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Resistively Loaded Radiating Dipole Based on a Transmission-Line Model for the Antenna

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Abstract

In order to smooth the waveform after the initial rise, as radiated from a long and thin pulse-radiating dipole, one can put resistive loading in series with the antenna conductors. In this note we consider a few forms of such resistive loading for which the resistance is continuously distributed along the antenna. The calculations are based on an approximate transmission-line model of the antenna. The results indicate some smoothing associated with a uniform resistance per unit length for the antenna. The waveform is further improved by the use of a special nonuniform resistance distribution for which the resistance per unit length goes to ∞ at the ends of the antenna.

I. Introduction

One of the problems of a pulse-radiating dipole antenna concerns the manner of impedance loading the structure so as to optimize the radiated waveform. Consider a general axially and lengthwise symmetric electric dipole antenna as in figure 1. We assume that it is driven by a capacitive generator; the generator capacitance is sometimes assumed to be large compared to the antenna capacitance. As discussed in two earlier notes[1,2] there are certain fundamental limitations imposed on the radiated waveform by such an antenna-pulser combination. The addition of series passive elements in the antenna will still give a radiated waveform consistent with these limitations. Such additional elements can still, however, have a significant effect on the radiated waveform, such as by damping resonances on the antenna.

As discussed in another two notes[3,4] lumped resistors can be introduced along the antenna structure to shape the waveform. In this note we consider some continuous distributions of resistive loading along the antenna. For these calculations we use an approximate transmission-line model for the antenna. The resistive loading is considered from the point of view of smoothing the waveform after the initial rise. The resistance attenuates the current wave propagating toward the ends of the antenna so that abrupt changes in the radiated waveform (associated with such reflections) are reduced. The case of uniform resistance distribution (per unit length of the antenna) is first considered. This is followed by consideration of a special form of nonuniform resistance distribution.

II. Transmission-Line Model

In figure 1 we have a somewhat general type of axially and lengthwise symmetric dipole antenna. The antenna is assumed to

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1. Capt Carl E. Baum, Sensor and Simulation Note 65, Some Limiting Low-Frequency Characteristics of a Pulse-Radiation Antenna, October 1968.

2. Capt Carl E. Baum, Sensor and Simulation Note 69, Design of a Pulse-Radiating Dipole Antenna as Related to High-Frequency and Low-Frequency Limits, January 1969.

3. David E. Merewether, Sensor and Simulation Note 70, Transient Pulse Transmission Using Impedance Loaded Cylindrical Antennas, February 1968.

4. D. E. Merewether, Sensor and Simulation Note 71, Transient Electromagnetic Fields Near a Cylindrical Antenna Multiply-Loaded with Lumped Resistors, August 1968.

