Sensor and Simulation Notes
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Some Considerations for Electrically-Small Multi-Turn Cylindrical Loops

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Abstract

This note discusses some considerations affecting the design of electrically-small multi-turn cylindrical loops. These considerations include symmetry in the loop turns, inaccuracy in the measurement of inhomogeneous magnetic fields, and the inductance and figure of merit of such a loop.

I. Introduction

A convenient geometry for an electrically-small $\dot{B}$ loop is a circular cylinder in which the currents flow in the azimuthal (ϕ) direction. As an idealization of such a structure consider a uniform cylindrical current sheet as illustrated in figure 1. This current sheet has length, ℓ, and radius, a, and is centered on a cylindrical (r,ϕ,z) coordinate system. The azimuthal surface current density is J_s which is independent of position on the current sheet. This current sheet is used to approximate the current distribution in an electrically-small multi-turn cylindrical loop. By electrically-small we mean that all frequencies of interest are small enough that the corresponding wavelengths or skin depths, as appropriate, in the media in the immediate vicinity of the loop are much larger than the loop dimensions. We also assume that the permeability is a uniform scalar constant in the immediate vicinity of the loop.

The turns of the cylindrical loop are assumed to be uniformly distributed along the axis of the loop and interconnected in a manner such that the current in the loop turns approximates the uniform cylindrical current sheet. There are various configurations of the turns which meet such criteria. Some of these configurations are presented in this note. These configurations have varying degrees of symmetry in them. Use of turn geometries with higher orders of symmetry may be advantageous in rejecting signals associated with unwanted electromagnetic field components.

There may be circumstances in which the z component of the magnetic field is significantly inhomogeneous in the vicinity of the loop. The response to such an inhomogeneous field is then considered so as to find an optimum length-to-diameter ratio which minimizes the error associated with the measurement of an inhomogeneous field.

The equivalent area of an electrically-small multi-turn cylindrical loop is just $N\pi a^2$ where N is the number of series-connected turns. For sufficiently low frequencies this equivalent area is quite accurate; inaccuracies are related to the dimensional inaccuracies of the loop. The loop inductance, however, is somewhat more complicated. In this note we approximate the inductance of a multi-turn cylindrical loop by using the inductance of a uniform cylindrical current sheet, an accurate formula being known for this. With the equivalent area and inductance we then calculate the equivalent volume and the spherical figure of merit. Maximizing the spherical figure of merit determines another optimum length-to-diameter ratio.

II. Geometry of Loop Turns

Consider first some of the possible configurations for the loop turns. Figure 2A shows an example of a 4 turn cylindrical loop. The loop turns are cut at a particular ϕ and laid out to give the expanded view in figure 2B. This type of loop winding is roughly a single helix except that the individual turns each lie in a plane; the signal terminals are connected to the ends of the helix. Such a turn configuration is a rather straightforward one, but it may have disadvantages. Ideally, the turn configuration is sensitive only to the time derivative of the component of the magnetic field parallel to the loop axis. However, if there is an electric field component

