Sensor and Simulation Notes

NOTE 54

MATCHING THE IMPEDANCE OF MULTIPLE TRANSITIONS TO A PARALLEL-PLATE TRANSMISSION LINE

by

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2 April 1968

Abstract

Multiple transitions can be used to launch a wave on a wide, parallel-plate transmission line. The problem of matching the impedance of the multiple transitions to the parallel-plate structure is considered. A variational method is used to solve the problem and appropriate design curves are presented.

I. Introduction

Multiple transitions can be used in the manner shown in figure 1 to connect multiple sources to a wide, parallel-plate transmission line for the purpose of launching a TEM wave on the parallel-plate structure. A detailed discussion of the concept of multiple transitions appears in EMP Sensor and Simulation Note XXXI where both the reasons for using multiple transitions and the problems associated with their use are presented.

In this present note the problem of matching the characteristic impedance of these multiple transitions to the impedance of corresponding sections of the parallel-plate transmission line is considered. That is, we want to design each transition section such that at any point along the length of a section $(0 \le z \le L)$ the input impedance seen looking back toward the parallel-plate transmission line is equal to the input impedance at $z = L$ of the section of parallel-plate transmission line to which the transition section is attached. By matching impedances in this way, one tends to optimize the transmission of power into the parallel-plate structure. It should be pointed out that, although matching impedances is an excellent first-order approximation, this approach is not rigorous in the sense that it does not attempt to match the electromagnetic fields on a transition section to those of a TEM wave on the parallel-plate structure. However, it is not clear that a more rigorous approach would be possible in a reasonable amount of time or that it would yield better results.

To make the problem more tractable we assume there are an infinite number of transitions arranged side by side in a periodic manner; thus, we need to consider only a single section or cell. In figure 2 the cross-section of a typical cell is shown at an arbitrary value of z. Assuming this cross-section is that of a uniform transmission line which extends to infinity along the z-axis, we can compute the impedance of the uniform line for a TEM wave by determining the capacitance per unit length and by relating the impedance to this capacitance. If we assume that the cross-sectional dimensions a/W and b/W of a transition section vary slowly with z and that a TEM wave propagates along the transition section, the impedance of this uniform line corresponds to the impedance Z_L of a transition section at an arbitrary value of z . In this note Z_L refers to the impedance of the entire cross-section of a cell, including the lower portion formed by the image below the ground plane defined by $y = 0$.

It turns out that Z_L depends on a/W and b/W , the characteristic dimensions of a cross-section, and that a relationship exists between a/W and b/W which allows us to compute a family of curves for b/W versus a/W for constant values of $f_g = Z_L/Z_o$ where Z_o is the impedance of free-space. These curves provide a means by which the impedance of the multiple transitions can be matched to the parallel-plate structure.

II. Formulation of the Impedance of a Transition Section

In the cross-section of a typical cell bounded by $|x| \leq W$ and $y \geq 0$ the potential $\phi(x,y)$ satisfies Poisson's equation

$$\nabla^2 \phi(x,y) = -\frac{1}{\varepsilon_o}\rho(x,y) \tag{1}$$

which is accompanied by the boundary conditions

$$\frac{\partial \phi}{\partial x} = 0 \quad \text{at} \quad x = \pm W$$

$$\phi = 0 \quad \text{at} \quad y = 0$$

and $\phi(x,y)$ is bounded for $y \to \infty$.

Green's function for the problem is defined by

$$\nabla^2 G(x,y|x',y') = -\delta(x-x')\delta(y-y') \tag{2}$$

with

$$\frac{\partial G}{\partial x} = 0 \quad \text{at} \quad x = \pm W$$

$$G = 0 \quad \text{at} \quad y = 0$$

and $G(x,y|x',y')$ is also bounded for $y \to \infty$.

Multiplication of (1) and (2) by $G(x,y|x',y')$ and $\phi(x,y)$ respectively, followed by subtraction and integration of corresponding members, yields

$$\phi(x',y') = \frac{1}{\varepsilon_o}\int_{cell} G(x,y|x',y')\rho(x,y)dS$$

$$+ \int_{cell}\left[G(x,y|x',y')\nabla^2\phi(x,y) - \phi(x,y)\nabla^2 G(x,y|x',y')\right]dS \quad . \tag{3}$$

A consideration of the boundary conditions for $\phi(x,y)$ and $G(x,y|x',y')$ shows that the second integral of (3) vanishes. Taking $y' = b$, we have

$$\phi(x',b) = \frac{1}{\varepsilon_o}\int_{cell} G(x,y|x',b)\rho(x,y)dS \quad . \tag{4}$$

The charge density for the problem is given by

$$\rho(x,y) = \sigma(x)\delta(y - b)$$

which, if substituted into (4), gives

$$\phi(x',b) = \frac{1}{\varepsilon_o}\int_{-W}^{W} G(x,b|x',b)\sigma(x)dx \quad . \tag{5}$$

From (2) and the subsequent boundary conditions, Green's function can be constructed, viz.,

$$G(x,b|x',b) = \frac{b}{2W} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1 - e^{-2\lambda_n b}\right)\cos\lambda_n x\cos\lambda_n x' \tag{6}$$

where $\lambda_n = \frac{n\pi}{W}$.