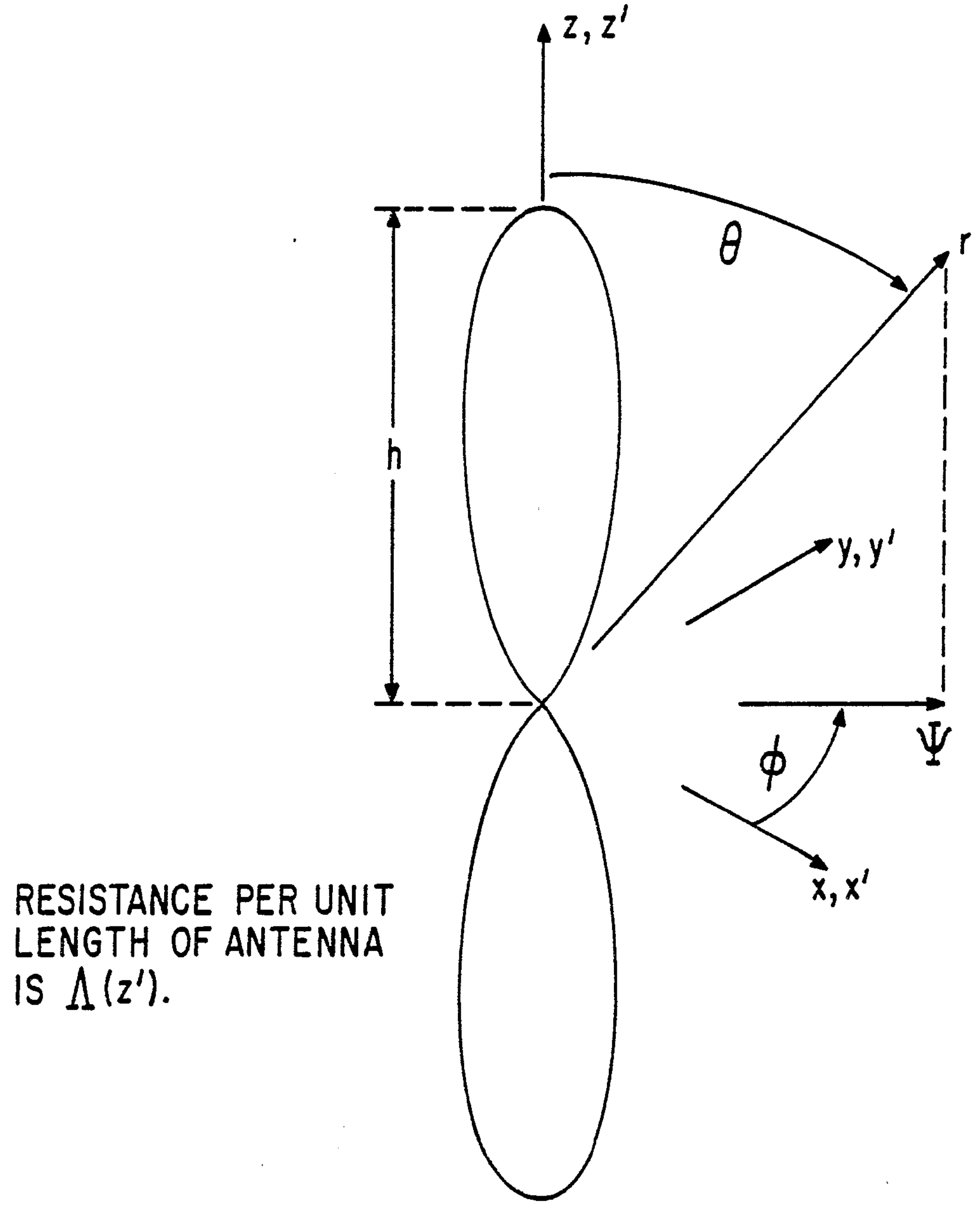


FIGURE 1. AXIALLY AND LENGTHWISE SYMMETRIC DIPOLE ANTENNA

be centered on the z' axis and located symmetrically with respect to the x', y' plane. Primed coordinates are used for currents and other quantities in the immediate vicinity of the antenna; unprimed coordinates are used for the position at which the fields are observed. In the cases of interest in this note the antenna is assumed to have a resistance per unit length $\Lambda(z')$ where for symmetry Λ is assumed to be even in z'. If one desires, Λ can be a more general type of impedance per unit length including inductance and capacitance.

In the transmission-line model of the antenna the antenna in figure 1 is approximated as a transmission line for purposes of calculating the current along the antenna and the voltage at the driving terminals. Figure 2A illustrates this concept. The generator has capacitance C_g and a time-domain voltage source $V_o u(t)$ where $u(t)$ is the unit step function. The generator drives a transmission line of length h equal to the antenna length; this transmission line is approximated as being terminated in an open circuit. The antenna current is $I(z')$ directed parallel to $\vec{e}_z$, the unit vector in the z' direction. $I(z')$ is even in z' by symmetry. For the transmission-line model as in figure 2A we use ζ as the coordinate along the transmission line and $I(\zeta)$ is equal and opposite along the two sides of the transmission line. Also there is a voltage $V(\zeta)$ along the transmission line.