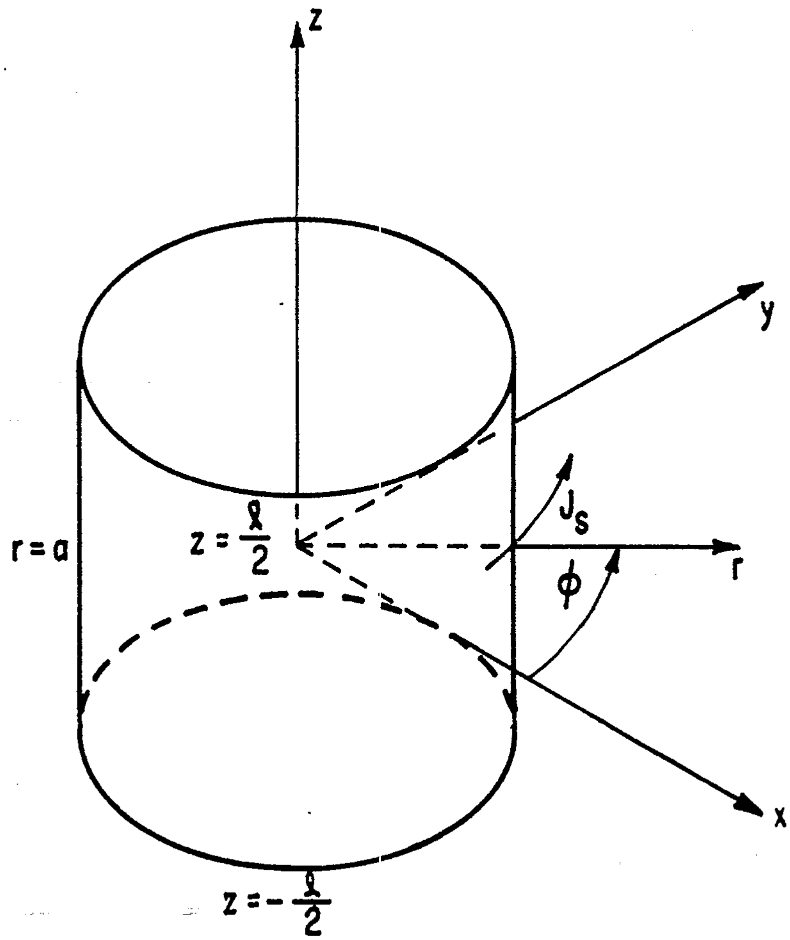
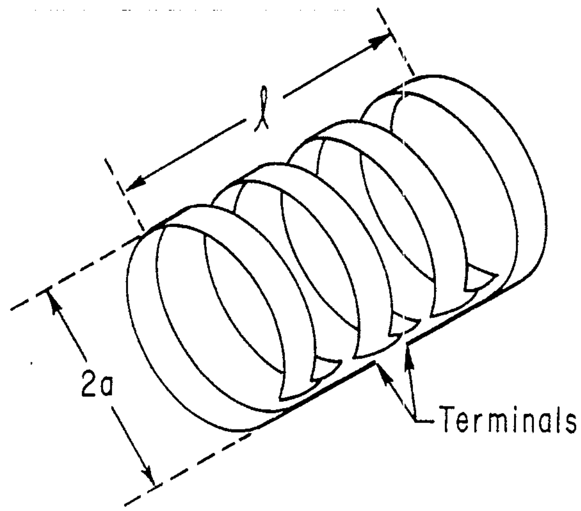
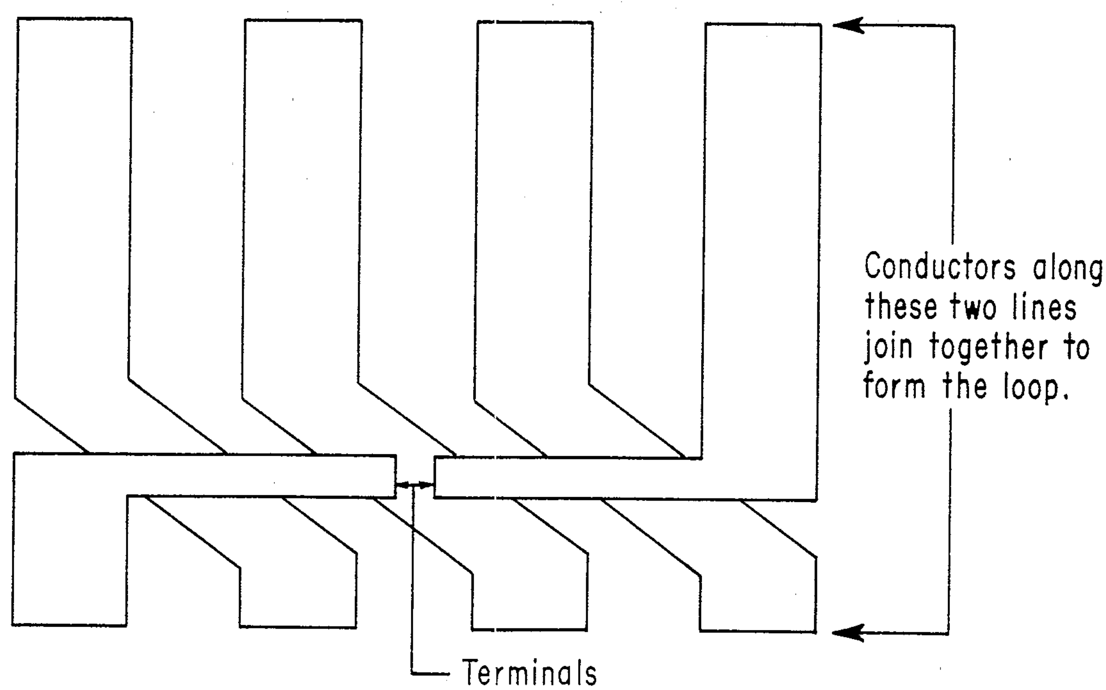


FIGURE 1 GEOMETRY OF UNIFORM CURRENT SHEET



A. Angular view



B. Expanded view

FIGURE 2. MULTI-TURN LOOP GEOMETRY: SINGLE HELIX

parallel to the loop axis, one would expect a signal associated with this field component to be produced at the loop terminals. Such a signal is produced because the ends of the loop winding are physically separated in the axial direction. The signal induced by the axial electric field can be shorted through the loop turns but there is an impedance associated with the loop inductance so that some signal is left to appear at the loop terminals.