Since $G(x,b|x',b)$ is symmetric in x and x' , (5) can be written as

$$\phi(x,b) = \frac{1}{\varepsilon_o} \int_{-W}^{W} G(x,b|x',b)\sigma(x')dx' \quad .$$

If we introduce

$$\frac{\sigma(x)}{\varepsilon_o} = \begin{cases} f(x) & |x| \leq a \\ 0 & a < |x| \leq W \end{cases} ,$$

(5) becomes

$$\phi(x,b) = \int_{-a}^{a} G(x,b|x',b)f(x')dx' \quad . \tag{7}$$

In this problem it is convenient to set $\phi(x,b) = 1$ for $-a \leq x \leq a$ so that (7) gives

$$1 = \int_{-a}^{a} G(x,b|x',b)f(x')dx' \quad . \tag{8}$$

Having $\phi(x,b) = 1$ and $\phi(x,0) = 0$, we can show that

$$\frac{Z_o}{Z_L} = \frac{1}{2}\frac{C}{\varepsilon_o}$$

where Z_L is the transmission line impedance of a transition section including the effect of the image below the ground plane which accounts for the factor of 1/2 and C is a capacitance determined by

$$C = \frac{\int_{-a}^{a} \sigma(x)dx}{\phi(x,b) - \phi(x,0)} = \int_{-a}^{a} \sigma(x)dx \quad .$$

Thus,

$$\frac{Z_o}{Z_L} = \frac{1}{2}\int_{-a}^{a} f(x)dx \quad . \tag{9}$$

A rapidly converging series for $f(x)$ with the correct form of singularity near the edges at $x = \pm a$ is

$$f(x) \approx \sum_{n=0}^{\infty} c_n (a^2 - x^2)^{n - 1/2} \quad . \tag{10}$$

For convenience we change the limits of integration of (8) and (9) to get

$$1 = \int_{-1}^{1} G(x,b|x',b)f(x)dx \tag{11}$$

and

$$\frac{Z_o}{Z_L} = \frac{1}{2}\int_{-1}^{1} f(x)dx \tag{12}$$

where

$$G(x,b|x',b) = \frac{b}{2W} + \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - e^{-2\lambda_n b}\right)\cos(\lambda_n ax)\cos(\lambda_n ax') \tag{13}$$

and

$$f(x) = \sum_{n=0}^{\infty} c_n (1 - x^2)^{n - 1/2} \quad . \tag{14}$$

Equations (11) and (12) can be combined to give the variational form

$$\frac{1}{2} f_g = \frac{\int_{-1}^{1}\int_{-1}^{1} f(x)G(x,b|x'b)f(x')dxdx'}{\left[\int_{-1}^{1} f(x)dx\right]^2} \tag{15}$$

where $f_g = Z_L/Z_o$. Substitution of (13) into (15) yields

$$f_g = \frac{b}{W} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1 - e^{-2\lambda_n b}}{n}\right)\left[\frac{\int_o^1 f(x)\cos(\lambda_n ax)dx}{\int_o^1 f(x)dx}\right]^2 \quad . \tag{16}$$

By virtue of the properties of the variational method an exact expression for $f(x)$ is not required to evaluate (16). We use the first term of (14) as a trial function for $f(x)$, viz.,

$$f(x) = \frac{1}{\sqrt{1 - x^2}}$$

and obtain for (16)

$$f_g = \frac{b}{W} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - e^{-2\lambda_n b}\right)J_o^2(\lambda_n a) \quad . \tag{17}$$

For given values of f_g and a/W , we use (17) to compute corresponding values of b/W . This computation can be performed by an iterative method in which the p-th iteration is

$$\left(\frac{b}{W}\right)_p = f_g - \frac{1}{\pi}\sum_{n=1}^{\infty} J_o^2\left(\frac{n\pi a}{W}\right) + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-2n\pi\left(\frac{b}{W}\right)_{p-1}} J_o^2\left(\frac{n\pi a}{W}\right) \qquad p = 1,2,3....$$

Ordinarily it is convenient to begin at

$$\left(\frac{b}{W}\right)_{p-1=0} = f_g - \frac{1}{\pi}\sum_{n=1}^{\infty} J_o^2\left(\frac{n\pi a}{W}\right)$$

and to continue, of course, until the difference $(b/W)_p - (b/W)_{p-1}$ between successive iterations is sufficiently small.

As a check, the results obtained by using the variational technique are compared with asymptotic expressions that are readily available for the cases in which (1) a is small compared to W and (2) a is nearly equal to W. In the first case the geometry of the problem is two charged parallel plates which can be considered to be isolated from the rest of the structure. This problem has been considered in a previous note[1] in which results are reported.