For calculating the various distributed elements of the equivalent transmission line the antenna can be approximated as an equivalent biconical antenna.[5] In this approximation the antenna (without the series impedance loading) has a characteristic impedance Z_∞ given by the characteristic impedance of an appropriate biconical antenna. A biconical antenna with cones at $\theta = \theta_1$ and $\theta = \pi - \theta_1$ has a characteristic impedance[6]

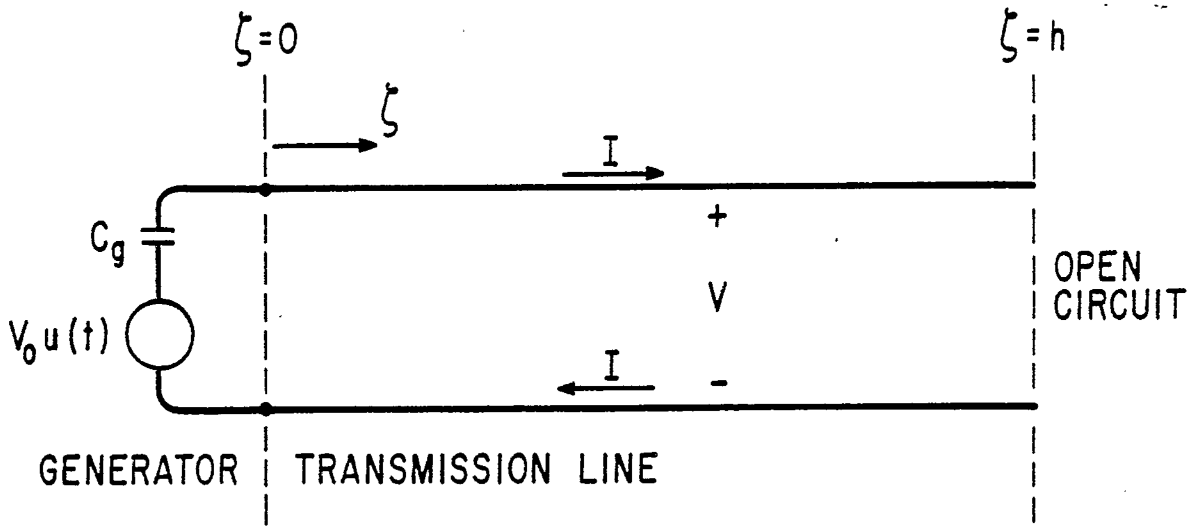
$$Z_\infty = \frac{Z_o}{\pi} \ln\left[\cot\left(\frac{\theta_1}{2}\right)\right] \tag{1}$$

where

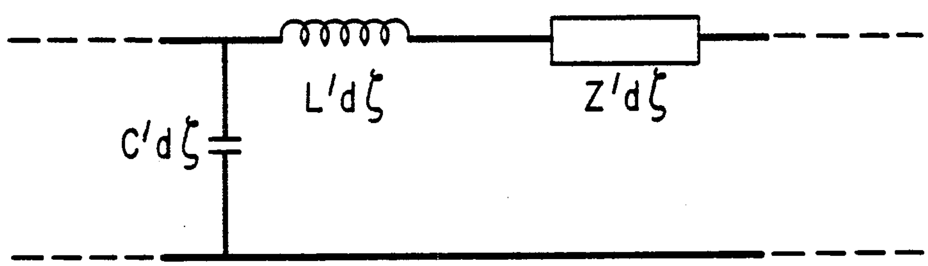
$$Z_o \equiv \sqrt{\frac{\mu_o}{\varepsilon_o}} \simeq 377\Omega \tag{2}$$

5. S. A. Schelkunoff and H. T. Friis, Antennas: Theory and Practice, Wiley, 1952, pp. 425-431.

6. All units are rationalized MKSA.



A. TRANSMISSION LINE WITH GENERATOR



B. INCREMENTAL SECTION OF TRANSMISSION LINE

FIGURE 2. TRANSMISSION-LINE MODEL OF ANTENNA

The corresponding geometric factor in the impedance is

$$f_g \equiv \frac{Z_\infty}{Z_o} = \frac{1}{\pi} \ln\left[\cot\left(\frac{\theta_1}{2}\right)\right] \tag{3}$$

If the biconical antenna has a half length h and a radius a at its ends ($z' = \pm h$) and if $a \ll h$ so that the antenna is thin then

$$f_g \simeq \frac{1}{\pi} \ln\left(\frac{2h}{a}\right) \tag{4}$$

The biconical antenna can then be assigned some value of a such that its mean radius is roughly the mean radius of the antenna of figure 1. Essentially a is chosen such that the approximate characteristic impedance of the antenna in figure 1 corresponds to that of the equivalent biconic.

