Sensor and Simulation Notes

Note L

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# The Buried Transmission Line Simulator With an Inductive Energy Source

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Abstract

The time history of the input current and voltage of a buried transmission line is calculated under the assumption that the line is excited by an inductive energy source. Results are displayed for both the case where the buried line is infinitely long and the case where it is of finite length and terminated by an open circuit.

## I. Introduction

In a previous note in this series[1] a buried parallel-plate transmission line was proposed as a device for simulating the longer time behavior of the electromagnetic pulse from a nuclear burst. Included in that note were computations of the response function of the buried line in both the frequency and time domains. In later notes of the series the current waveforms achievable by some realistic energy sources for this method of simulation were computed. One of these later notes[2] contains a discussion and a computation of the magnetic field and current density as a function of time and position which may be expected when the buried line is driven by a single capacitor. It was found that, for typical ground conductivities, a resistor is needed in series with the line in order that the current pulse should even roughly approximate the desired shape. In fact, without a series resistor the input current was found to be oscillatory. Since such a resistor is wasteful of the available energy, a subsequent note[3] described and analyzed an excitation scheme employing several capacitors which may be switched into a circuit at appropriate times in order to provide flexibility in the synthesis of pulse shapes without undue energy dissipation in the source.

Because it is desirable, in making design choices, to have available quantitative data on more than one excitation scheme, in this note computations are made and data displayed concerning the waveforms to be expected when the buried line is excited by an inductive energy source. For the very large energy levels that may be necessary in the future, an inductive energy storage method could indeed be preferable to a method employing capacitors. It is intended to discuss, in a future note, the increase in simulation possibilities and the reduction in switching problems that occur when the energy source contains both capacitive and inductive elements.

The same assumptions that have been made in previous analyses of the buried line simulator will be made in this note. Primarily these assumptions are:

1) Transmission-line theory provides an adequate description of events throughout the whole length of the line, including the excitation and "termination" regions.

2) The electromagnetic parameters of the ground are independant of frequency and position.

3) The times of interest, and ground conductivities of interest, are such that in the Laplace transform domain the characteristic impedance of the ground may be adequately approximated by

$$Z(s) = \sqrt{s\mu/\sigma}$$

Here and henceforth, MKS units and notation are used.

Based on these assumptions, a calculation is made in the next section of the input current on a buried line of infinite length with an inductive source, while in Section III similar computations are made for a finite-length "open-circuited" line. Among other comments Section IV contains a few remarks concerning the validity of the first assumption stated above.

## II. The Infinite Buried Line

The situation to be analysed is represented schematically in figures 1A and 1B. Initially the current through the inductor flows through the short circuit at the switch. At time zero the switch is opened, forcing the inductor current into the buried line. The practical problems involved in instantaneously opening the switch will not be considered in this note. After the switch has been opened the equivalent circuit of the structure takes the simple form shown in figure 1C, where some initial value of the current, i(0), must be assumed.

Only the current and the voltage across the line impedance in the equivalent circuit of figure 1C will be computed. The magnetic field at the surface of the ground will be proportional to the current calculated here. At any depth below the surface the magnetic field may be related to the surface field by a response function in the time domain[1]. Similar remarks apply to the electric field and the voltage.

The input impedance of the infinite buried line is directly proportional to the characteristic impedance of the ground. Denoting the proportionality constant, which depends on the geometry of the line, by $f_g$ one may write

$$Z_L(s) = f_g\sqrt{s\mu/\sigma} \quad . \tag{1}$$

Since the equation for the Laplace transform of the current in the circuit of figure 1C is clearly

$$sLI(s) + Z_L(s)I(s) = Li(0) \quad , \tag{2}$$

one may use equation (1) in conjunction with this to write the current in the time domain immediately in the form

$$i(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\left(\frac{Li(0)}{sL + f_g\sqrt{s\mu/\sigma}}\right)ds \quad . \tag{3}$$

Defining a normalized time by

$$\tau_L = t\left(\frac{f_g}{L}\right)^2 \frac{\mu}{\sigma}$$

and changing the variable of integration in (3), one may represent the normalized current by

$$h_L(\tau_L) \equiv \frac{i(t)}{i(0)} = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{e^{p\tau_L} dp}{p + \sqrt{p}} \quad . \tag{4}$$

This integral is a standard Laplace transorm[4] so $h_L(\tau_L)$ may be represented in terms of tabulated functions in the following manner

$$h_L(\tau_L) = e^{\tau_L} \mathrm{erfc}(\sqrt{\tau_L}) \tag{5}$$

where

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-y^2} dy \quad .$$

In a similar manner, normalizing the voltage across the load, $v(t)$, to a voltage defined by

$$v_L = \frac{f_g^2 \mu}{L\sigma} i(0)$$

it is straightforward to establish that

$$e_L(\tau_L) = \frac{v(t)}{v_L} = \frac{1}{\sqrt{\pi\tau_L}} - e^{\tau_L} \mathrm{erfc}(\sqrt{\tau_L}) \quad . \tag{6}$$

Curves of equations (5) and (6) are given in figures 2 and 3 along with some of the following asymptotic forms:

$$h_L(\tau_L) \to 1 \qquad \text{as} \quad \tau_L \to 0$$

$$\to \frac{1}{\sqrt{\pi\tau_L}} \qquad \text{as} \quad \tau_L \to \infty$$

$$e_L(\tau_L) \to \frac{1}{\sqrt{\pi\tau_L}} \qquad \text{as} \quad \tau_L \to 0$$

$$\to \frac{1}{2\tau_L\sqrt{\pi\tau_L}} \qquad \text{as} \quad \tau_L \to \infty \quad .$$

It can be seen from figure 2A that, although the overall shape of the current pulse is preferable to that from a source consisting of a single capacitor, there is a rapid drop for small times that is definitely undesirable.

## III. The Open-Circuited Line

In this section the buried line is assumed to be of finite length and the reflection from the bottom of the line is accounted for by assuming transmission-line theory to apply with the line terminating in an open circuit. This leads at once to the equation for the input impedance of the line in the Laplace transform domain,

$$Z_L(s) = f_g\sqrt{s\mu/\sigma}\ \coth\sqrt{\mu\sigma\ell^2 s} \quad . \tag{7}$$

Substituting this value in equation (2) and introducing a new dimensionless time variable by normalizing the actual time with respect to the diffusion time of the line (defined by $t_\ell = \mu\sigma\ell^2/4$) one is led to the equation analagous to (4)

$$h_\ell(\tau_\ell) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{\tau_\ell p/4}\,dp}{p + 4(t_\ell/t_o)\sqrt{p}\ \coth\sqrt{p}} \quad . \tag{8}$$

In equation (8)

$$\tau_\ell = t/t_\ell$$

and if $R_o$ is the D.C. resistance of the line neglecting end effects, i.e.

$$R_o = f_g/\ell\sigma \quad ,$$

then

$$t_o = L/R_o \quad .$$

Equation (8) may be integrated in more than one way, leading to various equivalent representations of $h_\ell(\tau_\ell)$ . Some of these representations are more suited to describe the small time behavior of the function while others are better adapted to describe the large time behavior. In the following two sub-sections examples of each type of representation are obtained.