In the second case the problem consists of an infinite array of charged parallel strips formed by periodically cutting slits in two infinite parallel plates. This problem has been considered previously[2,3] and the following result has been obtained. For values of a/W slightly less than and equal to unity

$$\frac{b}{W} = f_g - \frac{1}{\pi}\ln\left[\csc\left(\frac{\pi}{2}\, a/W\right)\right] \qquad . \tag{18}$$

Figure 3 illustrates how typically a curve computed by means of the variational method smoothly joins the asymptotic curves obtained from EMP Sensor and Simulation Note XXI and (18). In the regions $(a/W \to 0$ and $a/W \to 1)$ where the asymptotic forms are valid, the curve computed by using the variational method agrees with the asymptotic curve within 1 or 2 per cent. This excellent agreement in the region of the asymptotes gives strong credence to the rest of the curve joining the asymptotes. A statement that the overall accuracy is well within the limit of a 3% error is reasonable.

III. Results

Curves which show the relationship that exists among f_g, a/W, and b/W are presented in figures 4 through 7. These curves can be used to design the multiple transitions so that their impedance is matched to the corresponding sections of the parallel-plate transmission line.

A transition section is formed by gradually reducing the cross-sectional dimensions a/W and b/W from their values at $z = L$ where the transition section is connected to the parallel-plate structure to smaller values, approaching zero, at $z = 0$ where the transition is connected to a source. At $z = L$ the input impedance of the section of parallel-plate transmission line to which the transition is connected corresponds to

$$f_g = \frac{Z_L}{Z_o} = \frac{H}{W} \tag{19}$$

where H and W are depicted in figures 1 and 2. The curves in figures 4 through 7 can be used to determine a relationship between a/W and b/W such that, as these cross-sectional dimensions are reduced in size to form the transition, the value of f_g for the transition section is a constant along the length of the transition and is equal to the value of f_g given by (19).

The curves presented in figures 5, 6, and 7 are especially pertinent to the actual construction of a transition section because, if a/W is chosen to be a linear function of z, except for a constant multiplicative factor which will stretch or contract the abcissa coordinate, they show the physical shape of a typical transition as viewed from the side. Whether or not a/W should be chosen to be a linear function of z is not obvious from an electromagnetic viewpoint. Certainly a linear dependence of a/W on z would be easy to construct.

Just how a/W or b/W should vary as a function of z is not considered in this note. Clearly, if a dependence of a/W on z is defined, the curves contained in this note determine the dependence of b/W on z, and vice versa. Perhaps the problem of deciding the best dependence of a/W or b/W on z should be considered in a future note.

References

1. Capt. Carl E. Baum, Sensor and Simulation Note XXI, Impedances and Field Distributions for Parallel Plate Transmission Line Simulators, June 1966.
2. W. R. Smythe, Static and Dynamic Electricity, 2nd ed., 1950, p. 91.
3. This problem was worked out explicitly by Capt. Carl E. Baum, Air Force Institute of Technology, in a private communication, 22 March 1968.