The medium outside the antenna has permittivity ε_o, permeability μ_o, and zero conductivity. We then have inductance and capacitance per unit length for the equivalent transmission line given by

$$L' = \mu_o f_g$$
$$C' = \frac{\varepsilon_o}{f_g} \tag{5}$$

The series impedance $\Lambda(z')$ put into the antenna contributes an additional longitudinal impedance per unit length given by

$$Z'(\zeta) = 2\Lambda(\zeta) \tag{6}$$

Note the factor of 2 due to the presence of Λ in both arms of the antenna. For an incremental length $d\zeta$ we then have the lumped element representation of the transmission line as shown in figure 2B.

A tilde ~ over a quantity indicates the Laplace transform; the Laplace transform variable is s. For convenience we define a normalized retarded time as

$$\tau_h \equiv \frac{ct - r}{h} \tag{7}$$

where

$$c \equiv \frac{1}{\sqrt{\mu_o \varepsilon_o}} \tag{8}$$

The corresponding normalized Laplace transform variable is

$$s_h \equiv s t_h \tag{9}$$

where

$$t_h \equiv \frac{h}{c} \tag{10}$$

The local propagation constant on the transmission line is

$$\gamma = [(sL' + Z')sC']^{1/2} = \gamma_o \left[1 + \frac{Z'}{sL'}\right]^{1/2} \tag{11}$$

where

$$\gamma_o = s\sqrt{L'C'} = \frac{s}{c} = \frac{s_h}{h} \tag{12}$$

The local impedance is

$$Z = \left[\frac{sL' + Z'}{sC'}\right]^{1/2} = Z_\infty \left[1 + \frac{Z'}{sL'}\right]^{1/2} \tag{13}$$

The transmission-line equations for our case are

$$\begin{aligned} \frac{\partial \tilde{V}}{\partial \zeta} &= -(Z' + sL')\tilde{I} \\ \frac{\partial \tilde{I}}{\partial \zeta} &= -sC'\tilde{V} \end{aligned} \tag{14}$$

Differentiating the second equation with respect to ζ we obtain a one-dimensional wave equation for $\tilde{I}$ as

$$\frac{\partial^2 \tilde{I}}{\partial \zeta^2} - sC'(Z' + sL')\tilde{I} = 0 \tag{15}$$

This can also be written as

$$\frac{\partial^2 \tilde{I}}{\partial \zeta^2} - \left(\gamma_o^2 + sC'Z'\right)\tilde{I} = 0$$

$$\frac{\partial^2 \tilde{I}}{\partial \zeta^2} - \gamma^2 \tilde{I} = 0 \tag{16}$$

After solving for the current $I(\zeta)$ on the equivalent transmission line this current is used for $I(z')$ on the antenna with $\zeta = |z'|$. Then using a thin-antenna approximation I is assumed concentrated on the z' axis and the radiated waveform (as in ref. 1) is calculated as

$$\tilde{\xi}(\theta) = \sin(\theta)\, \frac{\mu_o}{4\pi}\, \frac{s}{V_o} \int_{-h}^{h} \tilde{I}(z') e^{\gamma_o z' \cos(\theta)} dz'$$

$$\xi(\theta) = \sin(\theta)\, \frac{\mu_o}{4\pi}\, \frac{1}{V_o}\, \frac{\partial}{\partial t} \int_{-h}^{h} I\left(z', t^* + \frac{z'\cos(\theta)}{c}\right) dz' \tag{17}$$

where t* is the retarded time given by

$$t^* \equiv t - \frac{r}{c} \tag{18}$$

The normalized waveform in equations 17 is related to the far or radiated electric field E_{f_θ} (only a θ component) by

$$\xi \equiv \frac{rE_{f_\theta}}{V_o}\,, \qquad \tilde{\xi} \equiv \frac{r\tilde{E}_{f_\theta}}{V_o}\, e^{\gamma_o r} \tag{19}$$

Note that ξ is considered using retarded time so that a current wave initiated at $t = 0$ at the center of the antenna will produce a waveform at the observer beginning at $t^* = 0$.