IIIA. Small Time Representation

As a result of substituting

$$p = z^2$$

$$\lambda = t_o/4t_\ell$$

in equation (8) one obtains

$$h_\ell(\tau_\ell) = \frac{\lambda}{\pi i} \int_{C_1} \frac{e^{\tau_\ell z^2/4} dz}{\lambda z + \coth z} \tag{9}$$

where the contour of integration, $C_1$ , is shown in figure 4A. Now expanding the integrand in powers of

$$\left(\frac{\lambda z - 1}{\lambda z + 1}\right) e^{-2z}$$

one obtains after a little manipulation

$$h_\ell(\tau_\ell) = \frac{\lambda}{\pi i} \int_{C_1} \frac{e^{\tau_\ell z^2/4}}{1 + \lambda z} dz + \frac{2\lambda}{\pi i} \int_{C_1} \frac{e^{\tau_\ell z^2/4}}{1 - \lambda^2 z^2} \left[\sum_{n=1}^{\infty} \left(\frac{\lambda z - 1}{\lambda z + 1}\right)^n e^{-2nz}\right] dz \quad . \tag{10}$$

If the further substitutions

$$v = \sqrt{\tau_\ell}\,\frac{z}{2\lambda} - \frac{2n}{\sqrt{\tau_\ell}}$$

are made in the $n^{th}$ term of the series in (10), that equation assumes a form in which the $z$ integral and the first integral of the series may be easily carried out in terms of well known functions. The result of performing these operations is

$$h_\ell(\tau_\ell) = e^{\tau_\ell/4\lambda^2} \operatorname{erfc}(\sqrt{\tau_\ell}/2\lambda) - \frac{\sqrt{\tau_\ell}}{\lambda} e^{-4/\tau_\ell}\left(\frac{2}{\sqrt{\pi}} - 2ae^{a^2}\operatorname{erfc}(a)\right) + O(e^{-16/\tau_\ell}) \quad (11)$$

where

$$a = \sqrt{\tau_\ell}/2\lambda + 2/\sqrt{\tau_\ell}$$

The first term on the right hand side of equation (11) describes the function with an error of less than one part in a thousand for all interesting values of $\lambda$ as long as $\tau_\ell$ is less than .4.

In a similar manner, normalizing the input voltage to the voltage generated by the initial current flowing through the D.C. resistance of the line, one may obtain to the accuracy stated above

$$e_\ell(\tau_\ell) = \frac{v(t)}{i(0)R_o} = \frac{1}{\lambda}\left(\frac{2\lambda}{\sqrt{\pi\tau_\ell}} - e^{\tau_\ell/4\lambda^2} \operatorname{erfc}(\sqrt{\tau_\ell}/2\ )\right) \quad (12)$$

## III B. Large Time Representation

Returning to equation (9) and examining the symmetry of the integrand one may quickly conclude that the contour $C_2$ of figure 4B is equivalent to $C_1$ . The closure of $C_2$ along the quarter circle at infinity may be assumed to exist since the integrand is exponentially small along that arc. The integral along $C_2$ may be evaluated in terms of the residues at the poles within $C_2$ determined by the roots of the denominator of the integrand of (9). These roots lie along the imaginary axis and have been tabulated[4] as a function of $\lambda$ .

These considerations lead to the following representations of current and voltage

$$h_\ell(\tau_\ell) = 2\sum_{n=1}^{\infty} \frac{e^{-y_n^2(\lambda)\tau_\ell/4}}{(1 + 1/\lambda) + \lambda y_n^2(\lambda)} \quad (13)$$

$$e_\ell(\tau_\ell) = 2\sum_{n=1}^{\infty} \frac{e^{-y_n^2(\lambda)\tau_\ell/4}}{1 + (\lambda + 1)/\lambda^2 y_n^2(\lambda)} \tag{14}$$

where $y_n(\lambda)$ is a root of the equation

$$\cot y_n = \lambda y_n$$

If $\tau_\ell$ is greater than .4 there is no need to take more than three terms in (13) or (14) to obtain an accuracy exceeding one part in a thousand.

Curves representing $h_\ell(\tau_\ell)$ and $e_\ell(\tau_\ell)$ with $t_o/t_\ell (= 4\lambda)$ as a parameter are given in figure 5. Figure 6 is a magnified version of figure 5A, included in order to exhibit more clearly the small time behavior of $h_\ell(\tau_\ell)$ .

It can be seen from equations (13) and (14) that for large time the expressions for the input current and voltage on the open-circuited line take the following asymptotic forms;

$$h_\ell(\tau_\ell) \rightarrow \alpha e^{-\gamma\tau_\ell}$$

$$e_\ell(\tau_\ell) \rightarrow \beta e^{-\gamma\tau_\ell}$$

In these equations $\alpha$ , $\beta$ , and $\gamma$ depend on the ratio $t_o/t_\ell$ . The following table presents the values of these asymptotic parameters for the $t_o/t_\ell$ ratios used in figures 5 and 6. Curves of these parameters are given in figure 7.

| $t_o/t_\ell$ | $\gamma$ | $\alpha$ | $\beta$ |
|---|---|---|---|
| .2 | .560 | .095 | .005 |
| .4 | .510 | .182 | .020 |
| 1.0 | .400 | .400 | .072 |
| 2.0 | .290 | .667 | .167 |
| 4.0 | .185 | 1.00 | .270 |
| 10.0 | .088 | 1.43 | .387 |
| 20.0 | .047 | 1.67 | .438 |

## IV. Comments

There are two general areas of the theoretical work on the buried line simulator that merit further comment. One area involves the mathematical technique used in computing certain functions describing the buried line while the other area involves the first assumption in the Introduction.

