Sensor and Simulation Notes
Note 65
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Some Limiting Low-Frequency Characteristics of a Pulse-Radiating Antenna

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Abstract

One method of generating an electromagnetic pulse involves radiating a pulse from an antenna to a somewhat distant observer. In this note we discuss some of the low-frequency limitations placed on the radiated waveform under the assumption that the late-time antenna currents go to zero. We find that the complete time integral of the radiated waveform must be zero and that if the antenna is designed to have a long-time electric dipole moment the radiated waveform may have only one zero crossing, while if the antenna does not have a long-time dipole moment the radiated waveform must have at least two zero crossings.

I. Introduction

In designing simulators for the nuclear electromagnetic pulse there are many cases to consider, depending on both the geometry of the nuclear burst being simulated and the type and location of the system on which the simulation is desired. For some cases of interest one is interested in producing a fast-rising pulse in a form which approximates a free-space plane wave over a rather large volume. One approach to this problem consists in radiating a pulse from an antenna.

Figure 1 illustrates this concept. The antenna is contained in a volume, V'. We use the position vector, $\vec{r}'$, for currents, charges, etc., on the antenna, while $\vec{r}$ is the position vector of the observer at which the fields are to be calculated. The origin of both sets of coordinates is taken at some common convenient point inside V'. Our interest will center on the case for which $|\vec{r}| >> |\vec{r}'|$ for all $\vec{r}'$ in V' so that we can use limiting expressions for the fields far away. If the overall dimensions of the system under test are also much less than $|\vec{r}|$ then the incident fields at the system approximate a uniform plane wave.

At the present time one such pulser with antenna is being built for AFWL by Physics International. It is designed to be operated while supported off the ground by some means such as a helicopter. We have decided to designate this general type of portable simulator with the acronym RES (for Radiating EMP Simulator). The above-mentioned pulser and others like it are designated RES I. The RES type of simulator has promise of being a rather flexible one. For ground-based systems the simulator can be brought to the system to be tested without disturbing the system's operational configuration.

In this note we consider some limitations on the low-frequency characteristics of such simulators and the corresponding implications for the radiated waveforms. The antenna is assumed to be placed in free space with no other media, scatterers, etc., present. The results then apply in the sense of an incident waveform at the observer. Using the formalism of the vector and scalar potentials we first develop the well-known antenna radiation in terms of the antenna currents. This is then applied to the low-frequency and late-time behavior of the pulsed antenna. The results are then specialized to the simpler case of a dipole type of antenna with axial and lengthwise symmetry. Finally we consider some illustrative examples of simple waveforms consistent with the low-frequency properties of such an antenna.

II. Radiation from Antenna

To begin our consideration of the pulse-radiating antenna we first formulate the fields in terms of the vector and scalar potentials using the source currents and charges and find the limiting expressions for large $|\vec{r}|$. A more complete discussion can be found in many tests.[1,2]

1. C. H. Papas, Theory of Electromagnetic Wave Propagation, McGraw Hill, 1965, chapters 1 and 2.
2. J. Van Bladel, Electromagnetic Fields, McGraw Hill, 1964, chapter 7.

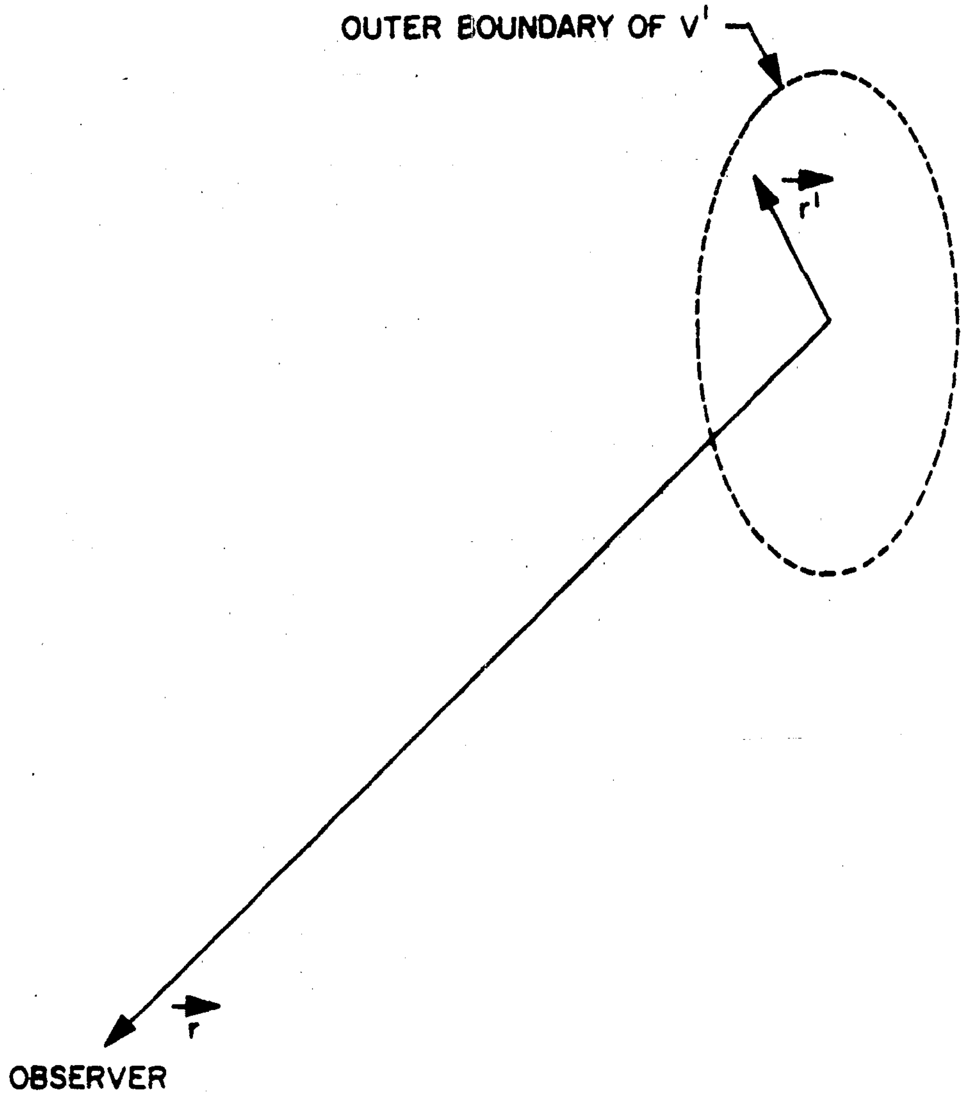


Fig. 1 GEOMETRY FOR RADIATING ANTENNA

Consider Maxwell's equations in free space[3]