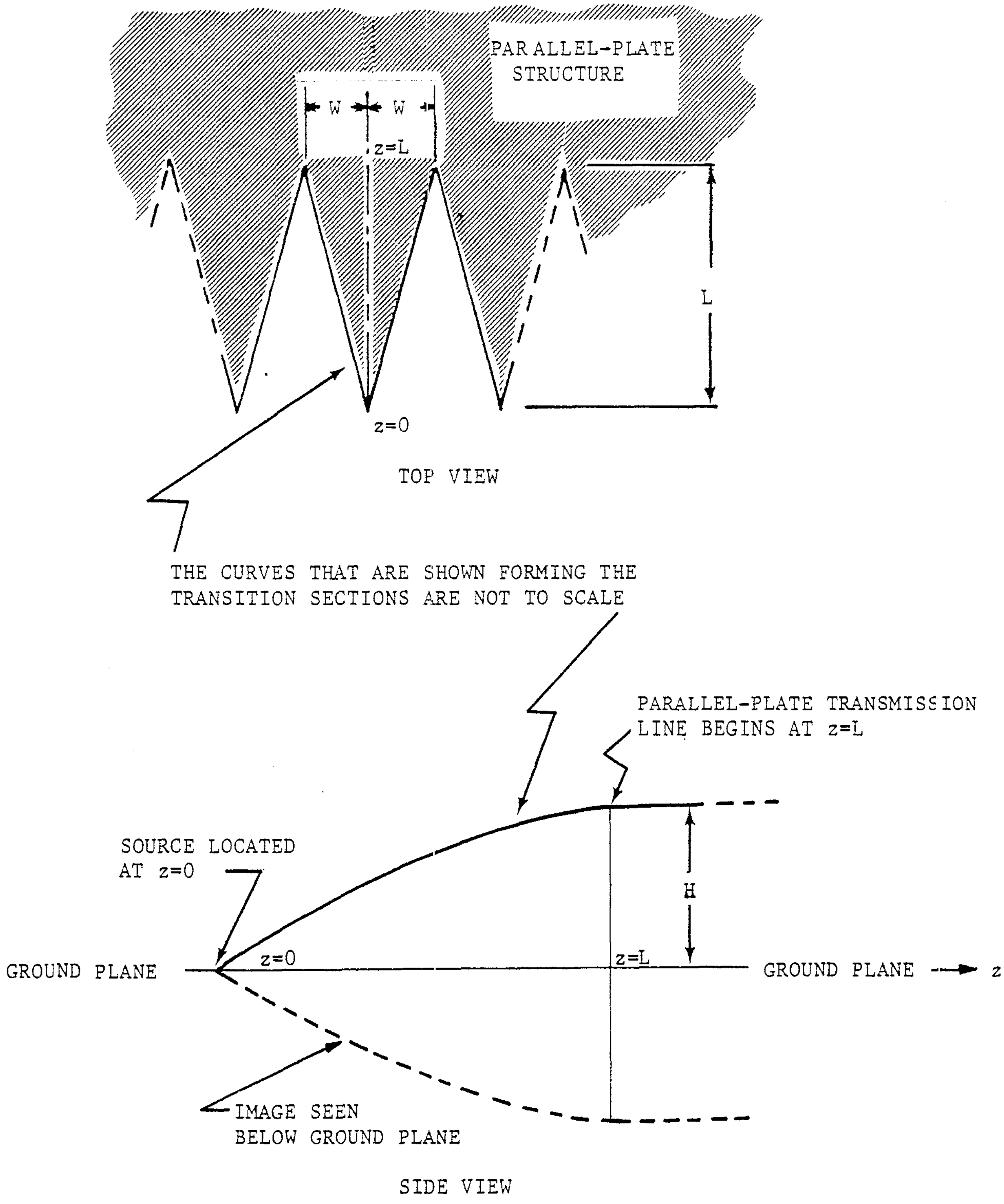

FIGURE 1: TOP AND SIDE VIEWS OF THE
TRANSITION SECTIONS

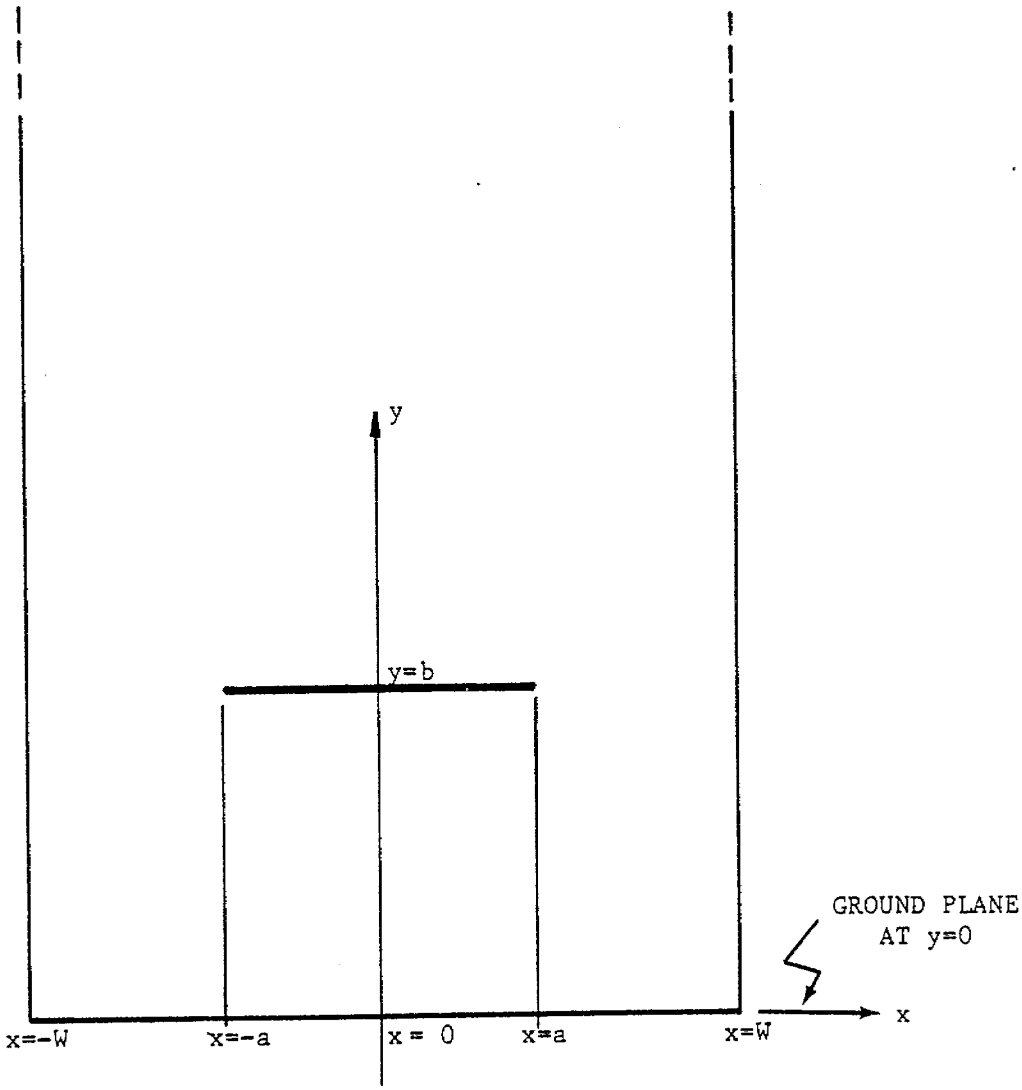

