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Parameters for Electrically-Small Loops and Dipoles Expressed in Terms of Current and Charge Distributions

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Abstract

This note formulates the basic low-frequency parameters of loop and dipole sensors in terms of appropriate current and charge distributions in space. The parameters include equivalent areas, equivalent lengths (or heights), and equivalent volumes. This formulation can be used to optimize the current and charge distributions for various applications.

I. Introduction

In a previous note[1] we discussed some of the basic parameters of loops and dipoles with the restriction that the frequencies of interest are low enough that radian wavelengths (or skin depths, as appropriate) in the medium of interest are much larger than the sensor dimensions. This allows one to use quasi-static techniques to calculate the low-frequency parameters of the sensor. For loops and dipoles we have equivalent areas and lengths and inductance or capacitance, as appropriate for the various Thevenin and Norton equivalent circuits. These parameters can be combined to give an equivalent volume which, when divided by an appropriate geometric volume containing the sensor, defines a figure of merit which can be used as a quantitative measure of the efficiency of a sensor design for a particular application.

In the present note we formulate these low-frequency loop and dipole parameters in terms of current and charge distributions, respectively. We assume static currents and charges in space with distributions appropriate to those for loops and dipoles and calculate the sensor parameters as appropriate integrals over these distributions. This formulation has the advantage of aiding one in looking for current and charge distributions which optimize various of these quasi-static sensor parameters. If one knows something about optimum current and charge distributions one can then consider sensor designs from the viewpoint of approximating these distributions.

II. Loop Parameters

The open circuit voltage (quasi-static) of a simple loop sensor is proportional to the time rate of change of some component of the incident $\vec{B}$ field.[2] If the loop is designed such that this particular component of the magnetic field through the loop (open circuited) is undistorted by the loop conductors, then we can calculate the open circuit voltage as

$$V_{oc} = -\oint_{C_o} \vec{E} \cdot d\vec{s} = \int_{S_o} \dot{\vec{B}} \cdot d\vec{S} \tag{1}$$

provided the line and surface integrals are defined such as to include the possibility of multiple-turn or fractional-turn loops. This equation suggests that we define an equivalent area $\vec{A}_{eq}$ for loops in general as

$$V_{oc} \equiv \dot{\vec{B}}_{inc} \cdot \vec{A}_{eq} \tag{2}$$

where $\vec{B}_{inc}$ is the incident static magnetic field. The equivalent area

1. Capt Carl E. Baum, Sensor and Simulation Note 38, Parameters for Some Electrically-Small Electromagnetic Sensors, March 1967.
2. All units are rationalized MKSA.

can be considered a vector to indicate the sensitivity of the loop to the component of $\vec{B}_{inc}$ in the direction parallel to $\vec{A}_{eq}$. From reference 1 we also have an equivalent length as

$$\vec{\ell}_{eq} = \frac{\mu}{L} \vec{A}_{eq} \tag{3}$$

where μ is the permeability of the medium and L is the loop inductance. This equivalent length is defined such that we have a relation between the short-circuit current and the incident field $\vec{H}_{inc}$ as

$$I_{sc} \equiv \vec{H}_{inc} \cdot \vec{\ell}_{eq} \tag{4}$$

There is an equivalent volume given by

$$V_{eq} = \frac{\mu}{L} \vec{A}_{eq} \cdot \vec{A}_{eq} = \frac{L}{\mu} \vec{\ell}_{eq} \cdot \vec{\ell}_{eq} \tag{5}$$

Defining some reference geometric volume V_g inside of which the sensor is contained we have a corresponding figure of merit given by

$$\eta \equiv \frac{V_{eq}}{V_g} \tag{6}$$

For convenience we also define

$$A_{eq} \equiv |\vec{A}_{eq}| \quad , \quad \ell_{eq} \equiv |\vec{\ell}_{eq}| \tag{7}$$

Now consider a static current distribution $\vec{J}(\vec{r}\,')$ in V_g. We have two position vectors, $\vec{r}\,'$ and $\vec{r}$. The corresponding volume integrals use dV' and dV to indicate integration over the primed and unprimed coordinates, respectively. With some static current density $\vec{J}(\vec{r}\,')$ which is required to be divergenceless, i.e.,

$$\nabla' \cdot \vec{J}(\vec{r}\,') = 0 \tag{8}$$

we can define a turns density (units meter^{-2}) as

$$\vec{n}(\vec{r}\,') \equiv \frac{1}{I} \vec{J}(\vec{r}\,') \tag{9}$$

where I is some convenient current used for the definition. For typical cases where there is one pair of sensor terminals we choose I to be the current into one of the terminals and out the other. In some cases it may be desirable to define $\vec{n}(\vec{r}\,')$ differently but we use equation 9 for our present purposes.