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad , \quad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \quad , \quad \nabla \cdot \vec{D} = \rho \tag{1}$$

together with the constitutive relations

$$\vec{B} = \mu_o \vec{H} \quad , \quad \vec{D} = \varepsilon_o \vec{E} \tag{2}$$

and the equation of continuity

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{3}$$

The fields can be derived from a vector potential, $\vec{A}$, and a scalar potential, Φ, as

$$\vec{B} = \nabla \times \vec{A} \quad , \quad \vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \tag{4}$$

where the potentials have the explicit solutions in terms of the currents and charges as

$$\vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi} \int_{V'} \frac{\vec{J}\left(\vec{r}\,',t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|} dV'$$

$$\Phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_o} \int_{V'} \frac{\rho\left(\vec{r}\,',t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|} dV' \tag{5}$$

where $c = 1/\sqrt{\varepsilon_o \mu_o}$ is the speed of light in vacuum. Note that $\vec{r}$ is the position at which $\vec{A}$ and Φ are calculated while $\vec{r}\,'$ contains the variables of integration over the antenna volume, V'. In this formulation the potentials are related by the Lorentz gauge as

$$\nabla \cdot \vec{A} + \varepsilon_o \mu_o \frac{\partial \Phi}{\partial t} = 0 \tag{6}$$

3. All units are rationalized MKSA.

For many purposes we use the Laplace transform (with respect to time) of the various quantities. This is denoted by the addition of a tilde, ∿, above the quantities. Since we are considering a pulse radiated from the antenna we can assume all electromagnetic quantities as initially zero (before $t = 0$) thereby making the initial conditions for the Laplace transform zero. The Laplace transform variable, s, can be set equal to $j\omega$, giving the Fourier transform. This will also be convenient for some purposes. For these transforms we have a propagation constant

$$\gamma = \frac{s}{c} \tag{7}$$

which in the frequency domain becomes

$$\gamma = jk \tag{8}$$

where

$$k = \frac{\omega}{c} \tag{9}$$

In the transform notation we have the potentials

$$\begin{aligned} \tilde{\vec{A}}(\vec{r}) &= \frac{\mu_o}{4\pi} \int_{V'} \tilde{\vec{J}}(\vec{r}\,') \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \, dV' \\ \tilde{\phi}(\vec{r}) &= \frac{1}{4\pi\epsilon_o} \int_{V'} \tilde{\rho}(\vec{r}\,') \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \, dV' \end{aligned} \tag{10}$$

the fields

$$\tilde{\vec{B}} = \nabla \times \tilde{\vec{A}} \quad , \quad \tilde{\vec{E}} = -\nabla\tilde{\phi} - s\tilde{\vec{A}} \tag{11}$$

and the equation of continuity

$$\nabla \cdot \tilde{\vec{J}} + s\tilde{\rho} = 0 \tag{12}$$

Now consider the limiting expressions for the far fields. Make some definitions as

$$|\vec{r}| \equiv r \quad , \quad |\vec{r}\,'| \equiv r' \tag{13}$$

and let ψ be the angle between $\vec{r}$ and $\vec{r}'$ so that

$$\cos(\psi) = \frac{\vec{r}\cdot\vec{r}'}{rr'} = \frac{\vec{e}_r\cdot\vec{r}'}{r'} \tag{14}$$

where $\vec{e}_r$ is the unit vector in the direction of $\vec{r}$. Then as $r\to\infty$ we have

$$\begin{aligned} |\vec{r}-\vec{r}'| &= r - \vec{e}_r\cdot\vec{r}' + O(r^{-1}) = r-\cos(\psi)r' + O(r^{-1}) \\ |\vec{r}-\vec{r}'|^{-1} &= r^{-1} + O(r^{-2}) \end{aligned} \tag{15}$$

From equation 10 as $r\to\infty$ the vector potential is then

$$\tilde{\vec{A}}(\vec{r}) = \frac{e^{-\gamma r}}{r}\frac{\mu_o}{4\pi}\int_{V'} e^{\gamma\vec{e}_r\cdot\vec{r}'}\,\tilde{\vec{J}}(\vec{r}')dV' + O(r^{-2}) \tag{16}$$

The first term is associated with the far fields and from it the radiation vector is defined as

$$\tilde{\vec{N}}(\vec{r}) \equiv \frac{\mu_o}{4\pi}\int_{V'} e^{\gamma\vec{e}_r\cdot\vec{r}'}\,\tilde{\vec{J}}(\vec{r}')dV' \tag{17}$$

so that for large r we have

$$\tilde{\vec{A}} = \frac{e^{-\gamma r}}{r}\tilde{\vec{N}} + O(r^{-2}) \tag{18}$$

In the time domain the radiation vector is

$$\vec{N}(\vec{r},t) = \frac{\mu_o}{4\pi}\int_{V'} \vec{J}(\vec{r}',t+\frac{\vec{e}_r\cdot\vec{r}'}{c})dV' \tag{19}$$

and the vector potential for large r is then

$$\vec{A}(\vec{r},t) = \frac{1}{r}\vec{N}(\vec{r},t-\frac{r}{c}) + O(r^{-2}) \tag{20}$$

Note in equation 19 that we assume that $\vec{N}$ is zero before $t = 0$ for all directions, $\vec{e}_r$. Thus we assume that for the source currents $\vec{J}$ is zero at a particular $\vec{r}'$ before $t = r'/c$. This gives no problem in that the definition of $t = 0$ can be arbitrarily chosen for our convenience. In turn this implies that the potentials and fields are zero before $t = r/c$.