A function which has been used previously[3] and will undoubtedly be used again in buried line calculations is what may be called the input admittance of the line in the time domain. This function is proportional to the input current on an open-circuited line due to an impulse of voltage and may be written[3]

$$g^{o}(\tau) = \frac{1}{2\sqrt{\pi\tau}}\left\{1 + 2\sum_{n=1}^{\infty} (-)^{n} e^{-4n^{2}/\tau}\right\} \tag{15}$$

This representation may be obtained from an integral analogous to (9) by using a method similar to that of Section III A. The point here is that the method of Section III B may also be employed to obtain the following equivalent representation;

$$g^{o}(\tau) = \frac{e^{-\pi^{2}\tau/16}}{2}\left\{1 + \sum_{n=1}^{\infty} e^{-n(n+1)\pi^{2}\tau/4}\right\} \quad . \tag{16}$$

Equation (16) is more useful than equation (15) if $\tau$ is greater than unity. It is also of interest to note that this function may be expressed in terms of tabulated functions in the form

$$g^{o}(\tau) = \frac{\theta_{4}(0,e^{-4/\tau})}{2\sqrt{\pi\tau}}$$

where the usual notation for Theta functions[4] has been used. For future reference it may be noted that the analogous function for the short-circuited line has the representations

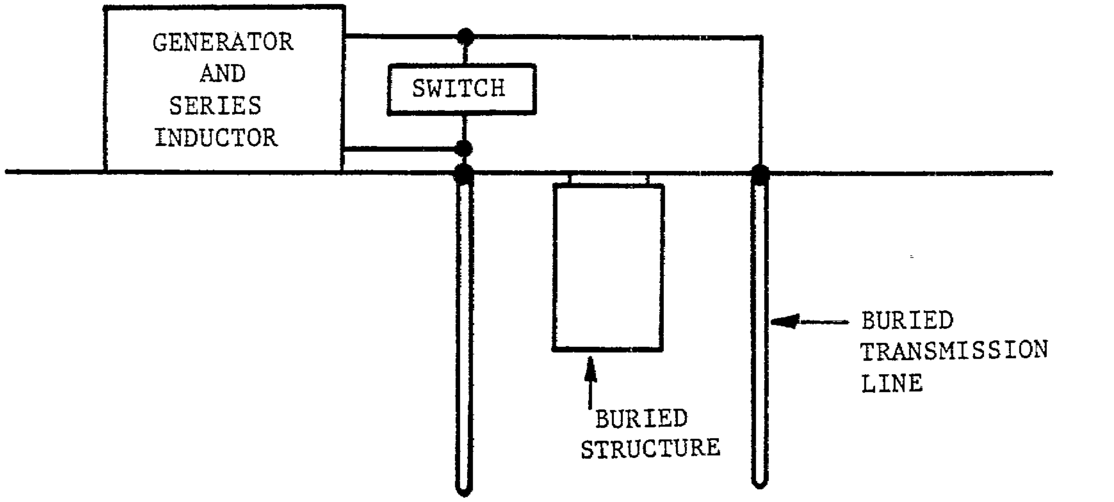
$$g^{s}(\tau) = \frac{1}{2\sqrt{\pi\tau}}\left\{1 + 2\sum_{n=1}^{\infty} e^{-4n^{2}/\tau}\right\}$$

$$= \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty} e^{-n^2\pi^2\tau/4}$$

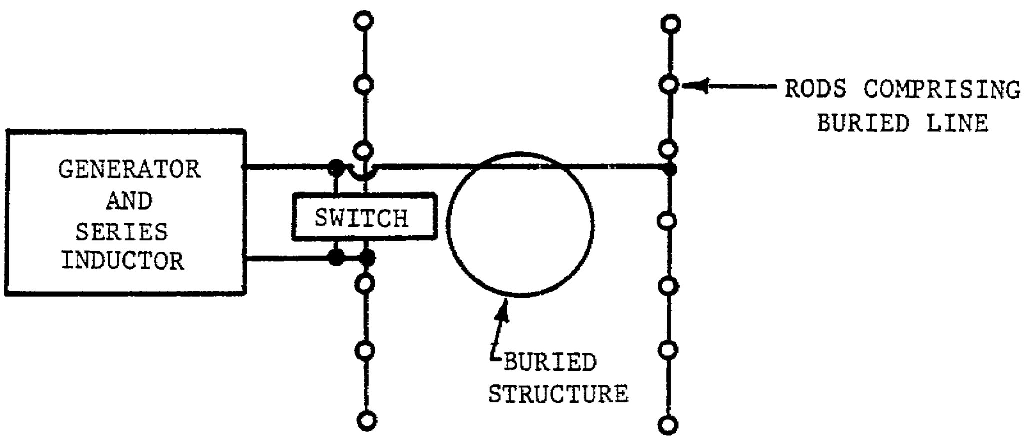
$$= \frac{\theta_4(\pi/2, e^{-4/\tau})}{2\sqrt{\pi\tau}} \quad .$$

Concerning the first assumption in the Introduction, that transmission-line theory provides an adequate description of the fields along the whole line, it is clear that this is invalid very near the source since it is impossible to excite the buried line structure so that only the TEM mode is present. If the ground is sufficiently conducting the assumption should be valid at all points slightly below the surface and slightly above the bottom of the structure. In fact, other assumptions may well be more restrictive than the present one for the ground conductivities that have been studied up to now. However, if in the future it becomes necessary to take account of ground conductivities of the order of $10^{-4}$ , the present considerations should be kept in mind.

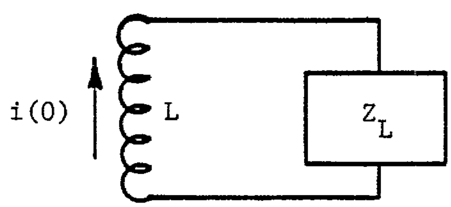
It should also be pointed out that the assumption of an open circuit at the other end of the line is physically impossible. In fact the actual structure is such that, if nothing is done at the lower end of the line to try to match impedances, the "termination" looks almost like the characteristic impedance of the line for the higher frequencies or shorter times. This conclusion results from a closer examination of the actual boundary-value problem involved, since for frequencies high enough for the skin depth in the ground to be comparable to the plate spacing of the line the fact that the guiding structure stops at a finite depth becomes irrelevant. Since in the real structures the plate spacings and depths are of the same order of magnitude this remark amounts to saying that if one wants to describe the actual situation up to times $\tau_\ell$ of the order of unity, then neglecting all but the dominant term of (11) is a more valid procedure than including any "correction" terms. It is left to a future note to determine up to what time the infinite line approximation is better than the finite line approximation.