The magnetic dipole moment $\vec{m}$ of the current distribution is defined by

$$\vec{m} \equiv \frac{1}{2} \int_{V_g} \vec{r}' \times \vec{J}(\vec{r}') dV' \qquad (10)$$

In terms of the turns density this becomes

$$\vec{m} = \frac{I}{2} \int_{V_g} \vec{r}' \times \vec{n}(\vec{r}') dV' \qquad (11)$$

If I is the current in one particular closed turn of wire (one turn being defined consistent with equation 9) the corresponding magnetic dipole moment can be written as a line integral in the form

$$\vec{m}_1 = \frac{I}{2} \oint_{C_1} \vec{r}' \times d\vec{s}' \qquad (12)$$

where C_1 is the contour of the wire in space.

Referring to figure 1 consider a contour C_{1_x} in the y'z' plane; let this be the projection of C_1 on the y'z' plane. The equivalent area of this wire turn has a component in the x direction given by

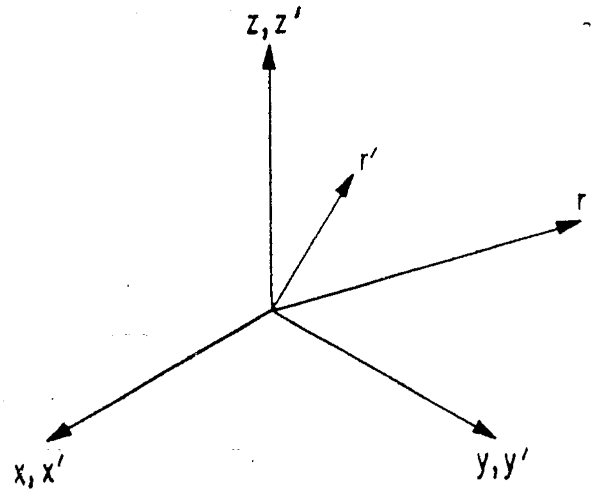
$$A_{1_x} = \oint_{C_x} \frac{1}{2} \vec{e}_x \cdot (\vec{r}' \times d\vec{s}') = \vec{e}_x \cdot \frac{1}{2} \oint_{C_x} \vec{r}' \times d\vec{s}' \qquad (13)$$

Note that $(1/2)|\vec{r}' \times d\vec{s}'|$ is just the incremental area covered by the radius vector $\vec{r}'$ in moving an incremental $d\vec{s}'$ along the curve. If C_{1x} is a simple closed contour as in figure 1 then A_{1_x} is just the geometric area enclosed by the curve. The equivalent area of the loop turn on contour C_1 is then just

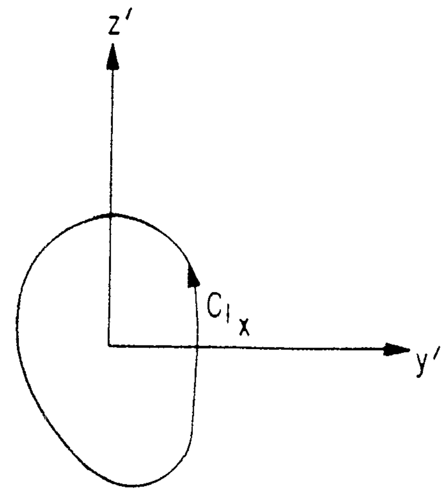
$$\vec{A}_1 = \frac{1}{2} \oint_{C_1} \vec{r}' \times d\vec{s}' \qquad (14)$$

Thus, the magnetic dipole moment and equivalent area of this loop turn are related by

$$\vec{A}_1 = \frac{\vec{m}_1}{I} \qquad (15)$$



A. COORDINATES



B. CONTOUR IN y'z' PLANE WITH POSITIVE CONVENTION

FIGURE 1. COORDINATES AND CONTOUR ILLUSTRATION

Since this relation holds for all the loop turns which go to make up the divergenceless current-density distribution we can calculate the equivalent area of this distribution by adding up all the magnetic dipoles giving

$$\vec{A}_{eq} = \frac{\vec{m}}{I} \tag{16}$$

which can be written as

$$\vec{A}_{eq} = \frac{1}{2I}\int_{V_g} \vec{r}\,' \times \vec{J}(\vec{r}\,')\, dV' = \frac{1}{2}\int_{V_g} \vec{r}\,' \times \vec{n}(\vec{r}\,')\, dV' \tag{17}$$

Thus, we have the equivalent area in terms of an integral over the current density or turns density.