For large r the fields also go to limiting forms which are called the far fields or radiation fields. Denoting these quantities with a subscript, f, we have after some manipulation

$$\tilde{\vec{E}}_f = s \frac{e^{-\gamma r}}{r} (\tilde{\vec{N}} \times \vec{e}_r) \times \vec{e}_r$$

$$\tilde{\vec{H}}_f = \frac{s}{Z_o} \frac{e^{-\gamma r}}{r} (\tilde{\vec{N}} \times \vec{e}_r) \tag{21}$$

where Z_o is the impedance of free space given by

$$Z_o \equiv \sqrt{\frac{\mu_o}{\varepsilon_o}} \tag{22}$$

In the time domain these far fields are

$$\vec{E}_f(\vec{r},t) = \frac{1}{r} \left(\frac{\partial \vec{N}(\vec{r},t-\frac{r}{c})}{\partial t} \times \vec{e}_r \right) \times \vec{e}_r$$

$$\vec{H}_f(\vec{r},t) = \frac{1}{Z_o r} \left(\frac{\partial \vec{N}(\vec{r},t-\frac{r}{c})}{\partial t} \times \vec{e}_r \right) \tag{23}$$

Note that $\vec{E}_f$ and $\vec{H}_f$ are mutually orthogonal and are each orthogonal to $\vec{e}_r$, giving an outward propagating wave. In a spherical coordinate system (r,θ,ϕ) centered on the antenna the far fields have only θ and ϕ components. The θ component of $\vec{E}_f$ has the same waveform as the ϕ component of $\vec{H}_f$, and the remaining components are similarly related. Thus, we need to consider the waveforms for one of the far fields.

III. Low-Frequency Behavior of Far Fields

Having the far fields expressed in terms of the antenna currents we now consider some of the low-frequency characteristics of the far fields. These low-frequency characteristics are reflected in certain features of the radiated waveform and the repeated integrals of the radiated waveform with respect to time. As a matter of notation we denote the nth repeated integral with respect to time as

$$i^n\vec{E}_f(t) \equiv \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \vec{E}_f(t_n) dt_n \cdots dt_1 \tag{24}$$

Also, for convenience, we sometimes use the retarded time

$$t^* \equiv t - \frac{r}{c} \tag{25}$$

The potentials and fields are zero before $t^* = 0$.

As a first simple result consider the full time integral of $\vec{E}_f$. From the first of equations 23 we have

$$i^{1}\vec{E}_f(\infty) = \int_0^\infty \vec{E}_f(t)dt = \frac{1}{r}\left(\vec{N}(\vec{r},t-\frac{r}{c})\times\vec{e}_r\right)\times\vec{e}_r\Big|_0^\infty \tag{26}$$

Now $\vec{N}(\vec{r},-\frac{r}{c}) = \vec{0}$ and assuming that $\vec{J}$ goes to zero at large times for a pulsed antenna, then we also have $\vec{N}(\vec{r},\infty) = \vec{0}$. Thus we have

$$i^{1}\vec{E}_f(\infty) = \vec{0} \tag{27}$$

Stated in another way the radiated waveform has zero net "area". If any component is initially strictly positive (for some $t^* \geq 0$) then it must go negative at some later time. The waveform, if not identically zero, must have both polarities.

Next we extend the above results by considering the low frequency behavior of the currents and charges and then applying the final-value theorem of the Laplace transform. Manipulate the radiation vector from equation 17 as

$$\tilde{\vec{N}}(\vec{r}) = \frac{\mu_o}{4\pi}\int_{V'}\tilde{\vec{J}}(\vec{r}\,')dV' + \frac{\mu_o}{4\pi}\int_{V'}\tilde{\vec{J}}(\vec{r}\,')\left[e^{\gamma\vec{e}_r\cdot\vec{r}\,'}-1\right]dV' \tag{28}$$

The first of these integrals can be integrated by parts giving[4]

$$\int_{V'}\tilde{\vec{J}}(\vec{r}\,')dV' = -\int_{V'}\vec{r}\,'(\nabla'\cdot\tilde{\vec{J}}(\vec{r}\,'))dV' = s\int_{V'}\vec{r}\,'\tilde{\rho}(\vec{r}\,')dV' \tag{29}$$

where the last integral is the electric dipole moment, $\tilde{\vec{p}}$. Thus we have

$$\int_{V'}\tilde{\vec{J}}(\vec{r}\,')dV' = s\tilde{\vec{p}} \quad , \quad \int_{V'}\vec{J}(\vec{r}\,')dV' = \frac{\partial\vec{p}}{\partial t} \tag{30}$$

For the second integral in equation 28 we have for small $|s|$

$$\int_{V'}\tilde{\vec{J}}(\vec{r}\,')\left[e^{\frac{s}{c}\vec{e}_r\cdot\vec{r}\,'}-1\right]dV' = \int_{V'}\tilde{\vec{J}}(\vec{r}\,')\,O(s)dV' \text{ as } s\to 0 \tag{31}$$

4. J.D. Jackson, Classical Electrodynamics, Wiley, 1962, p.271.

Now we expect $\vec{J}$ to go to zero for large times for our pulsed antenna. Then the final-value theorem of the Laplace transform implies

$$\vec{J}(\vec{r}',\infty) = 0 = \lim_{s\to 0} s\tilde{\vec{J}} \tag{32}$$

so that $\tilde{\vec{J}}$ must be less singular than 1/s or

$$\tilde{\vec{J}} = o\left(\frac{1}{s}\right) \quad \text{as } s\to 0 \tag{33}$$

Applying equation 32 to equation 31 we then have

$$\lim_{s\to 0}\int_{V'} \tilde{\vec{J}}(\vec{r}\,')O(s)dV' = 0 = \lim_{s\to 0}\int_{V'} \tilde{\vec{J}}(\vec{r}\,') \left[e^{\frac{s}{c}\vec{e}_r\cdot\vec{r}\,'}-1\right] dV' \tag{34}$$

Returning to equation 28 we then have for small $|s|$

$$\lim_{s\to 0} \tilde{\vec{N}}(\vec{r}) = \left[\lim_{s\to 0} \frac{\mu_o}{4\pi} s\tilde{\vec{p}}\right] + \left[\lim_{s\to 0}\int_{V'} \tilde{\vec{J}}(\vec{r}\,')\left[e^{\frac{s}{c}\vec{e}_r\cdot\vec{r}\,'} -1\right] dV'\right]$$

$$= \lim_{s\to 0} \frac{\mu_o}{4\pi} s\tilde{\vec{p}} = \frac{\mu_o}{4\pi}\vec{p}(\infty) \tag{35}$$

Thus, the late-time behavior of the electric dipole moment is brought to our attention. Depending on the design of the pulser and antenna, $\vec{p}(\infty)$ can be made to be either zero or nonzero. Even though the currents on the antenna go to zero at large t the antenna can remain in a state with charge separation at large times. A simple example is two pieces of metal with a generator to transfer charge from one to the other in such a manner as to produce a static electric dipole. One may wish to eventually discharge the antenna to prepare for another pulse, etc., but the discharge time can be made to be much larger than times of interest for the waveform. For purposes of our model then we can allow $\vec{p}(\infty)$ to be nonzero. Note that we have assumed $\vec{p}(0) = \vec{0}$, consistent with our other initial conditions.