A. SCHEMATIC SIDE VIEW




B. SCHEMATIC TOP VIEW




C. EQUIVALENT CIRCUIT

FIGURE 1 BURIED TRANSMISSION - LINE SIMULATOR
WITH INDUCTIVE ENERGY SOURCE

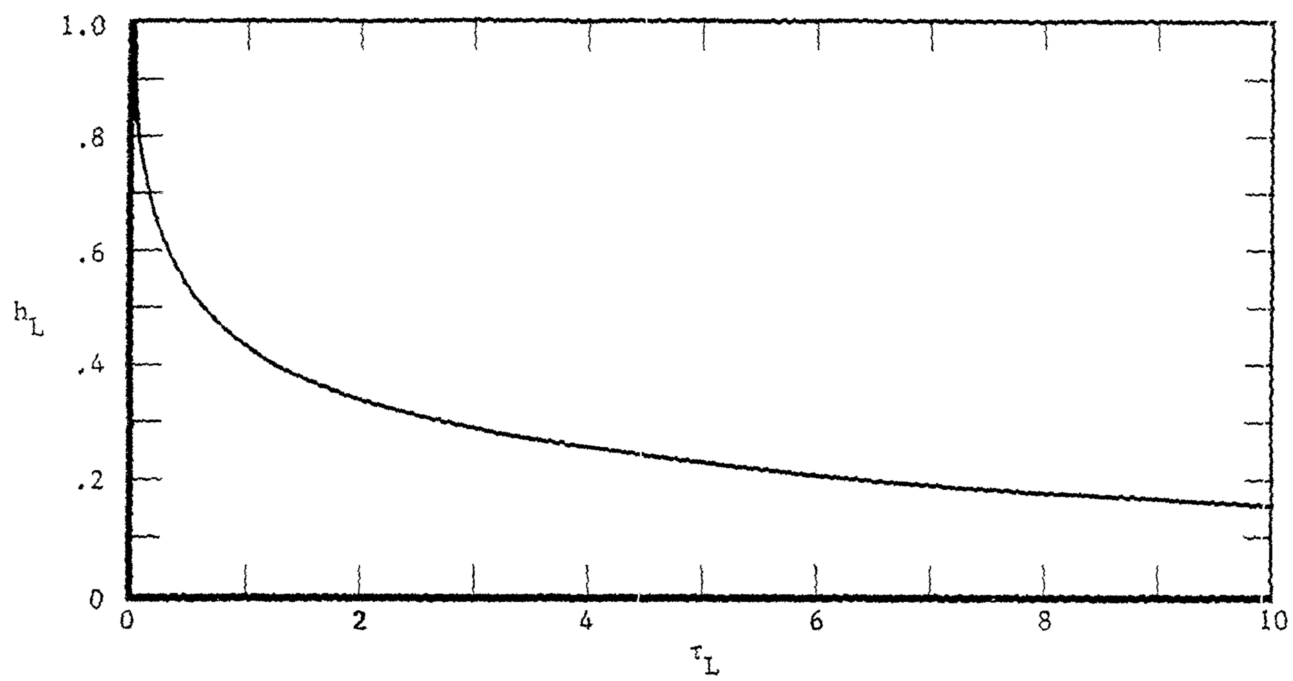



FIGURE 2A: CURRENT AT THE INPUT OF AN INFINITE BURIED LINE

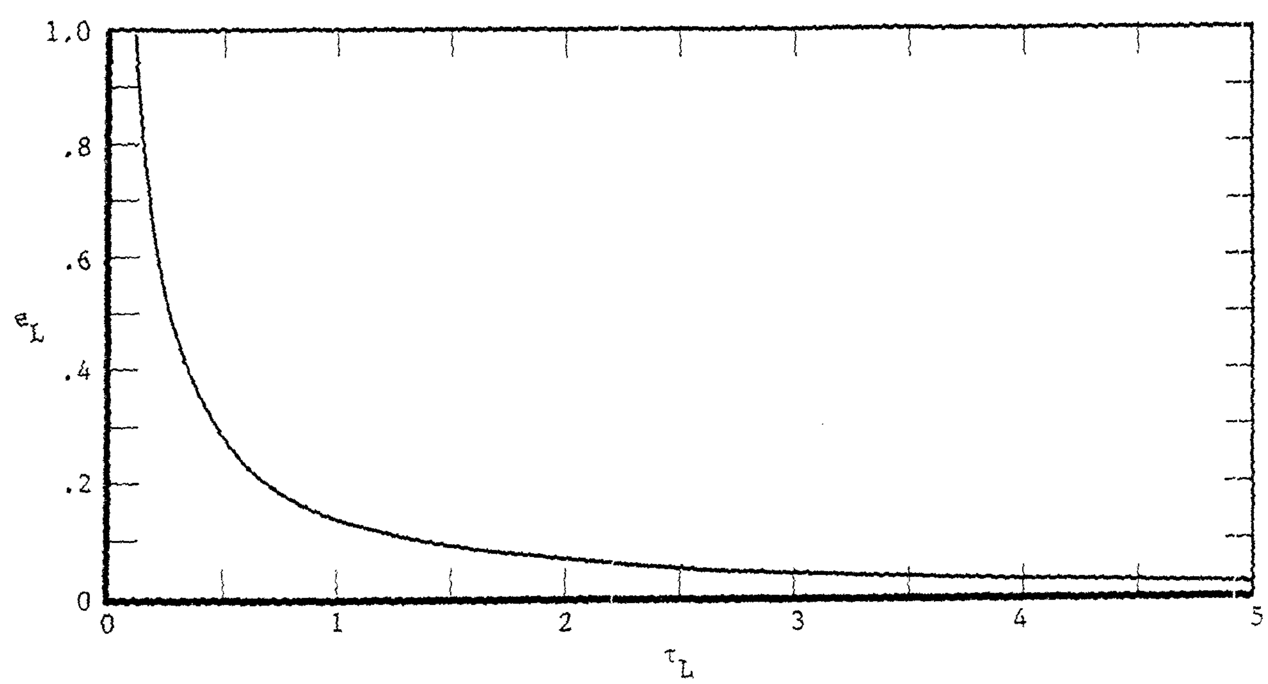