For convenience (as will become apparent later) we define other normalized waveforms by

$$\xi' \equiv 2\pi f_g \xi, \qquad \tilde{\xi}' = 2\pi f_g \frac{\tilde{\xi}}{t_h} \tag{20}$$

In this form $\tilde{\xi}'$ is the Laplace transform of ξ' with respect to τ_h (from equation 7). $\tilde{\xi}'$ is then a function of s_h which we take equal to $j\omega t_h$ to give a Fourier transform for the plots. Writing out these normalized waveforms we have

$$\begin{aligned} \tilde{\xi}'(\theta) &= \sin(\theta) \frac{\mu_o f_g s_h}{2V_o t_h^2} \int_{-h}^{h} \tilde{I}(z') e^{\gamma_o z' \cos(\theta)} dz' \\ \xi'(\theta) &= \sin(\theta) \frac{\mu_o f_g}{2V_o} \frac{\partial}{\partial t} \int_{-h}^{h} I\left(z', t^* + \frac{z' \cos(\theta)}{c}\right) dz' \end{aligned} \tag{21}$$

Assume that Z' is of a form such that at low frequencies ($\omega t_h << 1$) the charge distribution on the open-circuited transmission is not influenced by Z', but only by C'. Then the antenna capacitance is

$$C_a = C'h = \frac{\epsilon_o h}{f_g} \tag{22}$$

and the mean charge separation distance is

$$h_a = h \tag{23}$$

For example if Z' includes only resistors and/or inductors of finite magnitude then equations 22 and 23 apply. If on the other hand Z' were a single series capacitor these equations would not apply. Then for low frequencies ($s \to 0$) we have from reference 2 the result

$$\tilde{\xi} \approx t_h^2 f_\infty s \ , \qquad \tilde{\xi}' \approx 2\pi f_g f_\infty s_h \tag{24}$$

where

$$f_\infty(\theta) = \frac{1}{4\pi}\left[\frac{\varepsilon_o h}{C_a} + \frac{\varepsilon_o h}{C_g}\right]^{-1} \sin(\theta) \tag{25}$$

Define a capacitance parameter as

$$\alpha \equiv 1 + \frac{C_a}{C_g} \tag{26}$$

Then as $s \to 0$ we have asymptotically

$$\tilde{\xi}' \approx \frac{s_h}{2\alpha} \sin(\theta) \tag{27}$$

This general result for our restricted form of Z' is independent of Z' and can be used to check some of the results for the radiated waveforms in the low frequency limit.

The above discussion outlines the transmission-line model for the dipole antenna. This model will be used in the next two sections to calculate the radiated waveforms for the case where Λ is uniform with respect to z' and for the case that Λ is a particular function of z'. The reader should note that due to the limitations of this transmission-line model the results are rather approximate. Of course, one can calculate certain features of the waveform more accurately and use the results for comparison with the results of the transmission-line model. As discussed in reference 2 the amplitude of the initial rise of the radiated waveform can be accurately calculated if a biconical wave launcher is used. Also if the mean charge separation distance and capacitance of the antenna can be accurately calculated or measured we can calculate the low frequency content of the radiated waveform provided the generator meets certain requirements. In addition one can calculate the radiation from a perfectly conducting infinite cylindrical antenna[7] and use this for comparison to the results of the transmission-line model for a perfectly conducting antenna during part of the time-domain waveform. By such comparisons one can obtain some estimate of the accuracy of this transmission-line model for some specific antenna.

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7. R. W. Latham and K. S. H. Lee, Sensor and Simulation Note 73, Pulse Radiation and Synthesis by an Infinite Cylindrical Antenna, February 1969.