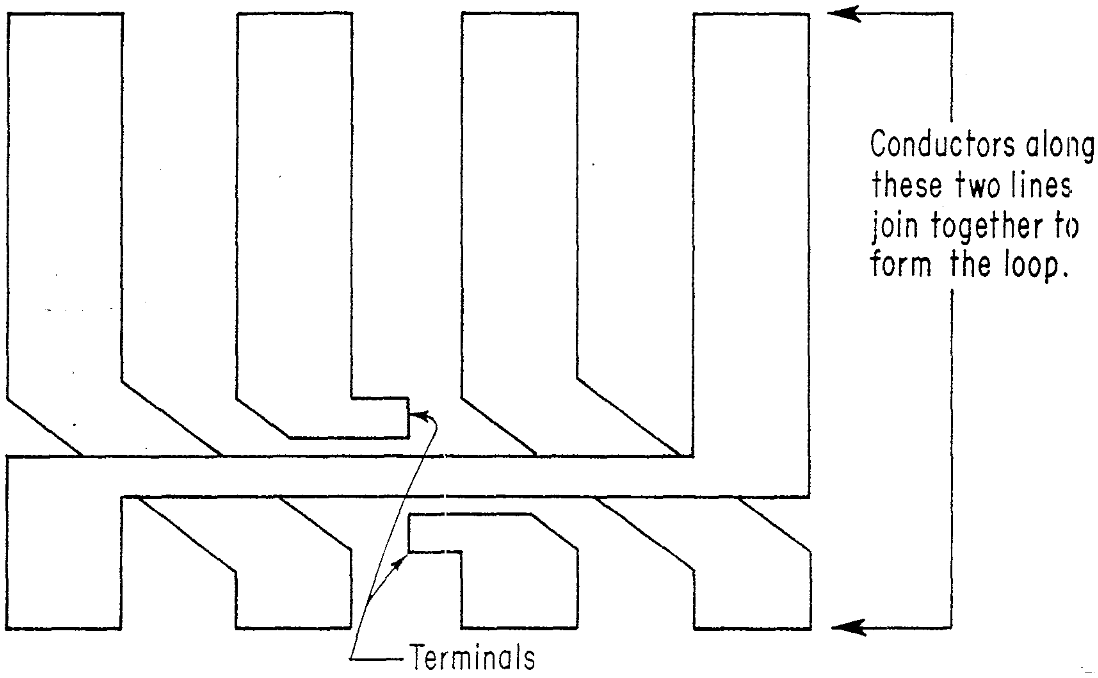
An alternative turn configuration is that in figure 3A. Here the ends of the helix are shorted together and the turns are separated in the center of the winding to give the two signal terminals. In this configuration the farthest separated loop turns are connected together with an impedance which is smaller in magnitude than the impedance associated with the loop inductance. The signal associated with an axial electric field should thereby be reduced.

Another turn configuration, with somewhat more complexity, is that shown in figure 3B. This case, as the previous two, is illustrated for 4 turns, but can be generalized to various numbers of turns. In this configuration a degree of symmetry has been added. Two helices are wound in opposite directions. Each end of one helix is joined to an end of the other helix. The two positions at which the two helices are thus joined at each end of the cylindrical structure are also connected together by another conductor. The turns in each helix are physically separated at their centers and the two helices joined to each other at this position to give the two signal terminals. We thus have what might be termed two counterwound helices with the signal removed from the centers of both helices.

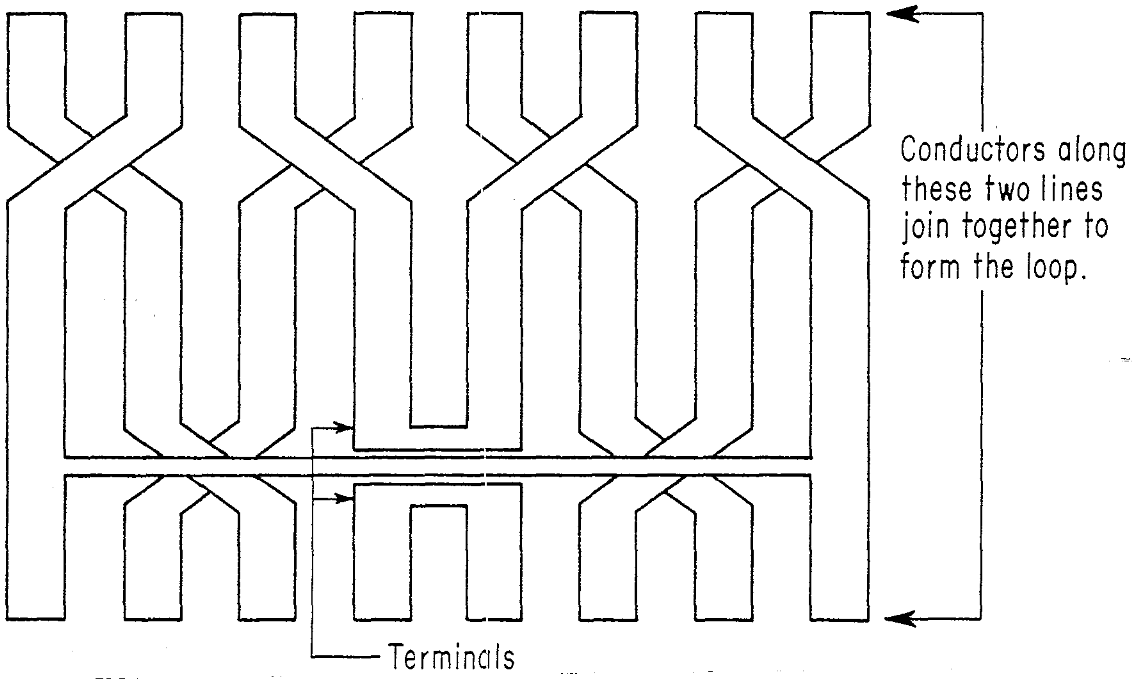
Another way to look at the winding geometry in figure 3B is to think of it as two parallel windings. Each winding starts in the center of the structure, progresses to one end, and returns to the center. Considered this way, the center position on each winding is at one end of the structure; these center positions on the two windings can be joined together as illustrated in figure 3B.