FIGURE 2B: VOLTAGE AT THE INPUT OF AN INFINITE BURIED LINE

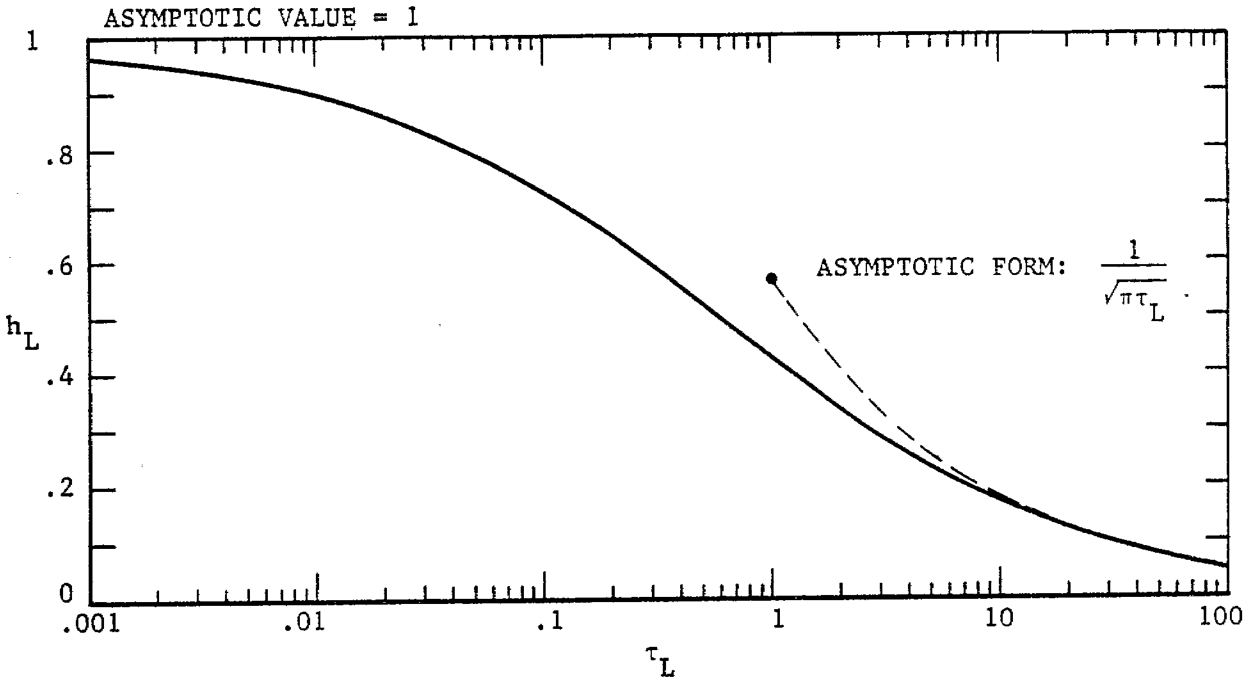



FIGURE 3A: CURRENT AT THE INPUT OF AN INFINITE BURIED LINE

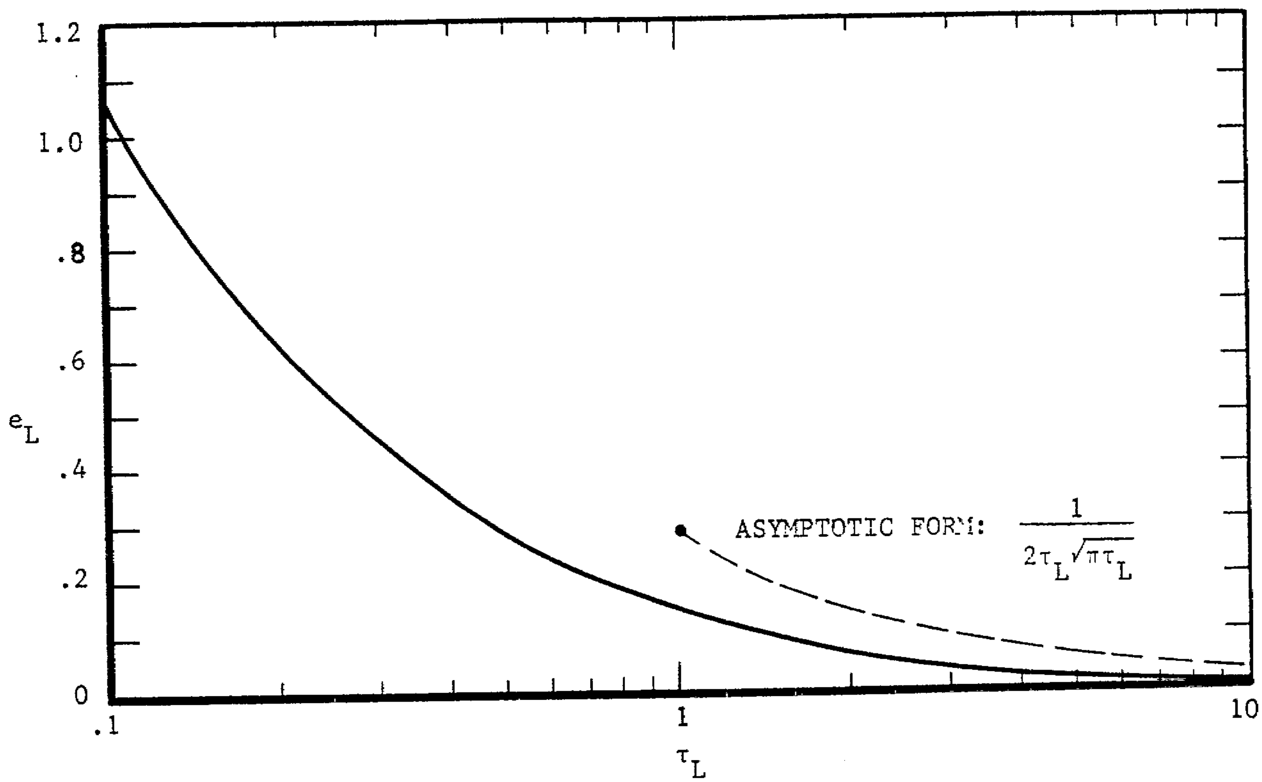
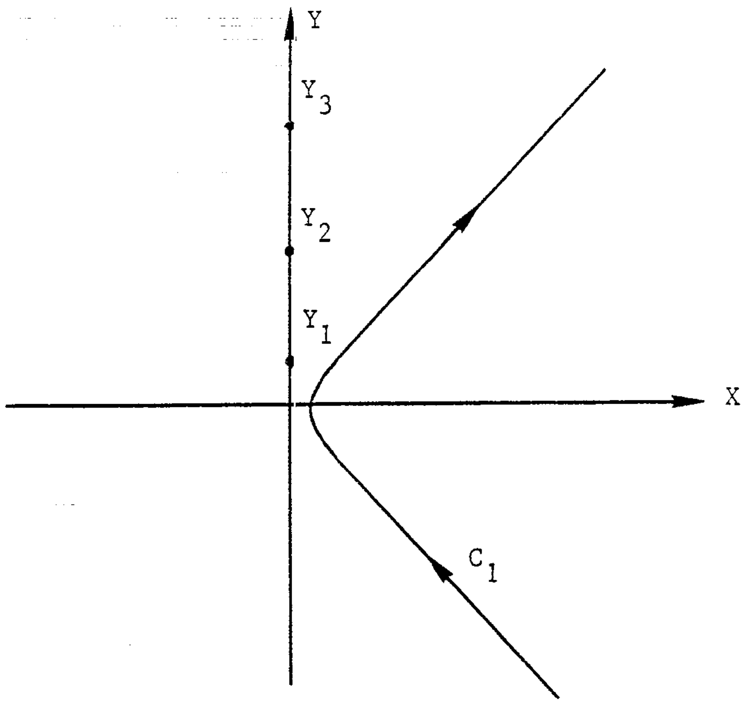
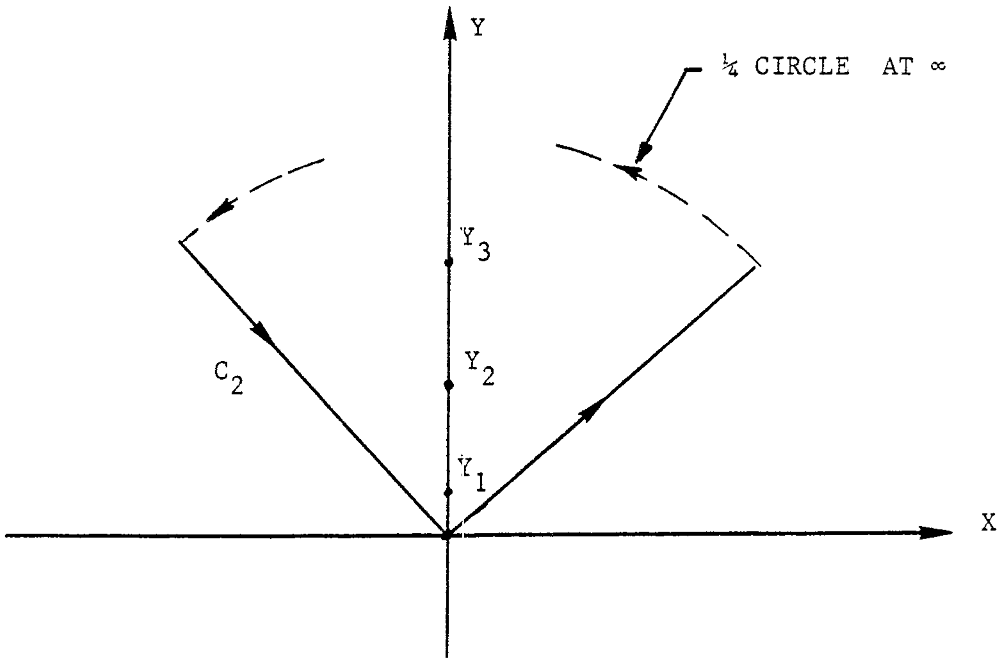