The utility of $\vec{p}(\infty)$ comes in the consideration of the time integral of the radiated fields. From equation 35 we have for the long-time limit of $i^1\vec{N}(t)$

$$i^1\vec{N}(\infty) = \lim_{s\to 0} s\left(\frac{\tilde{\vec{N}}}{s}\right) = \lim_{s\to 0} \tilde{\vec{N}} = \frac{\mu_o}{4\pi}\vec{p}(\infty) \tag{36}$$

Then from the first of equations 23 the second time integral of the radiated fields is given by

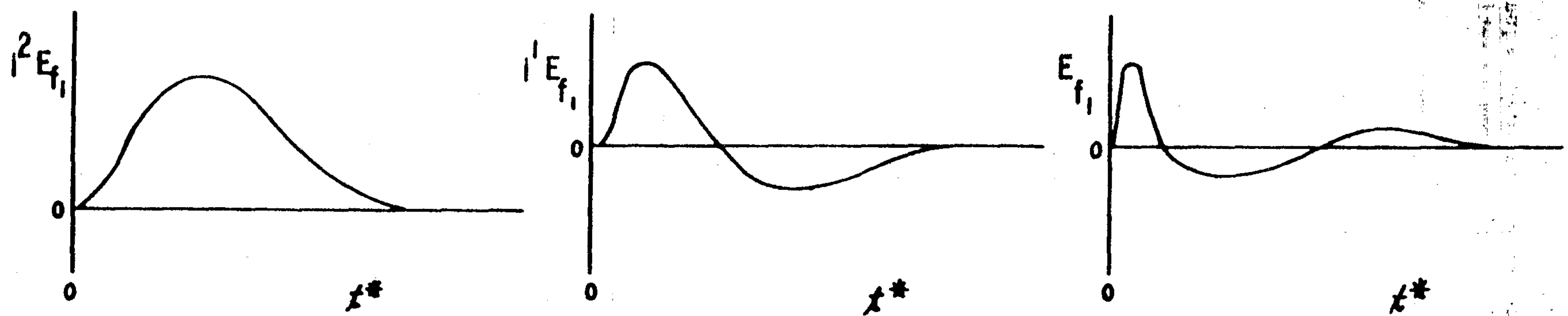
$$i^2\vec{E}_f(\infty) = \frac{1}{r}\left(\left(i^1\vec{N}(\infty)\right) \times \vec{e}_r\right) \times \vec{e}_r = \frac{\mu_o}{4\pi r}\left(\vec{p}(\infty) \times \vec{e}_r\right) \times \vec{e}_r \tag{37}$$

Thus, if $\vec{p}(\infty) = \vec{0}$ then $i^2\vec{E}_f(\infty) = \vec{0}$, while if $\vec{p}(\infty) \neq 0$ then $i^2\vec{E}_f(\infty) \neq 0$ except in the case that $\vec{e}_r$ (the direction vector to the observer) is parallel or anti-parallel to $\vec{p}(\infty)$.

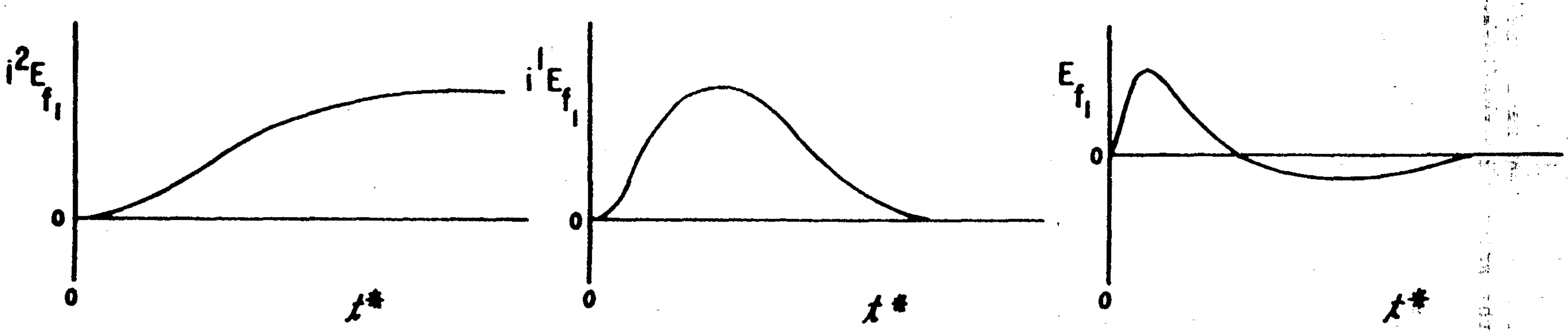
Figure 2 illustrates the effect of $\vec{p}(\infty)$ on the radiated waveform. Denote some particular component of $\vec{E}_f$ with the subscript, 1. First consider the case of $\vec{p}(\infty) = \vec{0}$ as illustrated in figure 2A. From equation 37 we have $i^2E_{f_1}(\infty) = 0$. Thus, $i^2E_{f_1}$ as a function of t* begins and ends at zero, and assuming that it is not identically zero then it can have a single polarity which we take as positive, as illustrated. Thus, $i^2E_{f_1}$ must increase and then decrease, implying that $i^1E_{f_1}$ is first positive and then negative. Thus, $i^1E_{f_1}$ must have at least one zero crossing, as illustrated. Since $i^1E_{f_1}$ must then increase, decrease, and increase again, then E_{f_1} is first positive, then negative, and finally again positive. Thus, E_{f_1} must have at least two zero crossings. Certainly E_{f_1} could have more than two zero crossings, but it must have a minimum of two if $\vec{p}(\infty) = \vec{0}$.

Second consider the case of $\vec{p}(\infty) \neq \vec{0}$ with the direction of $\vec{e}_r$ and choice of component 1 such that from equation 37, we have $i^2E_{f_1}(\infty) \neq 0$. Then taking $i^2E_{f_1}(\infty) > 0$ we have one possible behavior of $i^2E_{f_1}$ as illustrated in figure 2B, i.e., $i^2E_{f_1}$ can increase from zero to its final value without ever having a negative derivative. Thus, $i^1E_{f_1}$ can have a single polarity, as illustrated. Since $i^1E_{f_1}$ must then increase from zero and decrease back to zero, then E_{f_1} is first positive and then negative. Thus E_{f_1} must have at least one zero crossing. The radiated waveform could have more than one zero crossing, but it must have a minimum of one.