By using two counterwound helices we have more symmetry in the loop structure. This symmetry would not be perfect due to things such as dimensional imperfections. An axial electric field still sets up currents on the loop structure, but ideally, due to the symmetry, these do not appear as signals at the signal terminals. One way to reduce these currents would be to connect the two ends of the winding structure together as discussed above, thereby providing an additional path for such currents which is not directly connected to the signal terminals. There are various similar additional conductors which might be added to reduce unwanted signals, but these are not discussed here.

There are other techniques for improving the response of such multiturn cylindrical loops. In some cases one might remove the signal differentially and connect the position(s) midway on the loop turns to a third conductor such as a signal cable shield. One might also use various types of shields around the loop structure. In any event it may be desirable to put various types of symmetry into the loop turns. Counterwinding is one type of symmetry; perhaps other types of symmetry may be advantageous as well.



A. Single helix with terminals in center of winding



B. Two counterwound helices with terminals in center of winding

FIGURE 3. ALTERNATIVE MULTI-TURN LOOP GEOMETRIES: EXPANDED VIEWS

III. Response to an Inhomogeneous Magnetic Field

Referring back to figure 1, the cylindrical loop ideally responds only to the z component of the magnetic field. This loop is assumed to have N turns and a uniform number of turns per unit length, N/ℓ. Suppose, however, that the z component of the magnetic field is a significant function of position in the immediate vicinity of the loop. The loop responds to the time derivative of the magnetic flux linking all the loop turns; it does not necessarily respond to the time derivative of the magnetic field at a point. However, there is an optimum length-to-diameter ratio which allows one to consider the loop as approximately responding to the time derivative of the magnetic field at the center of the loop.

Assume then that the loop does not itself distort the magnetic field appreciably and that there are no sources of the magnetic field in the immediate vicinity of the loop. The frequencies are assumed low enough that the magnetic field in the immediate vicinity of the loop is a solution of Laplace's equation. The magnetic field can then be obtained from a potential function. In cylindrical coordinates this potential function can be represented as a sum of terms of the form

$$\psi_{n,\nu} = a_{n,\nu} \, J_n(\nu r) \begin{Bmatrix} \sin(n\phi) \\ \\ \cos(n\phi) \end{Bmatrix} e^{\nu z} \tag{1}$$

where n is zero or a positive integer and ν can be a general complex number. Some linear combination of $\sin(n\phi)$ and $\cos(n\phi)$ is assumed. For $\nu = 0$, equation (1) is replaced by a somewhat simpler form which is not of interest here. The z component of the magnetic field is then a sum of the form

$$b_{n,\nu} = \frac{\partial \psi_{n,\nu}}{\partial z} = a_{n,\nu} \, \nu J_n(\nu r) \begin{Bmatrix} \sin(n\phi) \\ \\ \cos(n\phi) \end{Bmatrix} e^{\nu z} \tag{2}$$

Note that the only terms which contribute to the magnetic field on the loop axis $(r = 0)$ are those for $n = 0$.

The flux linking the uniformly distributed turns of the cylindrical loop is then a sum of terms of the form

$$\Phi_{n,\nu} = \frac{N}{\ell} \int_0^a \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \int_0^{2\pi} b_{n,\nu} \, r \, d\phi \, dz \, dr \tag{3}$$

or

$$\Phi_{n,\nu} = \frac{N}{\ell} a_{n,\nu} \, \nu \int_0^a \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \int_0^{2\pi} J_n(\nu r) \begin{Bmatrix} \sin(n\phi) \\ \\ \cos(n\phi) \end{Bmatrix} e^{\nu z} \, r \, d\phi \, dz \, dr \tag{4}$$

Integrate first over ϕ. Note that the only nonzero result is for $n = 0$, for which case we define the trigonometric expression as one for convenience. Then for $n = 0$ we have

$$\Phi_{o,\nu} = \frac{N}{\ell} a_{o,\nu} \, 2\pi\nu \int_{o}^{a} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} J_o(\nu r) e^{\nu z} \, r \, dz \, dr \tag{5}$$

Integrating over z gives

$$\Phi_{o,\nu} = \frac{N}{\ell} a_{o,\nu} \, 4\pi \sinh\left(\nu\frac{\ell}{2}\right) \int_{o}^{a} J_o(\nu r) \, r \, dr \tag{6}$$

and integrating over r gives

$$\Phi_{o,\nu} = \frac{N}{\ell} a_{o,\nu} \, 4\pi \sinh\left(\nu\frac{\ell}{2}\right) a \frac{J_1(\nu a)}{\nu} \tag{7}$$

Note that these terms contributing to the flux are the same terms that are associated with the magnetic field on the loop axis.