FIGURE 3B: VOLTAGE AT THE INPUT OF AN INFINITE BURIED LINE




A. INITIAL PATH IN THE Z-PLANE




B. TRANSFORMED PATH IN THE Z-PLANE

FIGURE 4 INTEGRATION PATHS USED IN THE EVALUATION OF EQUATION (9)

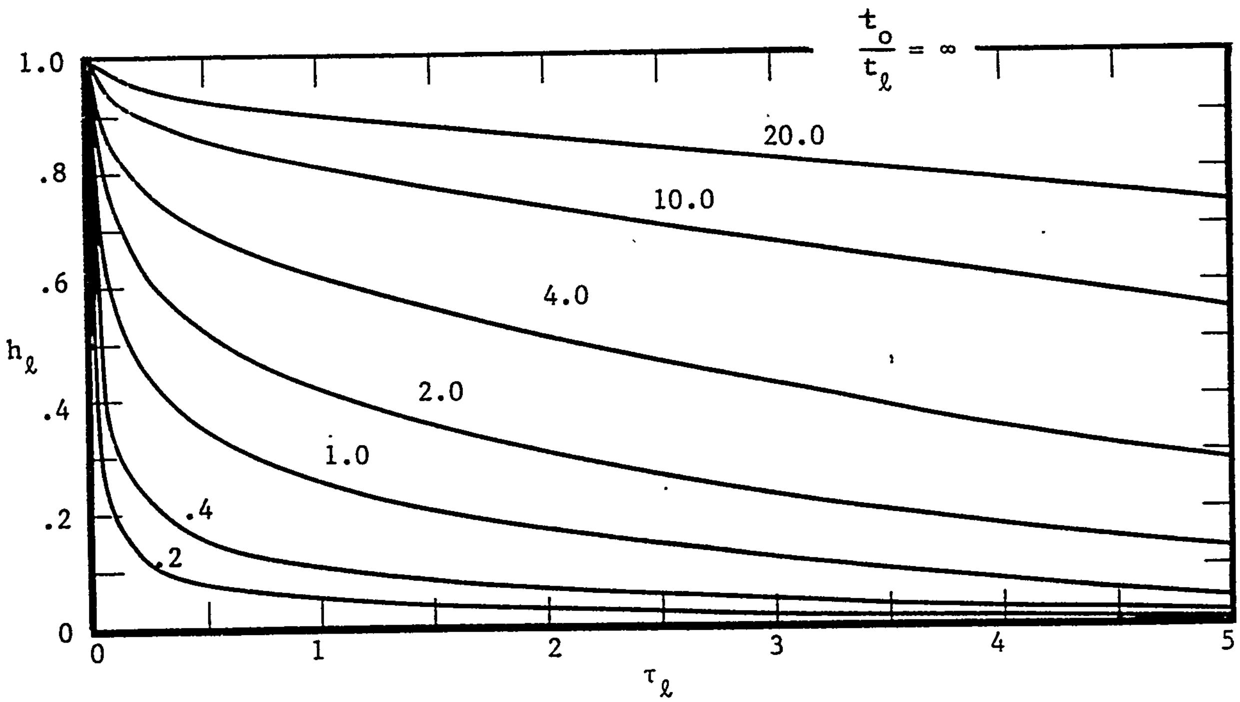



FIGURE 5A: CURRENT AT THE INPUT OF AN OPEN-CIRCUITED LINE

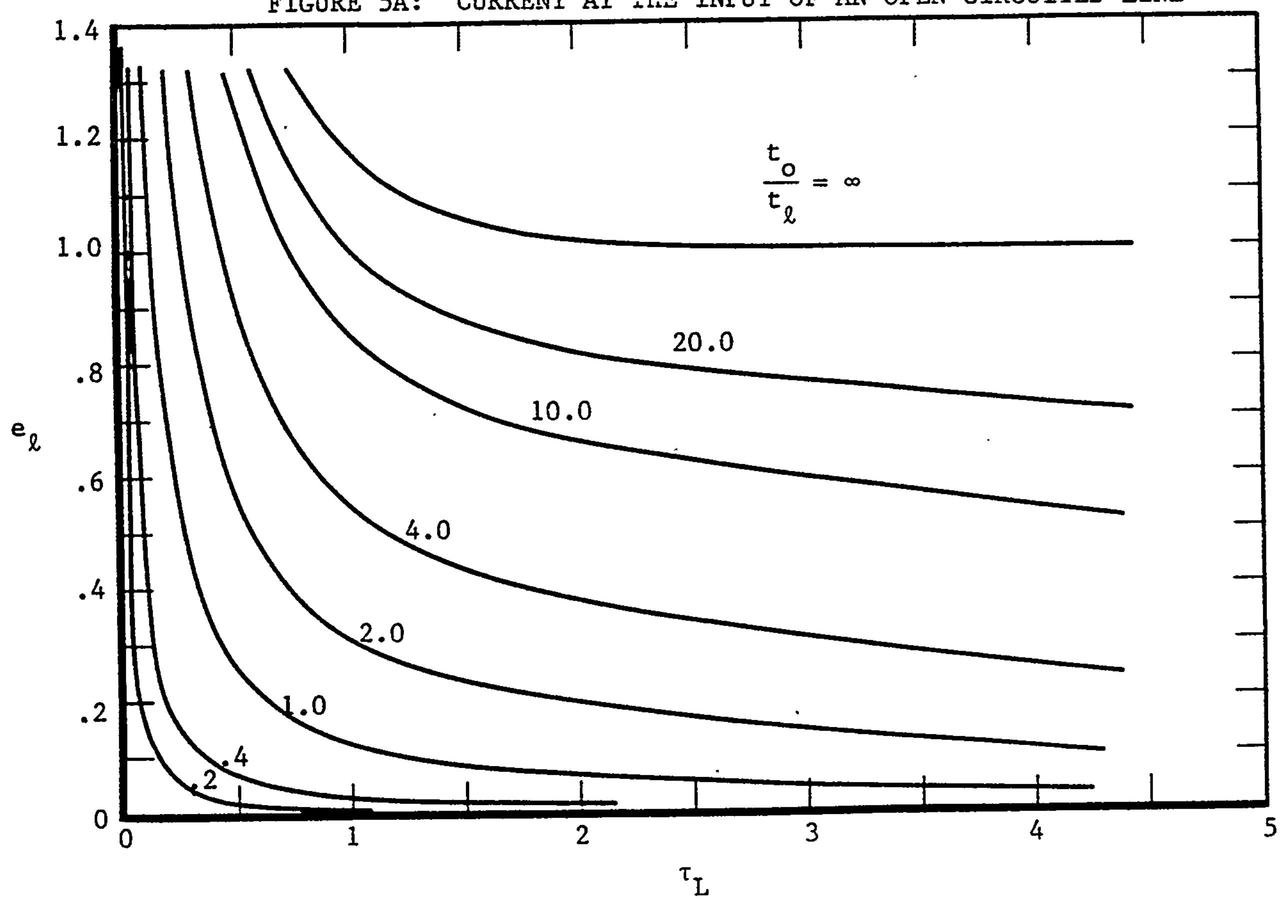



FIGURE 5B: VOLTAGE AT THE INPUT OF AN OPEN-CIRCUITED LINE

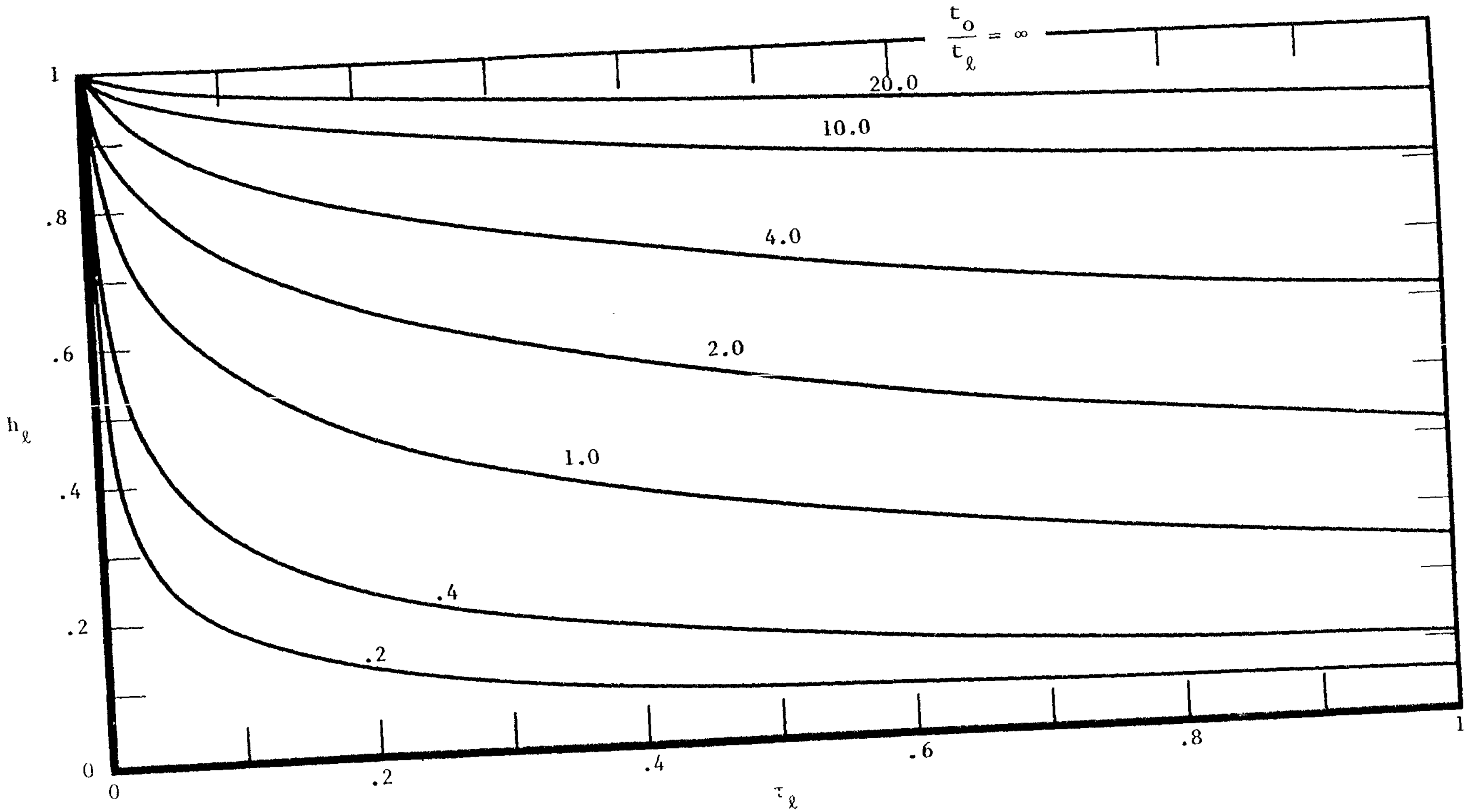