Suppose that one would like a single polarity (say nonnegative) radiated waveform. The above results show this to be impossible. There must be at least one zero crossing in the waveform and the time integral of the waveform must go to zero for large time. However, there need not be more than one zero crossing, but this requires, at a minimum, that $\vec{p}(\infty) \neq 0$. From the point of view of designing such an antenna one can then see the advantage of making the antenna have a charge separation with a net electric dipole moment at late times which are still of interest for the waveform.



A. POSSIBLE WAVEFORM FOR $\vec{p}(\infty) = \vec{0}$



B. POSSIBLE WAVEFORM FOR $\vec{p}(\infty) \neq \vec{0}$

FIG.2 INFLUENCE OF LATE-TIME ELECTRIC DIPOLE MOMENT ON RADIATED WAVEFORM

Another interesting result is the dependence of the low-frequency behavior of the antenna on the late-time dipole moment. From equation 35 we see that as $s \to 0$

$$\tilde{\vec{p}} \to \frac{\vec{p}(\infty)}{s} \tag{38}$$

provided that $\vec{p}(\infty) \neq \vec{0}$. Then from equations 28 and 35 we have as $s \to 0$

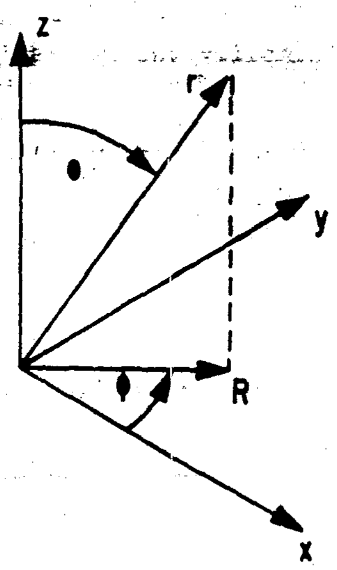
$$\tilde{\vec{E}}_f \to \frac{\mu_0}{4\pi} s \frac{1}{r} (\tilde{\vec{p}}(\infty) \times \vec{e}_r) \times \vec{e}_r \tag{39}$$

Thus for low frequencies the radiated waveform is proportional to the frequency. If the late-time dipole moment is zero, however, then $\tilde{\vec{E}}_f$ is proportional to some higher power of s for small $|s|$, thereby rolling off more rapidly for low frequencies.

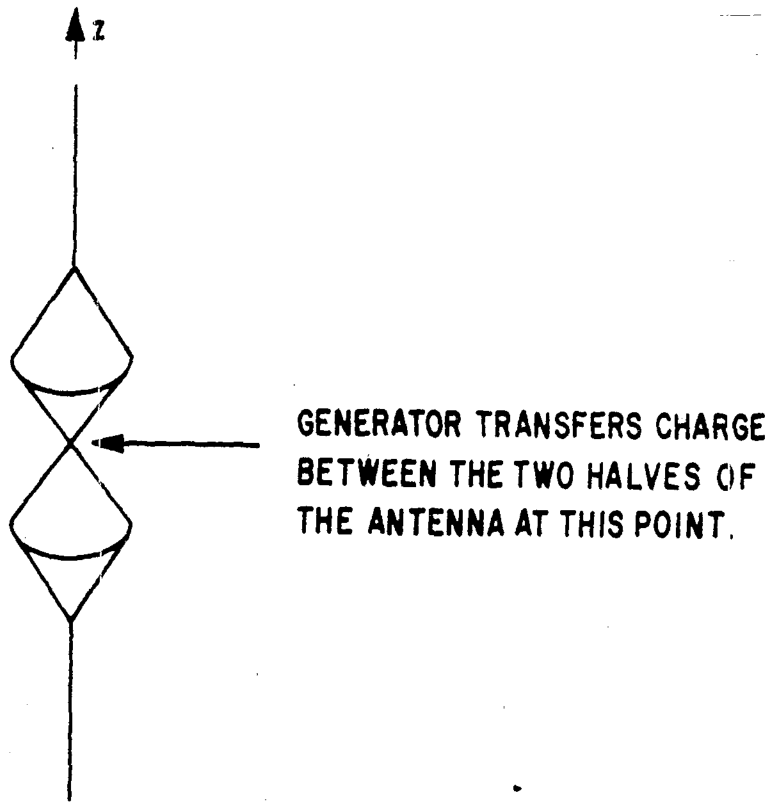
Remember that the above results for the shape of the radiated waveform and for its low-frequency behavior depend on certain assumptions regarding the late-time behavior of the currents on the antenna. Specifically we have assumed that the currents go to zero at late times and have allowed the possibility of a charge separation at late times with a net electric dipole moment, $\vec{p}(\infty)$. One might think of trying to design an antenna for which the currents do not go to zero at late times of interest for the waveform. For example, one might try to set up a steady current flow at late times which gave a net magnetic dipole moment. This could possibly be used to improve the waveform in some manner. However, such a design with steady currents at late times may be somewhat harder to achieve practically. In section V we consider some examples of radiated waveforms which are consistent with the above low-frequency limitations based on the late-time antenna currents going to zero, but with a net late-time electric dipole moment included.

IV. Axially and Lengthwise Symmetric Pulsed Dipole Antenna

We now consider the special case of an antenna which is axially symmetric and symmetric lengthwise. For reference consider the cylindrical (R,ϕ,z) and spherical (r,θ,ϕ) coordinate systems illustrated in figure 3A. Now constrain axial symmetry on the antenna by assuming that its geometry and sources are independent of ϕ. Also assume that sources only generate currents in the R and z directions. Then there are the remaining electromagnetic quantities referred to cylindrical coordinates: J_R, J_z, ρ, E_R, E_z, H_ϕ. These are all independent of ϕ. The far fields are more conveniently related to spherical coordinates, having E_{f_θ} and H_{f_ϕ} as components. Next constrain lengthwise symmetry by making the currents and charges have a certain symmetry with respect to the plane, $z = 0$. Specifically we require that J_z be an even function of z and that J_R and ρ be odd functions of z. This makes E_z and H_ϕ be even functions of z and E_R be an odd function of z. The far-field components, E_{f_θ} and H_{f_ϕ}, are now both even functions of $\theta-\pi/2$.