FIGURE 2: THE CROSS-SECTION OF A TYPICAL CELL

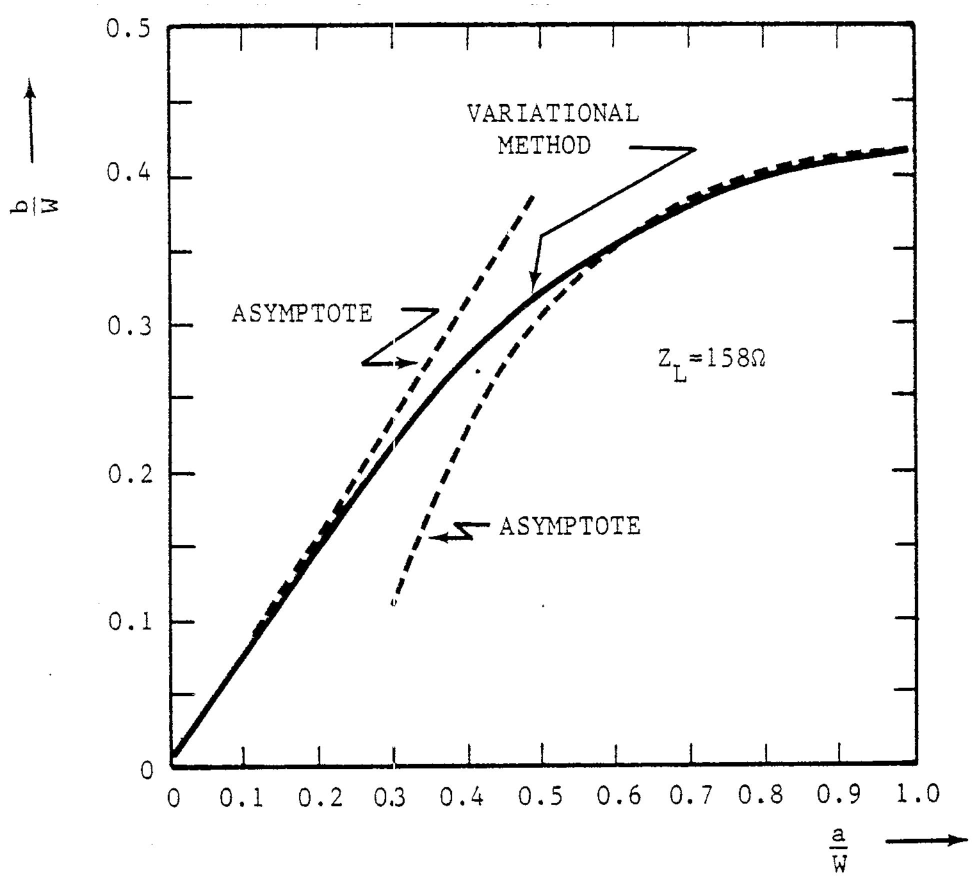

FIGURE 3: COMPARISON BETWEEN VARIATIONAL METHOD AND ASYMPTOTIC FORMS