III. Uniform Resistive Loading

As our first case for consideration let Λ be a real positive constant independent of z' and define

$$R_o \equiv \Lambda h \tag{28}$$

This is the case of uniform resistive loading of the antenna. R_o is the total resistance of one arm of the antenna, say between z' = 0 and z' = h. The wave equation for $\tilde{I}(\zeta)$ has the solution

$$\tilde{I}(\zeta) = \tilde{I}(0) \frac{e^{-\gamma\zeta} - e^{-\gamma(2h-\zeta)}}{1 - e^{-2\gamma h}} \tag{29}$$

where we have made $\tilde{I}(h) = 0$. The voltage from equations 14 is

$$\tilde{V}(\zeta) = \frac{\tilde{I}(0)}{sC'} \gamma \frac{e^{-\gamma\zeta} + e^{-\gamma(2h-\zeta)}}{1 - e^{-2\gamma h}} \tag{30}$$

The antenna impedance is then

$$Z_a \equiv \frac{\tilde{V}(0)}{\tilde{I}(0)} = Z \frac{1 + e^{-2\gamma h}}{1 - e^{-2\gamma h}} \tag{31}$$

Now $\tilde{I}(0)$ can be found from

$$\tilde{I}(0) = \frac{V_o}{s}\left[\frac{1}{sC_g} + Z_a\right]^{-1} \tag{32}$$

Then $\tilde{I}(\zeta)$ is given by

$$\tilde{I}(\zeta) = \frac{V_o}{s} \frac{e^{-\gamma\zeta} - e^{-\gamma(2h-\zeta)}}{Z[1 + e^{-2\gamma h}] + \frac{1}{sC_g}[1 - e^{-2\gamma h}]} \tag{33}$$

The resistance per unit length of the transmission line is

$$R' = Z' = 2\Lambda \tag{34}$$

giving

$$Z = Z_\infty\left[1 + \frac{R'}{sL'}\right]^{1/2} \tag{35}$$

Having $\tilde{I}(\zeta)$ we can substitute $\zeta = |z'|$ and use this current as the assumed antenna current. From equations 21 we have a normalized radiated waveform

$$\tilde{\xi}'(\theta) = \frac{\sin(\theta)}{2}\left\{\left[1 + \frac{R'}{sL'}\right]^{1/2}[1+e^{-2\gamma h}] + \frac{1}{sC_g Z_\infty}[1-e^{-2\gamma h}]\right\}^{-1}$$

$$\cdot \int_{-1}^{1} [e^{-\gamma h|\nu|} - e^{-\gamma h(2-|\nu|)}]e^{s_h \nu\cos(\theta)} d\nu \tag{36}$$

where we have used the substitution

$$\nu \equiv \frac{z'}{h} \tag{37}$$

Define the integral in equation 36 as Ξ. We then have

$$\Xi = \int_0^1 [e^{-\gamma h\nu} - e^{-2\gamma h}e^{\gamma h\nu}]\left[e^{s_h\nu\cos(\theta)} + e^{-s_h\nu\cos(\theta)}\right]d\nu$$

$$= \frac{e^{s_h\cos(\theta)-\gamma h} - 1}{s_h\cos(\theta) - \gamma h} - \frac{e^{-s_h\cos(\theta)-\gamma h} - 1}{s_h\cos(\theta) + \gamma h}$$

$$- \frac{e^{s_h\cos(\theta)+\gamma h} - 1}{s_h\cos(\theta) + \gamma h}e^{-2\gamma h} + \frac{e^{-s_h\cos(\theta)+\gamma h} - 1}{s_h\cos(\theta) - \gamma h}e^{-2\gamma h} \tag{38}$$

So that the normalized radiated waveform is

$$\tilde{\xi}'(\theta) = \frac{\sin(\theta)}{2}\,\Xi\left\{\left[1+\frac{R'}{sL'}\right]^{1/2}[1+e^{-2\gamma h}] + \frac{1}{sC_g Z_\infty}[1-e^{-2\gamma h}]\right\}^{-1}$$

$$= \frac{2\pi f_g}{t_h} \frac{r\tilde{E}_{f_\theta}}{V_o} e^{\gamma_o r} \tag{39}$$

From equations 38 and 39 one can rather approximately calculate the radiated waveform as a function of ω and θ for various choices of resistance and other antenna parameters. Taking the inverse transform the corresponding time-domain waveforms can also be calculated.