A. COORDINATE SYSTEMS



B. EXAMPLE: ONE CONFIGURATION OF RES I

FIG. 3 AXIALLY AND LENGTHWISE SYMMETRIC PULSED DIPOLE ANTENNA

These symmetry conditions simplify the expressions for the radiated fields and define a class of antennas of practical importance. A common example of such an antenna is a cylindrical rod which is divided in the center of its length and driven there with an electrical energy source which transfers charge between the two halves of the rod. The first model of RES I uses an antenna with the above symmetry restrictions as illustrated in figure 3B. The resistive impedances in the antenna are not shown. The generator (located inside one of the antenna halves) includes a capacitor which discharges into the antenna at the midpoint, transferring charge between the two antenna halves.

Considering the single component of the far electric field the first of equations 21 gives

$$\tilde{E}_{f_\theta} = s \frac{e^{-\gamma r}}{r} \vec{e}_\theta \cdot \left[(\vec{\tilde{N}} \times \vec{e}_r) \times \vec{e}_r\right] \qquad (40)$$

Using the scalar triple product relations gives

$$\tilde{E}_{f_\theta} = -s \frac{e^{-\gamma r}}{r} \vec{\tilde{N}} \cdot \vec{e}_\theta = -\frac{\mu_o}{4\pi} s \frac{e^{-\gamma r}}{r} \int_{V'} \vec{\tilde{J}}(\vec{r}') \cdot \vec{e}_\theta e^{\gamma \vec{e}_r \cdot \vec{r}'} dV' \qquad (41)$$

or

$$E_{f_\theta}(\vec{r},t) = -\frac{\mu_o}{4\pi r} \frac{\partial}{\partial t} \int_{V'} \vec{J}(\vec{r}', t - \frac{r - \vec{e}_r \cdot \vec{r}'}{c}) \cdot \vec{e}_\theta \, dV' \qquad (42)$$

For convenience in analysis of such pulsed antennas we define a normalized waveform as

$$\tilde{\xi}(\theta,s) \equiv \frac{r\tilde{E}_{f_\theta}}{V_o} e^{\gamma r} = -\frac{\mu_o}{4\pi} \frac{s}{V_o} \int_{V'} \vec{\tilde{J}}(r') \cdot \vec{e}_\theta e^{\gamma \vec{e}_r \cdot \vec{r}'} dV' \qquad (43)$$

where V_o is a scaling voltage, such as the voltage on a capacitor before it is discharged into the antenna. The corresponding expression in the time domain is

$$\xi(\theta,t^*) = \frac{rE_{f_\theta}(\vec{r},t)}{V_o} = -\frac{\mu_o}{4\pi} \frac{1}{V_o} \frac{\partial}{\partial t} \int_{V'} \vec{J}(\vec{r}', t^* + \frac{\vec{e}_r \cdot \vec{r}'}{c}) \cdot \vec{e}_\theta dV' \qquad (44)$$

Note that we consider ξ as a function of t^* to explicitly show the dependence of the normalized waveform on t^*, independent of r. Then as we have defined $\tilde{\xi}$, it is the Laplace transform of ξ with respect to t^*. Also note that ξ is dimensionless and can be used as one indication of the efficiency of a particular design of antenna with generator(s) if V_o is appropriately chosen. Another point to note is that ξ is zero for $\theta = 0$ and for $\theta = \pi$ because of the symmetry in the antenna currents.

If one desires he can perform part of the integral over V' in equation 41 by integrating over ϕ' and using the ϕ' independence of the components of $\vec{J}$ as expressed in cylindrical coordinates. An approximation that is often used to calculate the radiation from antennas of the present type, if they are long and thin is to take some estimate of the current and consider it lumped on the z axis for $|z'| < z_{max}$ where $2\, z_{max}$ is the length of the antenna. In such a case we have

$$\vec{J}(\vec{r}\,',t) = I(z',t)\ \delta(x')\ \delta(y')\ \vec{e}_z \tag{45}$$

which gives

$$\tilde{\xi}(\theta,s) = \sin(\theta)\ \frac{\mu_o}{4\pi}\ \frac{s}{V_o}\int_{-z_{max}}^{z_{max}} \tilde{I}(z')e^{\gamma z'\cos(\theta)}\ dz'$$

$$\xi(\theta,t^*) = \sin(\theta)\ \frac{\mu_o}{4\pi}\ \frac{1}{V_o}\frac{\partial}{\partial t}\int_{-z_{max}}^{z_{max}} I\left(z',t^*+\frac{z'\cos(\theta)}{c}\right)\ dz' \tag{46}$$

For the special case of $\theta = \frac{\pi}{2}$ (and applying the second of equations 30) the second of equations 46 becomes

$$\xi\left(\frac{\pi}{2},t^*\right) = \frac{\mu_o}{4\pi}\,\frac{1}{V_o}\,\frac{\partial}{\partial t}\int_{-z_{max}}^{z_{max}} I(z',t^*)\,dz' = \frac{\mu_o}{4\pi}\,\frac{1}{V_o}\,\frac{\partial^2 p_z(t^*)}{\partial t^2} \tag{47}$$

Thus, at this observation angle we need only know the dipole moment for the thin-wire approximation. While the thin-wire results are only approximate they may be useful in pointing out some qualitative features of various designs of antennas with pulsers.

Note that while the results of the present section apply to an antenna in free space, they can also be applied to the case that the plane, $z = 0$, is taken as a perfectly conducting sheet and only that part of the antenna for $z \geq 0$ remains. Standard image theory applies here with appropriate factors of 2 introduced. For experimental purposes one can use the free-space antenna design, except cut at the $z = 0$ plane where a conducting ground plane (much larger than the antenna) is placed. With appropriate corrections on any sources placed at the $z = 0$ plane to account for the imaging, the normalized waveform can be conveniently measured at low source levels.