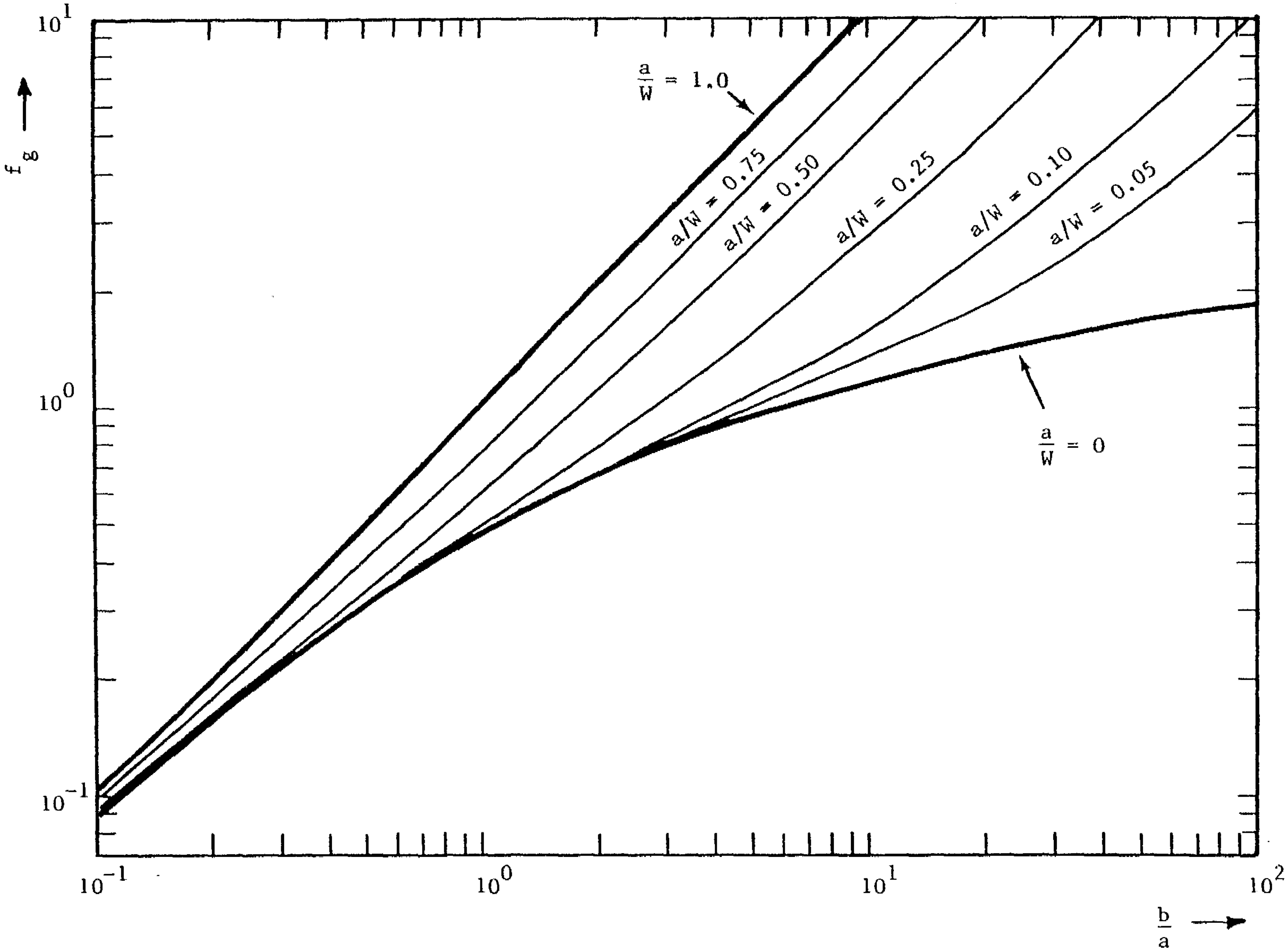

FIGURE 4: CURVES OF f_g VERSUS b/a WITH a/W AS A PARAMETER

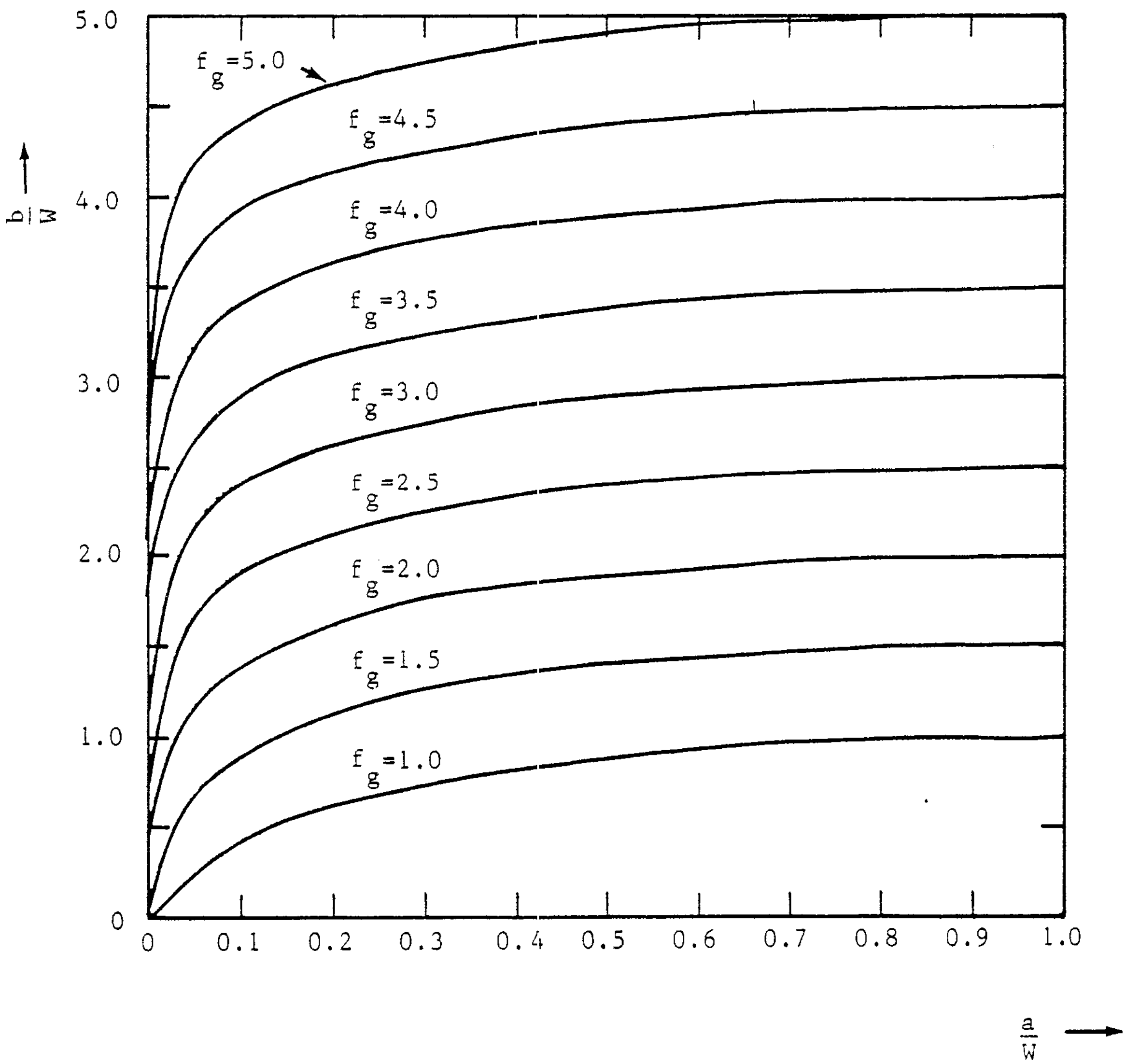