Note the relation between the equivalent area and the magnetic dipole moment. This is similar to the relation between the equivalent height and electric dipole moment for a dipole antenna[3] if current is replaced by charge. The result of equation 16 could also be derived from reciprocity in a manner very similar to the derivation in reference 3 (section IV) using loops instead of electric dipoles.

The vector potential associated with the static current distribution is given by

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\int_{V_g} \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dV' \tag{18}$$

The magnetic energy is given by

$$U_m = \frac{1}{2} LI^2 = \frac{1}{2\mu}\int_{V_\infty} \vec{B}\cdot\vec{B}\, dV = \frac{1}{2}\int_{V_\infty} \vec{B}\cdot\vec{H}\, dV \tag{19}$$

where the volume of integration is all space. As shown in Smythe[4] this magnetic energy can be written as

$$U_m = \frac{1}{2}\int_{V_g} \vec{A}(\vec{r})\cdot\vec{J}(\vec{r})\, dV \tag{20}$$

3. Capt Carl E. Baum, Sensor and Simulation Note 69, Design of a Pulse-radiating Dipole Antenna as Related to High-Frequency and Low-Frequency Limits, January 1969.
4. W. R. Smythe, Static and Dynamic Electricity, 2nd ed., McGraw Hill, 1950, p.315.

The inductance of the sensor is then

$$L = \frac{1}{I^2} \int_{V_g} \vec{A}(\vec{r}) \cdot \vec{J}(\vec{r})\, dV$$

$$= \frac{\mu}{4\pi I^2} \int_{V_g} \int_{V_g} \frac{\vec{J}(\vec{r}\,') \cdot \vec{J}(\vec{r})}{|\vec{r}-\vec{r}\,'|}\, dV' dV$$

$$= \frac{\mu}{4\pi} \int_{V_g} \int_{V_g} \frac{\vec{n}(\vec{r}\,') \cdot \vec{n}(\vec{r})}{|\vec{r}-\vec{r}\,'|}\, dV' dV \qquad (21)$$

Having the equivalent area and inductance of the loop sensor we can calculate an equivalent length as

$$\vec{\ell}_{eq} = \frac{\mu}{L} \vec{A}_{eq} = 2\pi I \frac{\int_{V_g} \vec{r}\,' \times \vec{J}(\vec{r}\,')\, dV'}{\int_{V_g}\int_{V_g} \frac{\vec{J}(\vec{r}) \cdot \vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dV' dV}$$

$$= 2\pi \frac{\int_{V_g} \vec{r}\,' \times \vec{n}(\vec{r}\,')\, dV'}{\int_{V_g}\int_{V_g} \frac{\vec{n}(\vec{r}) \cdot \vec{n}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dV' dV} \qquad (22)$$

The figure of merit comes from the equivalent volume as

$$\eta = \frac{V_{eq}}{V_g} = \frac{\mu}{V_g L}\vec{A}_{eq}\cdot\vec{A}_{eq} = \frac{\pi}{V_g}\frac{\left\{\int_{V_g} \vec{r}\,' \times \vec{J}(\vec{r}\,')dV'\right\}\cdot\left\{\int_{V_g} \vec{r}\,' \times \vec{J}(\vec{r}\,')dV'\right\}}{\int_{V_g}\int_{V_g}\frac{\vec{J}(\vec{r}\,')\cdot\vec{J}(\vec{r})}{|\vec{r}-\vec{r}\,'|}dV'dV}$$

$$= \frac{\pi}{V_g}\frac{\left\{\int_{V_g} \vec{r}\,' \times \vec{n}(\vec{r}\,')\,dV'\right\}\cdot\left\{\int_{V_g} \vec{r}\,' \times \vec{n}(\vec{r}\,')\,dV'\right\}}{\int_{V_g}\int_{V_g}\frac{\vec{n}(\vec{r}\,')\cdot\vec{n}(\vec{r})}{|\vec{r}-\vec{r}\,'|}dV'dV} \qquad (23)$$

Note that these expressions for the figure of merit (and the corresponding expressions for the equivalent volume) are homogeneous in $\vec{J}$ or n as appropriate; the equivalent volume is only a function of the relative distribution of the current density (or turns density) in space. Knowing the relative current distribution for some loop design one can calculate the figure of merit (and other parameters) from appropriate integrals in this section. For some types of loops the current distribution may be known by inspection because of the special interconnection of the loop turns, symmetry considerations, etc. However, for other types of loops, finding the current distribution may require more elaborate calculations such as the solution of boundary value problems.

