

$$h_1 = h_2 = 1 \; , \; h_3 = \frac{\Psi}{\Psi_{max}} \tag{7.3}$$

satisfying the constraints (5.4) and (6.7).

For convenience set

$$v' \equiv c = [\mu_0\varepsilon_0]^{-1/2} = [\mu'\varepsilon']^{-1/2} \tag{7.4}$$

giving

$$v = h_3\, v' = h_3\, c = \frac{\Psi}{\Psi_{max}} c \le c \quad \text{for} \quad 0 < \Psi \le \Psi_{max} \tag{7.5}$$

So Ψ_{max} is the maximum Ψ for our lens domain. With our assumption of permeability μ_0 we have

$$\begin{aligned} \mu' &= h_3\,\mu_0 = \frac{\Psi}{\Psi_{max}}\,\mu_0 \\ \varepsilon' &= \frac{\mu_0}{\mu'}\varepsilon_0 = h_3^{-1}\,\varepsilon_0 = \frac{\Psi_{max}}{\Psi}\,\varepsilon_0 \\ \varepsilon &= h_3^{-1}\,\varepsilon' = h_3^{-2}\,\varepsilon_0 = \left[\frac{\Psi_{max}}{\Psi}\right]\varepsilon_0 \ge \varepsilon_0 \quad \text{for} \quad 0 < \Psi \le \Psi_{max} \\ Z &= Z' = \left[\frac{\mu_0}{\varepsilon}\right]^{1/2} = h_3 Z_0 = \frac{\Psi}{\Psi_{max}} Z_0 \le Z_0 \quad \text{for} \quad 0 < \Psi \le \Psi_{max} \end{aligned} \tag{7.6}$$

Note that while $\mu' < \mu_0$ in the lens region this is not important since here μ' is merely formal.

Now the formal fields can be found from an electric potential as

$$\begin{aligned} \vec{E}_0' &= -\nabla_t' \Phi_e' \quad , \quad \vec{H}_0' = \frac{1}{Z'} \vec{1}_\phi \times \vec{E}_0' \\ 0 &= \nabla_t' \cdot \left[\varepsilon' \nabla_t' \Phi_e' \right] \end{aligned} \tag{7.7}$$

With

$$\nabla_t' = \vec{1}_z \frac{\partial}{\partial z} + \vec{1}_\Psi \frac{\partial}{\partial \Psi} \tag{7.8}$$

the potential equation becomes

$$\begin{aligned} 0 &= \varepsilon_0 \frac{\Psi_{max}}{\Psi} \frac{\partial^2 \Phi_e'}{\partial z^2} + \frac{\partial}{\partial \Psi} \left[\varepsilon_0 \frac{\Psi_{max}}{\Psi} \frac{\partial \Phi_e'}{\partial \Psi} \right] \\ 0 &= \varepsilon \frac{\partial^2 \Phi_e'}{\partial z^2} + \Psi \frac{\partial}{\partial \Psi} \left[\Psi^{-1} \frac{\partial \Phi_e'}{\partial \Psi} \right] \end{aligned} \tag{7.9}$$

This is precisely the same equation as for ΨA_{0_ϕ} in [5], derived by an entirely different procedure. From (7.3) we also have the real fields

$$\vec{E}_0 = \vec{E}_0' \quad , \quad \vec{H}_0 = \vec{H}_0' \tag{7.10}$$

So the present differential-geometry scaling method gives the same results as the rotationally symmetric static-vector-potential.

Installing (perfect) conductors in a ϕ-independent manner over an appropriate range $\phi_1 \leq \phi \leq \phi_2$ then gives a TEM waveguiding structure through the inhomogeneous dielectric lens. From (7.8) one can solve for Φ_e' subject to choice of appropriate potentials (two or more) on the conductors. The fields then follow directly. In [5] this was discussed in the context of a bend in a coaxial cable, although it can apply to other forms of TEM structures. See this, for example, in the context of a different kind of bending lens with both μ and ε variable [7 (Section E.5)].

8. Concluding Remarks

The generalized form of a TEM plane wave involving inhomogeneous constitutive parameters can then be used for the formal fields for generating lens designs based on differential-geometric scaling. Among other things this allows one to have purely dielectric lenses (free space permeability). Of course one can have a dual case with a variable-permeability, constant-permittivity lens.

The current procedure has essentially given an extra degree of freedom on transient lens design via differential-geometric scaling. Perhaps other interesting examples of such lenses can also be developed utilizing this new kind of formal field.

References

1. C. E. Baum, D. V. Giri, and R. D. Gonzales, Electromagnetic Field Distribution of the TEM Mode in a symmetrical Two-Parallel-Plate Transmission Line, Sensor and Simulation Note 219, April 1976.

2. C. E. Baum, Two-Dimensional Inhomogeneous Dielectric Lenses for E-Plane Bends of TEM Waves Guided Between Perfectly Conducting Sheets, Sensor and Simulation Note 388, October 1995.

3. C. E. Baum, Dielectric Body-of-Revolution Lenses with Azimuthal Propagation, Sensor and Simulation Note 393, March 1996.

4. C. E. Baum, Dielectric Jackets as Lenses and Application to Generalized Coaxes and Bends in Coaxial Cables, Sensor and Simulation Note 394, March 1996.

5. C. E. Baum, Azimuthal TEM Waveguides in Dielectric Media, Sensor and Simulation Note 397, March 1996.

6. L. P. Eisenhart, *An Introduction to Differential Geometry*, Princeton U. Press, 1947.

7. C. E. Baum and A. P. Stone, *Transient Lens Synthesis: Differential Geometry in Electromagnetic Theory*, Hemisphere Publishing Corp. (Taylor & Francis), 1991.

8. E. G. Farr and C. E. Baum, Radiation from Self-Reciprocal Apertures, ch. 6, pp. 281–308, in C. E. Baum and H. N. Kritikos (eds.), *Electromagnetic Symmetry*, Taylor & Francis, 1995.