FIGURE 6: SHORT TIME BEHAVIOUR OF THE CURRENT ON AN OPEN-CIRCUITED LINE

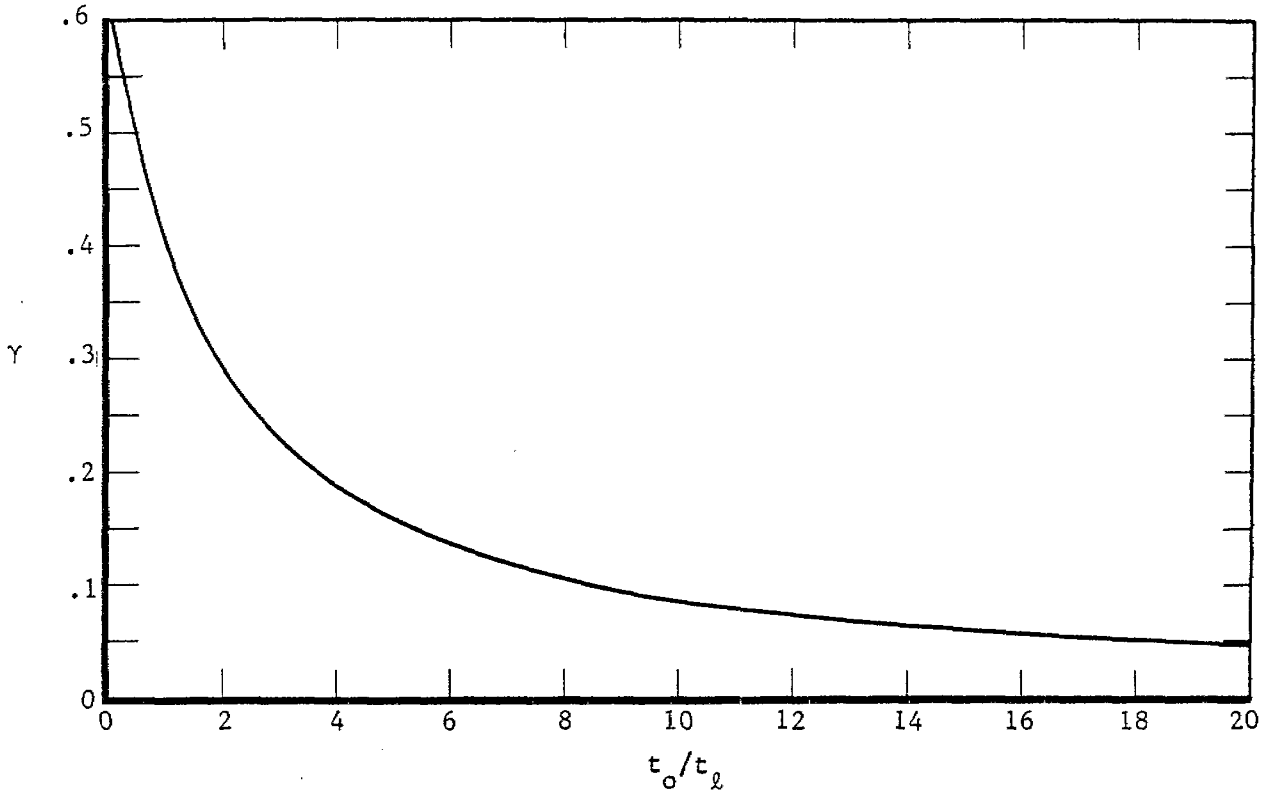



FIGURE 7A: ASYMPTOTIC DECAY CONSTANT OF FINITE LINE

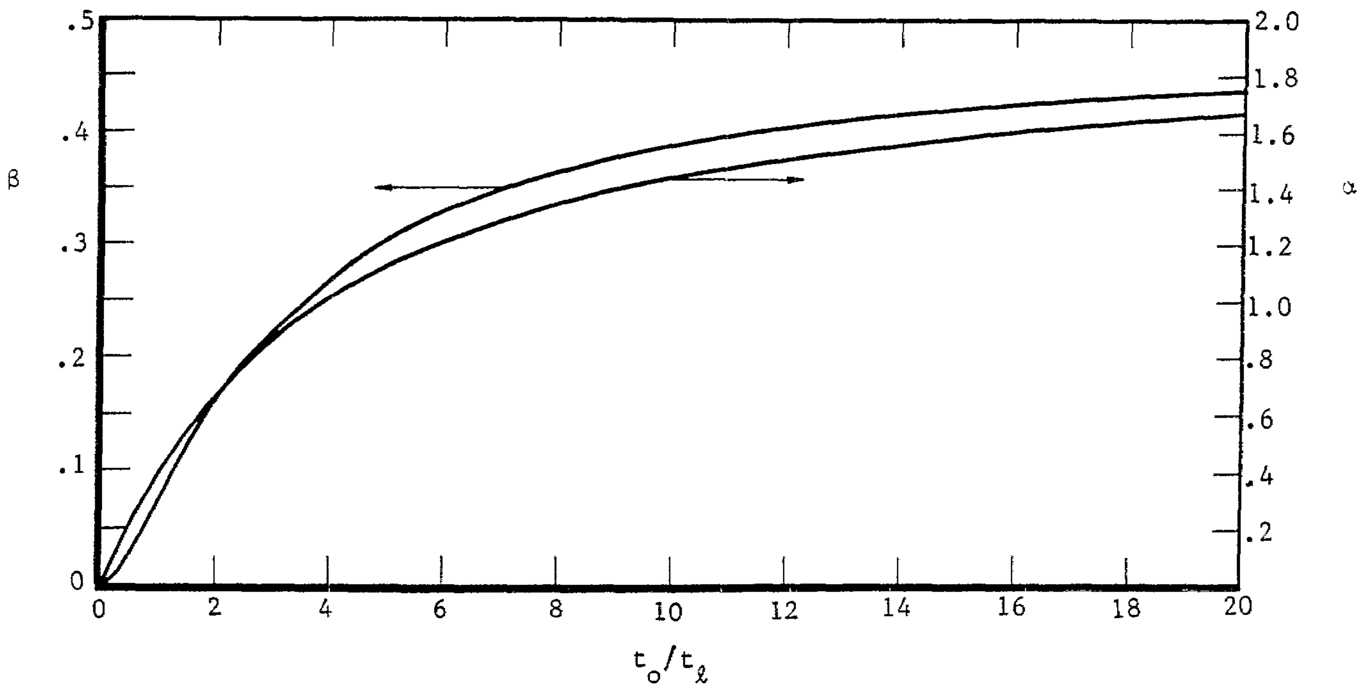



FIGURE 7B: ASYMPTOTIC PARAMETERS USEFUL IN FINITE LINE CALCULATIONS

## References


1. Carl E. Baum, Sensor and Simulation Note XXII, "A Transmission-Line EMP Simulation Technique for Buried Structures", June, 1966.
2. Carl E. Baum, Sensor and Simulation Note XLIV, "The Capacitor Driven, Open-Circuited, Buried-Transmission-Line Simulator", June, 1967.
3. Carl E. Baum, Sensor and Simulation Note XLIX, "The Buried-Transmission-Line Simulator Driven by Multiple Capacitive Sources", August, 1967.
4. Milton Abromowitz and Irene A. Stegun, Editors, Handbook of Mathematical Functions, National Bureau of Standards, AMS-55, June, 1964.