III. Dipole Parameters

For an electric dipole the open circuit voltage (quasi static) is proportional to some component of the incident electric field. If $\vec{E}$ is the resultant static electric field (including the presence of the sensor) then the open circuit voltage is given by

$$V_{oc} = -\int_{C_o} \vec{E}\cdot d\vec{s} \qquad (24)$$

where the line integral is along the curve C_o extending between two positions on appropriate separate conductors. Equation 24 suggests the definition of an equivalent height $\vec{h}_{eq}$ as

$$V_{oc} \equiv -\vec{E}_{inc}\cdot\vec{h}_{eq} \qquad (25)$$

where $\vec{E}_{inc}$ is the incident electric field. The sensor is then sensitive to the component of the incident electric field parallel to $\vec{h}_{eq}$. We also have (from reference 1) an equivalent area given by

$$\vec{A}_{eq} = \frac{C}{\varepsilon}\vec{h}_{eq} \qquad (26)$$

where ε is the permittivity of the medium and C is the dipole capacitance. For the present discussion we assume the medium is nonconducting, although as discussed in reference 1 the equivalent height and area can be extended to the case of a conducting medium as well, with certain limitations. The equivalent area is defined so that the incident displacement current density $\vec{D}_{inc}$ and the short circuit current are related by

$$I_{sc} = - \vec{D}_{inc} \cdot \vec{A}_{eq} \tag{27}$$

The equivalent volume is given by

$$V_{eq} = \frac{C}{\varepsilon} \vec{h}_{eq} \cdot \vec{h}_{eq} = \frac{\varepsilon}{C} \vec{A}_{eq} \cdot \vec{A}_{eq} \tag{28}$$

With the sensor contained in some reference volume V_g we have a figure of merit given by

$$\eta \equiv \frac{V_{eq}}{V_g} \tag{29}$$

We also define

$$h_{eq} \equiv |\vec{h}_{eq}| \quad , \quad A_{eq} \equiv |\vec{A}_{eq}| \tag{30}$$

Next consider a static charge distribution $\rho(\vec{r}\,')$ in V_g with

$$\int_{V_g} \rho(\vec{r}\,')\, dV' = 0 \tag{31}$$

We use the two position vectors $\vec{r}\,'$ and $\vec{r}$ as in the previous section. The electric dipole moment is

$$\vec{p} = \int_{V_g} \vec{r}\,' \rho(\vec{r}\,')\, dV' \tag{32}$$

Let $\rho(\vec{r}\,')$ be the charge density on the sensor when we have put some charge Q into one of the sensor terminals and out the other (in the absence of an incident electric field). The mean charge separation distance is then given by

$$\vec{h}_a \equiv \frac{1}{Q} \vec{p} \tag{33}$$

In reference 3 we have shown that the mean charge separation distance is the same as the equivalent height. Thus, we have

$$\vec{h}_{eq} = \frac{1}{Q}\int_{V_g} \vec{r}\,'\,\rho(\vec{r}\,')\,dV' \tag{34}$$

The equivalent height is thus given by an integral over the static charge distribution.

The scalar potential associated with $\rho(r')$ is given by

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon}\int_{V_g} \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV' \tag{35}$$

The electric energy is given by

$$U_e = \frac{1}{2}\,C\,V^2 = \frac{1}{2}\,\varepsilon\int_{V_\infty} \vec{E}\cdot\vec{E}\,dV = \frac{1}{2}\int_{V_\infty}\vec{D}\cdot\vec{E}\,dV \tag{36}$$

where V is the voltage at the antenna terminals and the volume of integration is all space. Next we use

$$\vec{E} = -\nabla\Phi \qquad , \qquad \nabla\cdot\vec{D} = \rho \tag{37}$$

so that we have

$$\begin{aligned} U_e &= -\frac{1}{2}\int_{V_\infty}\vec{D}\cdot\nabla\Phi\,dV = -\frac{1}{2}\int_{V_\infty}[\nabla\cdot(\Phi\vec{D}) - \Phi\nabla\cdot\vec{D}]\,dV \\ &= \frac{1}{2}\int_{V_g}\Phi\rho\,dV - \frac{1}{2}\int_{V_\infty}\nabla\cdot(\Phi\vec{D})\,dV \end{aligned} \tag{38}$$