Ideally the flux could be simply related to the magnetic field at the center of the loop. For $(r,z) = (0,0)$ the nonzero terms in the magnetic field expansion are of the form

$$b_{o,\nu}\Big|_{\substack{r=0\\z=0}} = a_{o,\nu} \, \nu \tag{8}$$

Dividing this last expression into $\Phi_{o,\nu}$ gives an equivalent area of the form

$$A_{o,\nu} = \frac{\Phi_{o,\nu}}{a_{o,\nu}\nu} = \frac{N4\pi a}{\ell\nu^2} \sinh\left(\nu\frac{\ell}{2}\right) J_1(\nu a) \tag{9}$$

If $A_{o,\nu}$ were independent of ν, then the total flux linked by the loop would be simply related to the magnetic field at the center of the loop. If the magnetic field in the vicinity of the loop is not too inhomogeneous, i.e., if $|\nu\ell/2| \ll 1$ and $|\nu a| \ll 1$ for significant terms in the field expansion, then the transcendental functions in equation (9) can be expanded for small arguments giving

$$A_{o,\nu} \simeq \frac{N4\pi a}{\ell\nu^2}\left[\left(\frac{\nu\ell}{2}\right)+\frac{1}{6}\left(\frac{\nu\ell}{2}\right)^3\cdots\right]\left[\left(\frac{\nu a}{2}\right)-\frac{1}{2}\left(\frac{\nu a}{2}\right)^3\cdots\right]$$

$$\simeq N\pi a^2\left[1+\frac{1}{6}\left(\frac{\nu\ell}{2}\right)^2\cdots\right]\left[1-\frac{1}{2}\left(\frac{\nu a}{2}\right)^2\cdots\right]$$

$$\simeq N\pi a^2\left[1+\frac{1}{6}\left(\frac{\nu\ell}{2}\right)^2-\frac{1}{2}\left(\frac{\nu a}{2}\right)^2\cdots\right] \qquad (10)$$

This expansion includes terms up through ν^2. There are no odd powers of ν present. Defining ξ as the length-to-diameter ratio, note that if we choose the special case of

$$\xi \equiv \frac{\ell}{2a} = \frac{\sqrt{3}}{2} \simeq .866 \qquad (11)$$

then the ν^2 term is also zero. This choice of ξ can then be considered an optimum choice for reducing the dependence of the loop sensitivity on the degree of homogeneity of the magnetic field.

IV. Inductance and Figure of Merit

Next consider the loop inductance. Idealize the current distribution in the multi-turn cylindrical loop as a uniform cylindrical current sheet as illustrated in figure 1. The inductance of such a loop is given by[1,2,3]

$$L = N^2\frac{\mu}{3}\sqrt{\ell^2+(2a)^2}\left[K(m)-E(m)+\left(\frac{2a}{\ell}\right)^2(E(m)-\sqrt{m})\right] \qquad (12)$$

where

$$m = 1-m_1 = \left[1+\left(\frac{\ell}{2a}\right)^2\right]^{-1} \qquad (13)$$

and where μ is the permeability of the medium in which the loop is placed. Using the notation for the length-to-diameter ratio, as in equation (11), the inductance is

$$L = N^2\mu\pi a\,\frac{2}{3\pi\xi^2}\left[(1+\xi^2)^{1/2}(1-\xi^2)E(m)+\xi^2(1+\xi^2)^{1/2}K(m)-1\right] \qquad (14)$$

where

$$m = 1-m_1 = \left[1+\xi^2\right]^{-1} \qquad (15)$$

1. C. Snow, NBS Circular 544, Formulas for Computing Capacitance and Inductance, p. 31, Sept 1954.
2. See AMS 55, Handbook of Mathematical Functions, National Bureau of Standards, 1964, for the notation regarding the elliptic integrals.
3. All units are rationalized MKSA.

The inductance can be put into a convenient form as

$$L = N^2\mu \frac{\pi a^2}{\ell} f(\xi) \tag{16}$$

where

$$f(\xi) = \frac{4}{3\pi\xi} \left[(1+\xi^2)^{1/2} (1-\xi^2)E(m)+\xi^2(1+\xi^2)^{1/2} K(m)-1\right] \tag{17}$$

and where $f(\xi)$ is a dimensionless factor which is graphed in figure 4.