V. Examples of Simple Waveforms

Suppose that our desired radiated waveform has a relatively fast rise and a much longer decay time which approximates an exponential decay. Thus, we might define an ideal normalized waveform something like

$$\xi_o(t^*) = e^{-\beta t^*}\, u(t^*) \tag{48}$$

where $\beta > 0$. We could include another exponential term to give a nonzero rise time but this would not significantly affect the low-frequency behavior of ξ_o and is not of interest to the discussion in this section. Note that the complete time integral of ξ_o is given by

$$i^1\xi_o(\infty) = \frac{1}{\beta} \neq 0 \tag{49}$$

However, as shown in section III, this result is impossible for a radiated waveform with the present restrictions on the antenna currents. Taking the Laplace transform of equation 48 gives

$$\tilde{\xi}_o = \frac{1}{s+\beta} \tag{50}$$

With the restriction of section III that the late-time antenna currents go to zero we found in equation 39 that for low frequencies the waveform is proportional to s, provided the late-time electric dipole moment is nonzero, except for two particular directions to the observer. If $\vec{p}(\infty) = \vec{0}$ then the waveform is proportional to some higher power of s for small $|s|$. Assume then for low frequencies we have a normalized waveform, $\tilde{\xi}$, proportional to s. For our present examples we multiply $\tilde{\xi}_o$ by $s/(s+\gamma)$ with $\gamma > 0$, thereby achieving the required low frequency behavior. Thus, we have

$$\tilde{\xi} = \frac{s}{(s+\beta)(s+\gamma)} \tag{51}$$

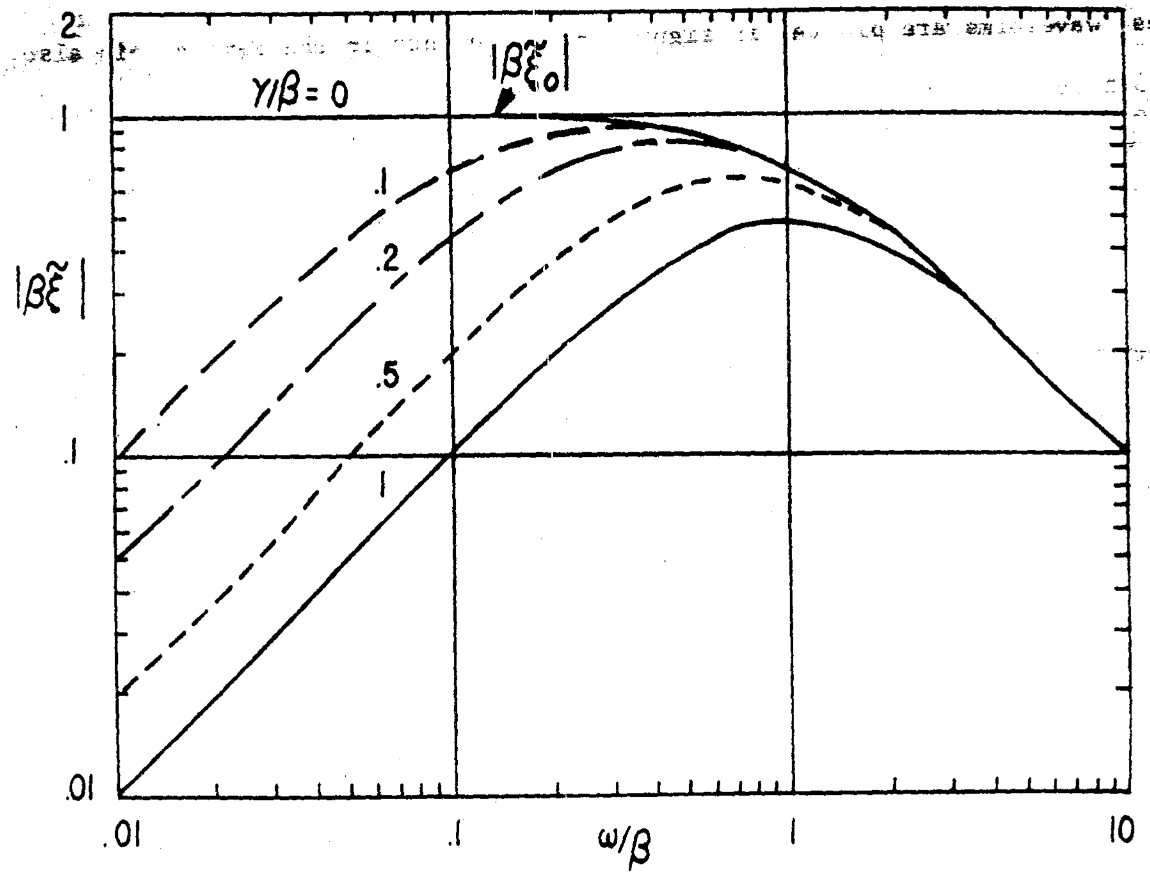
Setting $s = j\omega$ we see in figure 4A the effect of this additional factor on $|\tilde{\xi}|$ for $\gamma \leq \beta$. The high frequency rolloff occurs at $\omega/\beta \simeq 1$ and the low frequency rolloff occurs at $\omega/\beta \simeq \gamma/\beta$. The ratio of γ/β is then an indication of the distortion introduced into the ideal waveform, $\tilde{\xi}_o$. This distortion is minimized by making γ/β as small as possible.

In the time domain the normalized waveform for $\gamma \neq \beta$ is given by

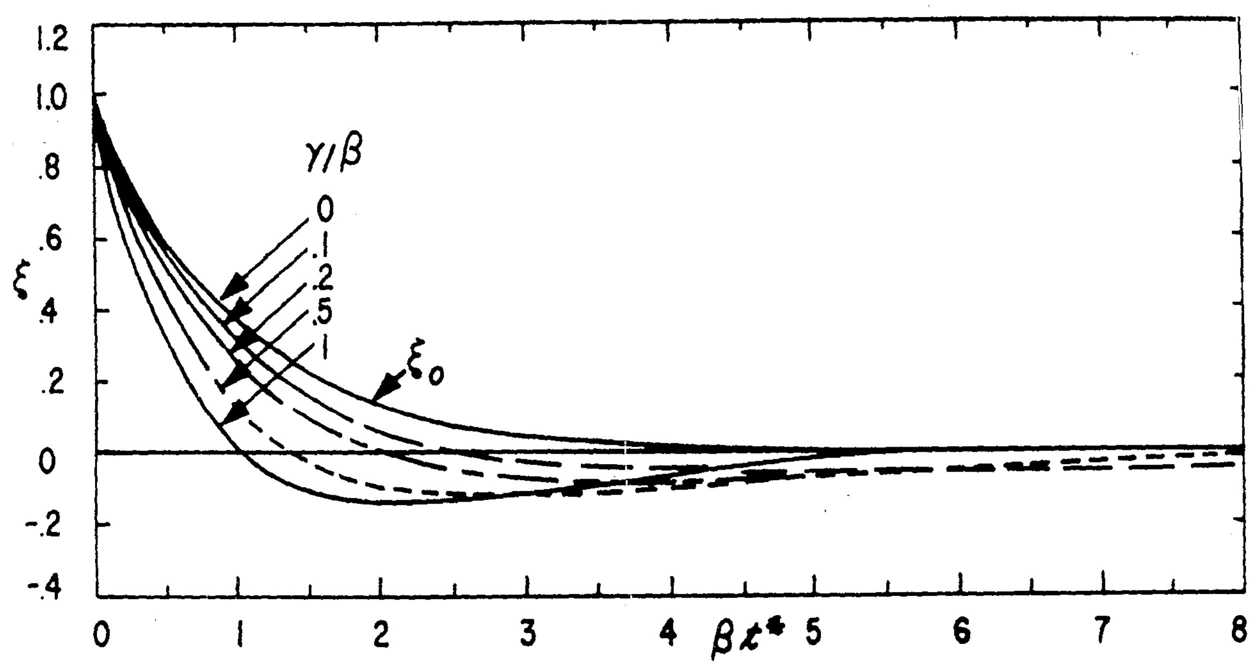
$$\xi(t^*) = \frac{1}{1-\frac{\gamma}{\beta}} \left[e^{-\beta t^*} - \frac{\gamma}{\beta}\, e^{-\gamma t^*}\right] u(t^*) \tag{52}$$