The last of these integrals can be written as a surface integral over a surface S_∞ which we take as a surface of constant $r = |\vec{r}|$ with $r \to \infty$. Thus, we write

$$\int_{V_\infty}\nabla\cdot(\Phi\vec{D})\,dV = \lim_{r\to\infty}\int_{S_\infty}\Phi\vec{D}\cdot d\vec{S} \tag{39}$$

where $d\vec{S}$ points away from the origin. Now in another note[5] we show with the restriction given by equation 31 that $\Phi = O(r^{-2})$ as $r \to \infty$. The area of S_∞ is $O(r^2)$ as $r \to \infty$. Then since $|\vec{D}| \to 0$ as $r \to \infty$ we have

$$\int_{V_\infty} \nabla \cdot (\Phi \vec{D})\, dV = 0 \tag{40}$$

and thus we have

$$U_e = \frac{1}{2} \int_{V_g} \phi(\vec{r})\, \rho(\vec{r})\, dV \tag{41}$$

This result is the analog of equation 20 in the previous section. The capacitance of the sensor is then

$$C = \frac{1}{V^2} \int_{V_g} \Phi(\vec{r})\, \rho(\vec{r})\, dV$$

$$= \frac{1}{4\pi\varepsilon V^2} \int_{V_g} \int_{V_g} \frac{\rho(\vec{r}\,')\rho(\vec{r})}{|\vec{r}-\vec{r}\,'|}\, dV' dV \tag{42}$$

Now the antenna voltage is given by

$$V = \frac{Q}{C} \tag{43}$$

which, when substituted into equation 42 gives

$$C = 4\pi\varepsilon Q^2 \left\{ \int_{V_g} \int_{V_g} \frac{\rho(\vec{r}\,')\rho(\vec{r})}{|\vec{r}-\vec{r}\,'|}\, dV' dV \right\}^{-1} \tag{44}$$

so that the capacitance is expressed in terms of the charge distribution. Note that Q (the antenna charge) is just an integral of $\rho(\vec{r}\,')$ over some appropriate part of the sensor.

5. Capt Carl E. Baum, Sensor and Simulation Note 72, An Equivalent-Charge Method for Defining Geometries of Dipole Antennas, January 1969.

The equivalent area is given in terms of the equivalent height and capacitance as

$$\vec{A}_{eq} = \frac{C}{\varepsilon}\vec{h}_{eq} = 4\pi Q \frac{\int_{V_g} \vec{r}\,' \rho(\vec{r}\,')\, dV'}{\int_{V_g}\int_{V_g} \frac{\rho(\vec{r}\,')\rho(\vec{r})}{|\vec{r}-\vec{r}\,'|}\, dV' dV} \tag{45}$$

The figure of merit comes from the equivalent volume as

$$\eta = \frac{V_{eq}}{V_g} = \frac{C}{V_g \varepsilon}\vec{h}_{eq} \cdot \vec{h}_{eq} = \frac{4\pi}{V_g} \frac{\left\{\int_{V_g} \vec{r}\,'\rho(\vec{r}\,')dV'\right\} \cdot \left\{\int_{V_g} \vec{r}\,'\rho(\vec{r}\,')dV'\right\}}{\int_{V_g}\int_{V_g} \frac{\rho(\vec{r}\,')\rho(\vec{r})}{|\vec{r}-\vec{r}\,'|}\, dV' dV} \tag{46}$$

Note that the figure of merit and equivalent volume are homogeneous in ρ. Knowing $\rho(\vec{r}\,')$ one can then calculate the various sensor parameters as integrals over this distribution. However, for a particular sensor design, one may need to solve the Laplace equation in Φ to determine this distribution, making the problem somewhat more difficult. On the other hand, the formulation of the figure of merit in equation 46 (as also in equation 23) may be used to look for optimum charge (or current) distributions for various sensor design applications. Then one might try to design electrically-small sensors which approximate these distributions.