FIGURE 5: CURVES OF b/W VERSUS a/W WITH $1 \leq f_g \leq 5$ AS A PARAMETER

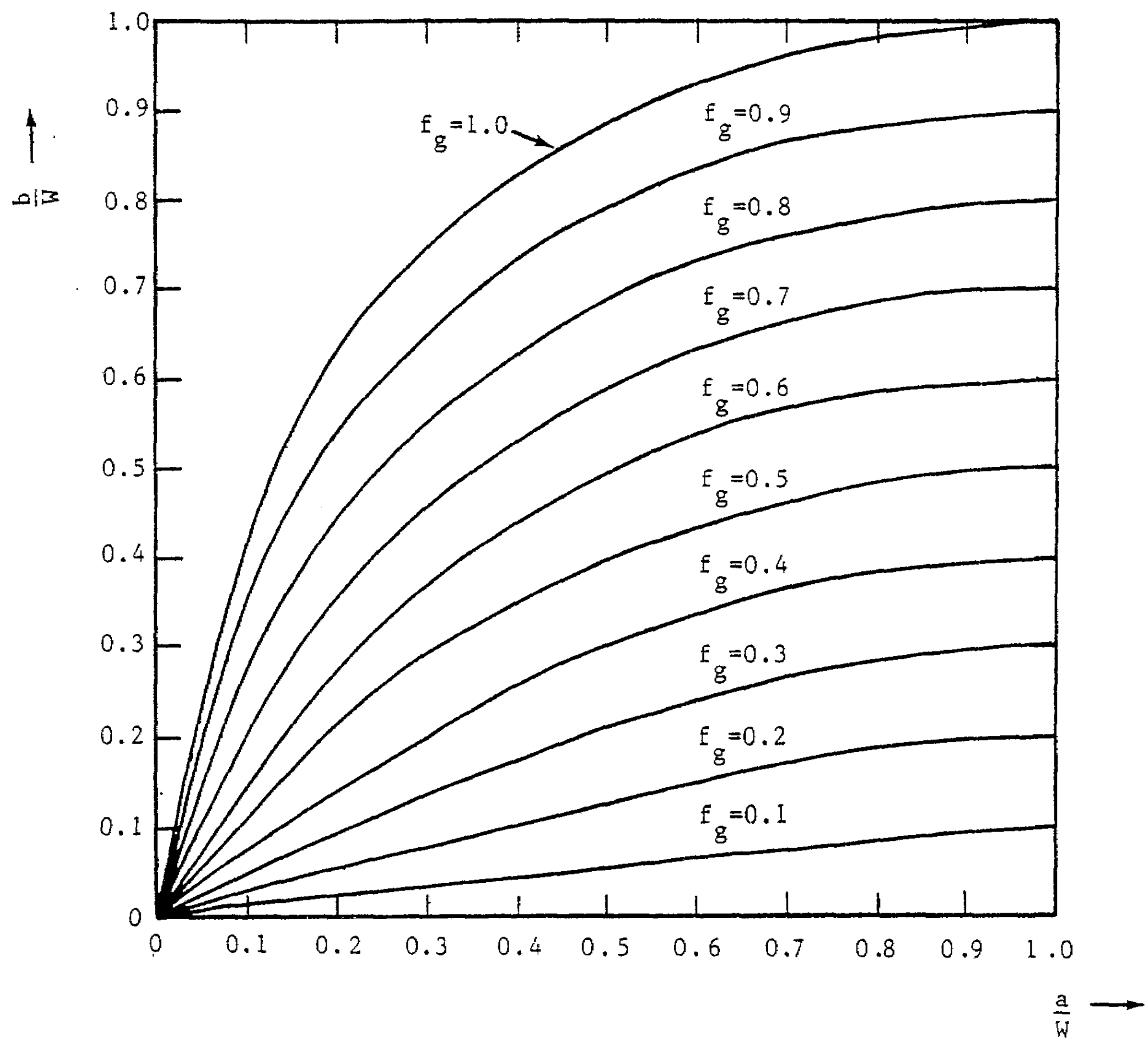

FIGURE 6: CURVES OF b/W VERSUS a/W WITH