The mathematical form of the inductance simplifies somewhat for both large and small ξ. Consider first the case of large ξ, or small m, for which we expand $f(\xi)$ in powers of m. Rewrite $f(\xi)$ as

$$f(\xi) = \frac{4}{3\pi\xi} \left[\frac{2m-1}{m^{3/2}} E(m) + \frac{1-m}{m^{3/2}} K(m) -1\right] \tag{18}$$

Then expand the elliptic integrals for small m, giving

$$\begin{aligned} f(\xi) &\approx \frac{4}{3\pi\xi}\left\{m^{-3/2}\left[(2m-1)\frac{\pi}{2}(1-\tfrac{1}{4}m\ldots)+(1-m)\frac{\pi}{2}(1+\tfrac{1}{4}m\ldots)\right]-1\right\} \\ &\approx \frac{4}{3\pi\xi}\left\{\frac{\pi}{2}m^{-3/2}\left[\frac{3}{2}m\ldots\right]-1\right\} \\ &= \frac{4}{3\pi\xi}\left[\frac{3\pi}{4m^{1/2}}-1\right] \\ &= \frac{4}{3\pi\xi}\left[\frac{3\pi}{4}(1+\xi^2)^{1/2}-1\right] \\ &\approx \frac{4}{3\pi\xi}\left[\frac{3}{4}\pi\xi-1\right] \\ &= 1-\frac{4}{3\pi\xi} \end{aligned} \tag{19}$$

This is included for its appropriate range of ξ in figure 4. For large ξ the inductance is then approximately

$$L \approx N^2\mu\frac{\pi a^2}{\ell}\left[1-\frac{8a}{3\pi\ell}\right] \tag{20}$$

Rewrite $f(\xi)$ in terms of m_1 as

$$f(\xi) = \frac{4}{3\pi}\frac{1}{\sqrt{m_1}}\left[\frac{1-2m_1}{1-m_1}E(m)+\frac{m_1}{1-m_1}K(m)-(1-m_1)^{1/2}\right] \tag{21}$$

Expanding the Elliptic integrals for small ξ, or small m_1, gives[4]

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4. These expressions can be found in H. B. Dwight, Tables of Integrals and Other Mathematical Data, 4th ed., pp. 180-181, 1965.