while for $\gamma = \beta$ we have

$$\xi(t^*) = [1 - \beta t^*]e^{-\beta t^*}\, u(t^*) \tag{53}$$



A. FREQUENCY DOMAIN



B. TIME DOMAIN

FIG.4 EXAMPLES OF RADIATED WAVEFORMS

These waveforms are plotted in figure 4B. Note that in the time domain also, small γ/β minimizes the waveform distortion. The initial fast rise and smooth late-time behavior are maintained in going from ξ_o to ξ. One of the effects of the distortion is to decrease the slope after the initial rise. The derivative of the waveform for $t^* > 0$ is

$$\frac{\partial \xi}{\partial t} = \frac{1}{1-\frac{\gamma}{\beta}}\left[-\beta e^{-\beta t^*} + \frac{\gamma^2}{\beta} \quad e^{-\gamma t^*}\right] \tag{54}$$

Then the slope after the initial rise is

$$\left.\frac{\partial \xi}{\partial t}\right|_{t^*=0^+} = \frac{-\beta}{1-\frac{\gamma}{\beta}}\left[1 - \left(\frac{\gamma}{\beta}\right)^2\right] = -\beta\left[1 + \frac{\gamma}{\beta}\right] \tag{55}$$

Here again γ/β is a measure of the waveform distortion. Another effect of the distortion is the undershoot of the resulting waveform. The minimum of ξ occurs at a retarded time, t^*_{min}, found by setting the right side of equation 54 to zero, thereby giving

$$\beta t^*_{min} = \frac{2}{1-\frac{\gamma}{\beta}} \quad \ln\left(\frac{\beta}{\gamma}\right) \tag{56}$$

From this we find the minimum of ξ (for $\gamma \neq \beta$) as

$$\begin{aligned}\xi_{min} &= \frac{1}{1-\frac{\gamma}{\beta}}\left[\left(\frac{\beta}{\gamma}\right)^{-\frac{2}{1-\gamma/\beta}} - \frac{\gamma}{\beta}\left(\frac{\beta}{\gamma}\right)^{-\frac{2\gamma/\beta}{1-\gamma/\beta}}\right] \\ &= \frac{1}{1-\frac{\gamma}{\beta}}\left(\frac{\beta}{\gamma}\right)^{-\frac{2}{1-\gamma/\beta}}\left[1 - \frac{\beta}{\gamma}\right] = -\frac{\beta}{\gamma}\left(\frac{\beta}{\gamma}\right)^{-\frac{2}{1-\gamma/\beta}} \\ &= -\left(\frac{\gamma}{\beta}\right)^{\frac{1+\gamma/\beta}{1-\gamma/\beta}}\end{aligned} \tag{57}$$

For small γ/β this becomes

$$\xi_{min} \simeq -\frac{\gamma}{\beta} \tag{58}$$

so that γ/β is here also a measure of the waveform distortion.

For the present waveform examples we have chosen ξ in a way that approximately preserves some of the features of ξ_o. This choice of ξ satisfies the requirement that $i^1\xi(\infty) = 0$ and it takes advantage of the assumption that $\vec{p}(\infty) \neq 0$ by having only one zero crossing. Of course there are many other ways one could choose ξ and still retain these properties. The actual ξ, of course, depends on the detailed design of the antenna and associated pulsers. A practical antenna and pulser design may give a radiated waveform that looks roughly of the form given in equation 52. Assuming that we have a given desired β one would like to make γ/β as small as possible to get the best waveform. However, this means increasing the low-frequency output of the antenna for a given high-frequency output. This implies an increase in $|\vec{p}(\infty)|$ which may place more stringent requirements on the antenna, e.g., one may decide to increase its size.

VI. Summary

There are many possible designs for a pulse-radiating antenna. These involve various spatial distributions of conductors, dielectrics, magnetic materials, electrical energy sources, etc. Different designs may be used to try to optimize the radiated waveform in terms of rise time, peak amplitude, and/or various other parameters. However, as long as we restrict the antenna current to a volume of space with limited dimensions, require the antenna currents to go to zero for long times of interest, and consider a distant observer so that a far field can be defined as the radiated waveform, then the complete time integral of the far field must go to zero. If we design the antenna to have no late-time electric dipole moment then the radiated waveform must have at least two zero crossings. But if we design the antenna to have a significant late-time electric dipole moment the radiated waveform may have as few as one zero crossing, depending on other features of the antenna and pulser design.

The calculation of radiated waveforms for some realistic antenna geometry may be rather complicated. However, certain features of the waveforms may be obtained somewhat more easily, especially in the high-frequency and low-frequency limits. The results of the present note are based on the low-frequency limit and point out the importance of the late-time electric dipole moment. This is given by things like the low-frequency limits of the capacitance and mean charge separation distance of the antenna and by the generator voltage and capacitance (assuming a generator which is basically a charged capacitor). Parameters such as these can be used to characterize the low-frequency properties of a pulse-radiating antenna, thereby characterizing some of the long-time features of the waveform.

We would like to thank A1C Henry J. McDermott, Jr., for the graphs in this note.