$0.1 \leq f_g \leq 1.0$ AS A PARAMETER

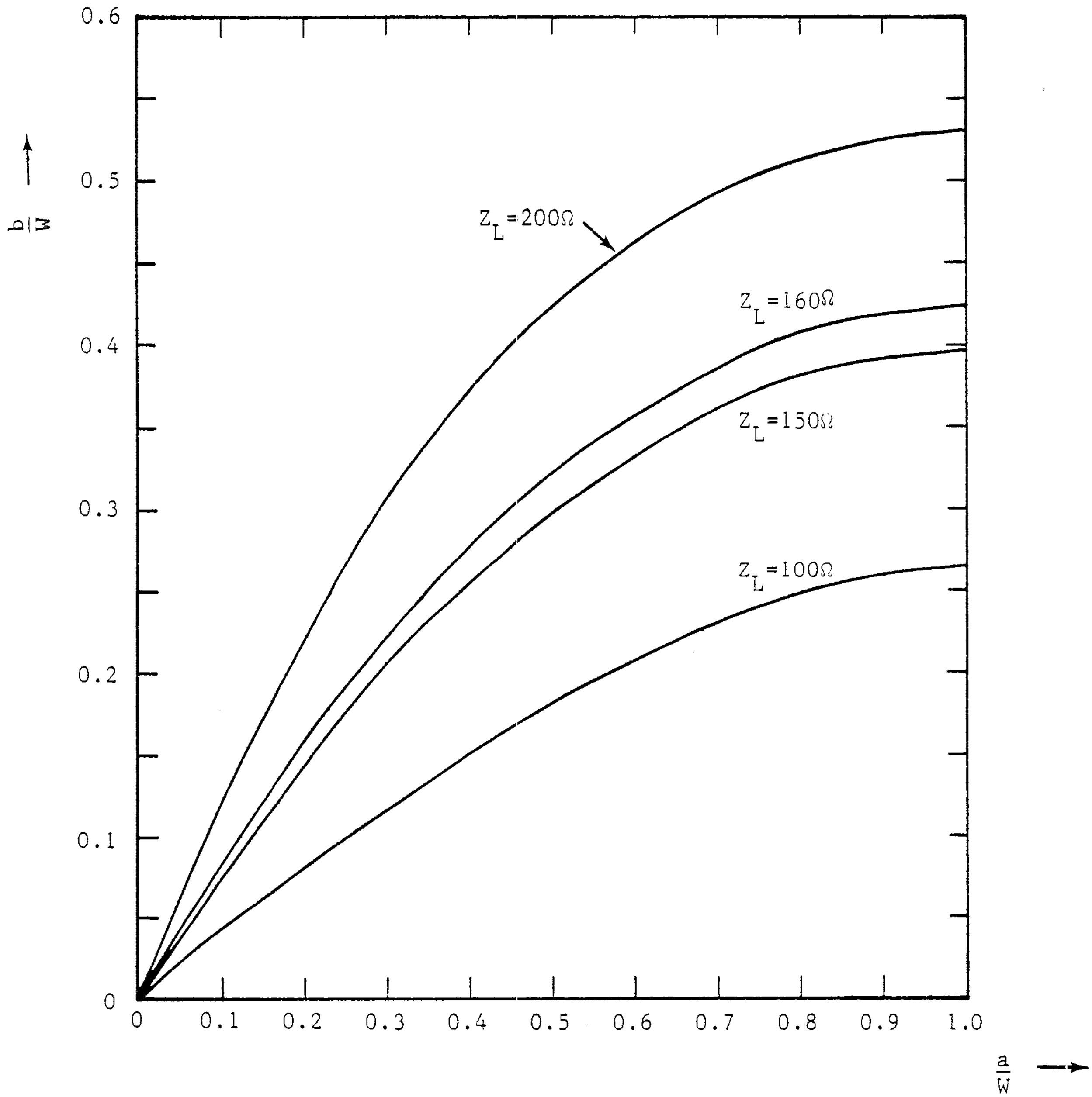

FIGURE 7: CURVES OF b/W VERSUS a/W WITH TYPICAL VALUES OF Z_L AS A PARAMETER