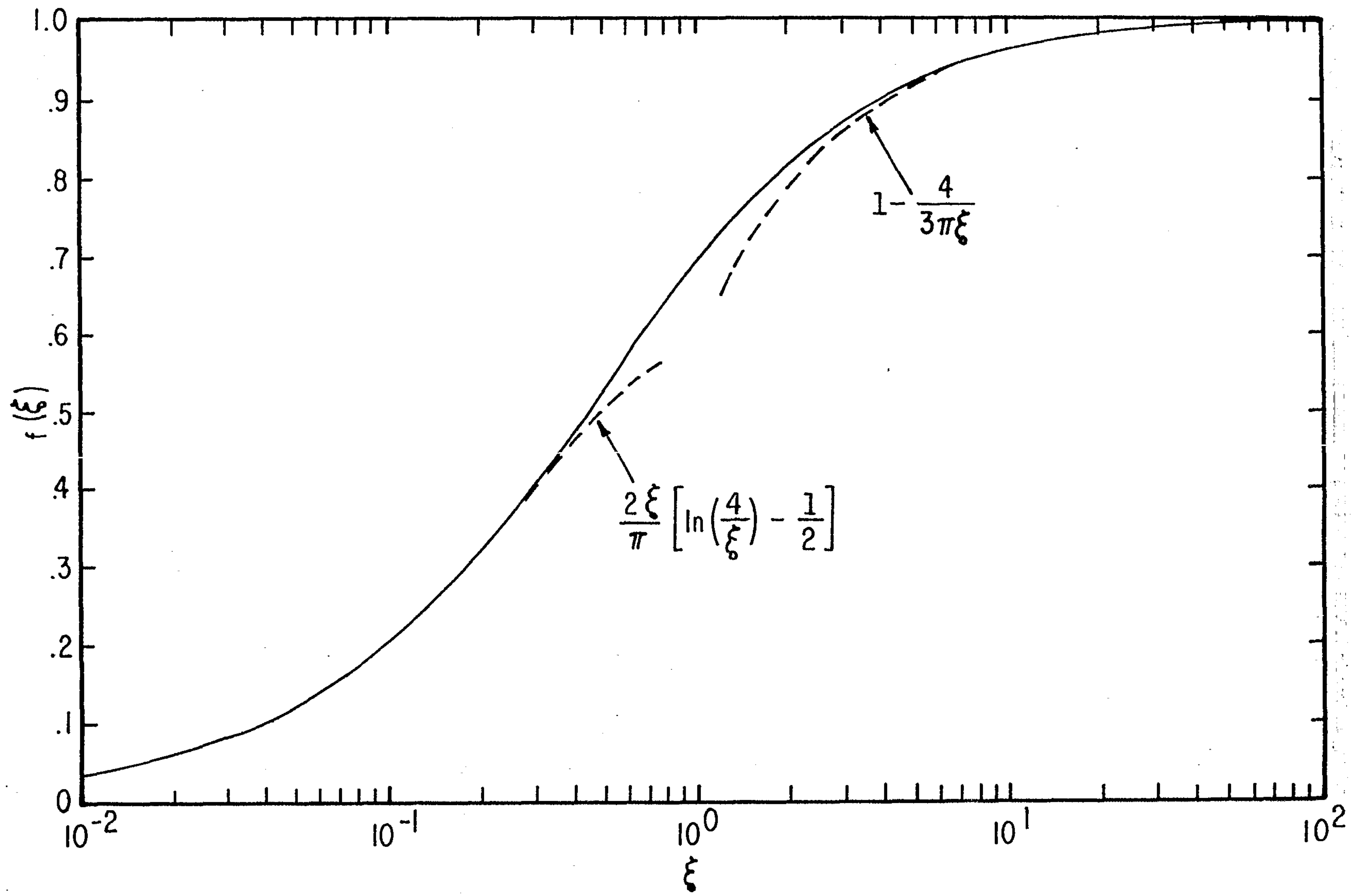

FIGURE 4 DIMENSIONLESS FACTOR IN LOOP INDUCTANCE

$$f(\xi) \simeq \frac{4}{3\pi}\frac{1}{\sqrt{m_1}}\left\{\frac{1-2m_1}{1-m_1}\left[1+\frac{m_1}{4}\left[\ln\left(\frac{16}{m_1}\right)-1\right]\ldots\right]+\frac{m_1}{1-m_1}\left[\frac{1}{2}\ln\left(\frac{16}{m_1}\right)\ldots\right]-(1-m_1)^{1/2}\right\}$$

$$\simeq \frac{4}{3\pi}\frac{1}{\sqrt{m_1}}\left\{(1-m_1\ldots)\left[1+\frac{m_1}{4}\left[\ln\frac{16}{m_1}-1\right]\right]+(m_1\ldots)\frac{1}{2}\ln\left(\frac{16}{m_1}\right)-(1-\frac{m_1}{2}\ldots)\right\}$$

$$\simeq \frac{4}{3\pi}\frac{1}{\sqrt{m_1}}\left\{1+\frac{m_1}{4}\ln\left(\frac{16}{m_1}\right)-\frac{5}{4}m_1+\frac{m_1}{2}\ln\left(\frac{16}{m_1}\right)-1+\frac{m_1}{2}\right\}$$

$$= \frac{4}{3\pi}\sqrt{m_1}\left[\frac{3}{4}\ln\left(\frac{16}{m_1}\right)-\frac{3}{4}\right]$$

$$= \frac{\sqrt{m_1}}{\pi}\left[\ln\left(\frac{16}{m_1}\right)-1\right]$$

$$\simeq \frac{\xi}{\pi}\left[\ln\left(\frac{16}{\xi^2}\right)-1\right]$$

$$= \frac{2\xi}{\pi}\left[\ln\left(\frac{4}{\xi}\right)-\frac{1}{2}\right] \tag{22}$$

This is also included for its appropriate range of ξ in figure 4. For small ξ the inductance is then approximately

$$L \simeq N^2\mu a\left[\ln\left(\frac{8a}{\ell}\right)-\frac{1}{2}\right] \tag{23}$$

The equivalent area of the multi-turn cylindrical loop is given by

$$A_{eq} = N\pi a^2 \tag{24}$$

With the equivalent area and the inductance we can now calculate the equivalent volume as[5]

$$V_{eq} = \frac{\mu A_{eq}^2}{L} = \frac{\pi a^2 \ell}{f(\xi)} \tag{25}$$

This equivalent volume is a measure of the efficiency of the sensor in removing energy from the electromagnetic fields. Note in equation (25) that $\pi a^2 \ell$ is just the geometric volume of the cylindrical loop.

5. Capt Carl E. Baum, Sensor and Simulation Note XXXVIII, Parameters for Some Electrically-Small Electromagnetic Sensors, March 1967.

Consider a sphere of radius, r_o, with a volume $\frac{4}{3}\pi r_o^3$. Make r_o just large enough so that the cylindrical loop can fit inside it. This gives

$$r_o = \left(a^2 + \left(\frac{\ell}{2}\right)^2\right)^{1/2} = a\left(1 + \xi^2\right)^{1/2} \tag{26}$$

We then have a figure of merit based on a sphere as

$$\eta_s = \frac{V_{eq}}{\frac{4}{3}\pi r_o^3} = \frac{3}{2}\,\frac{\xi}{(1+\xi^2)^{3/2}\,f(\xi)} \tag{27}$$

One can think of this spherical figure of merit as a measure of the efficiency of the cylindrical loop in filling a given volume of space. Using a sphere for this volume is convenient but somewhat arbitrary. In figure 5 there is plotted η_s versus ξ. At $\xi \simeq .459$ then $\eta_s \simeq 1.024$ and it is at a maximum. Thus, an optimum spherical figure of merit occurs with the length about equal to the radius. At $\xi = \frac{\sqrt{3}}{2}$, an optimum value from the point of view of measurements of inhomogeneous magnetic fields, then $\eta_s \simeq .855$, a little less than the maximum value.

Note that the formula for the inductance of a uniform cylindrical current sheet only approximately applies to a practical multi-turn cylindrical loop. The same inaccuracy carries over into the equivalent volume and figure of merit.

V. Summary

This note has discussed some considerations in the design of electrically-small multi-turn cylindrical loops. The geometry of the loop turns can be designed with various degrees of symmetry to minimize the sensitivity of the loop to unwanted electromagnetic field components. The error introduced in measurements of inhomogeneous magnetic fields can be minimized by an appropriate choice of the length-to-diameter ratio. The inductance has been calculated using the approximation that the currents in the loop are uniformly distributed in a cylindrical sheet. Using this inductance the spherical figure of merit has been calculated. The spherical figure of merit has a maximum for the length about equal to the radius of the loop.

We would like to thank Mr. John N. Wood for the numerical calculations and the resulting graphs.

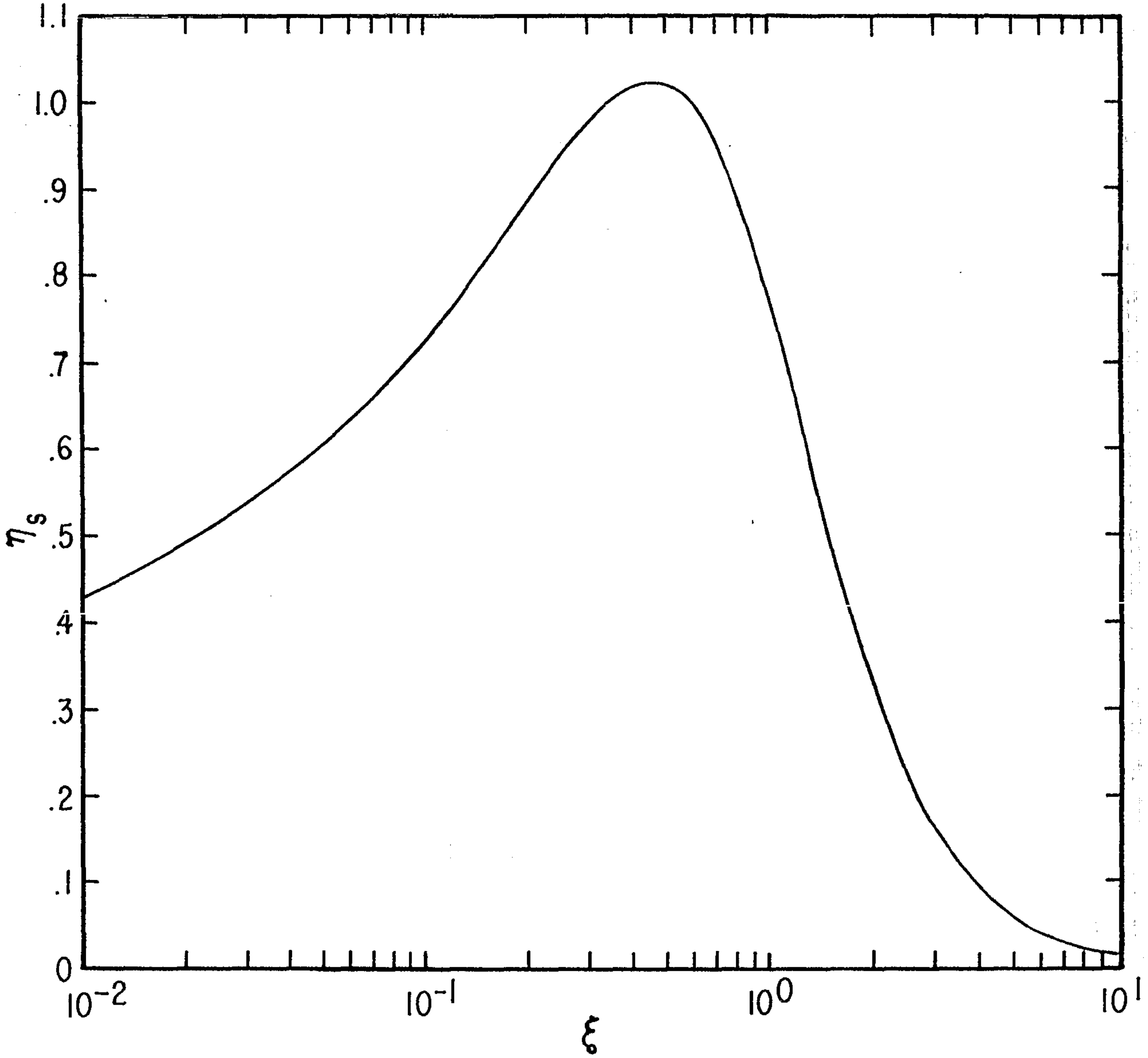

FIGURE 5 SPHERICAL FIGURE OF MERIT