For our present purposes we only consider the waveform for this case of uniform resistive loading for the specific angle $\theta = \pi/2$ so that the observer is located on the x, y plane, a plane of symmetry. Then we have

$$\Xi = \frac{2}{\gamma h}[1 - 2e^{-\gamma h} + e^{-2\gamma h}]$$

$$= \frac{2}{\gamma h}[1 - e^{-\gamma h}]^2 \tag{40}$$

Defining a dimensionless parameter as

$$\beta \equiv \frac{R' t_h}{L'} = \frac{2\Lambda t_h}{L'} = \frac{2R_o}{Z_\infty} \tag{41}$$

then γh can be written as

$$\gamma h = s_h \left[1 + \frac{\beta}{s_h}\right]^{1/2} = [s_h(s_h + \beta)]^{1/2} \tag{42}$$

Also note the relation

$$sC_g Z_\infty = sC_a Z_\infty \frac{C_g}{C_a} = s_h[\alpha - 1]^{-1} \tag{43}$$

The normalized waveform can then be written as

$$\tilde{\xi}'\left(\frac{\pi}{2}\right) = \frac{1}{\gamma h}[1 - e^{-\gamma h}]^2 \left\{\left[\left[1 + \frac{\beta}{s_h}\right]^{1/2} + \frac{\alpha - 1}{s_h}\right]\right.$$

$$+ \left[\left[1 + \frac{\beta}{s_h}\right]^{1/2} - \frac{\alpha - 1}{s_h}\right]e^{-2\gamma h}\Bigg\}^{-1} \tag{44}$$

Expanding this result as a geometric series we have

$$\tilde{\xi}'\left(\frac{\pi}{2}\right) = \frac{1}{\gamma h}\left\{\left[1 + \frac{\beta}{s_h}\right]^{1/2} + \frac{\alpha - 1}{s_h}\right\}^{-1} [1 - 2e^{-\gamma h} + e^{-2\gamma h}]$$

$$\cdot \sum_{n=o}^{\infty} (-1)^n \left[\frac{\left[1 + \frac{\beta}{s_h}\right]^{1/2} - \frac{\alpha - 1}{s_h}}{\left[1 + \frac{\beta}{s_h}\right]^{1/2} + \frac{\alpha - 1}{s_h}}\right]^n e^{-2n\gamma h} \tag{45}$$

In this form we can separate the initial wave and the successive reflections by considering each term associated with a particular power of $e^{-\gamma h}$.

If we make a further simplification by assuming that the generator capacitance is arbitrarily large so that we can set $\alpha = 1$ we then have

$$\tilde{\xi}'\left(\frac{\pi}{2}\right) = \frac{1}{s_h + \beta}[1 - 2e^{-\gamma h} + e^{-2\gamma h}] \sum_{n=o}^{\infty} (-1)^n e^{-2n\gamma h}$$

$$= \frac{1}{s_h + \beta}\left\{\sum_{m=o}^{\infty} (-1)^m e^{-2m\gamma h} - 2\sum_{m=o}^{\infty} (-1)^m e^{-(2m+1)\gamma h} + \sum_{m=1}^{\infty} (-1)^{m-1} e^{-2m\gamma h}\right\}$$

$$= \frac{1}{s_h + \beta}\left\{1 + 2\sum_{n=1}^{\infty} (-1)^n e^{-(2n-1)\gamma h}\right\} \tag{46}$$

For convenience define

$$\tilde{\xi}'_o \equiv \frac{1}{s_h + \beta}$$

$$\tilde{\xi}_n' \equiv \frac{2(-1)^n}{s_h + \beta} e^{-(2n-1)\gamma h} \qquad \text{for } n = 1, 2, \ldots \tag{47}$$

so that we can write

$$\tilde{\xi}'\left(\frac{\pi}{2}\right) = \sum_{n=o}^{\infty} \tilde{\xi}_n' \tag{48}$$

Now $\tilde{\xi}_o'$ is associated with the initial signal which reaches the observer and the $\tilde{\xi}_n'$ for $n \geq 1$ can be considered successive reflection terms.

For the time-domain waveform at $\theta = \pi/2$ (and for large C_g) we can take the inverse Laplace transforms of the various terms with respect to s_h. For the $n = 0$ term we have

$$\xi_o' = e^{-\beta\tau_h} u(\tau_h) \tag{49}$$

For the terms for $n \geq 1$ first use the inverse Laplace transform (with respect to s_h)[8]

$$L^{-1}\left\{\frac{e^{-m'\sqrt{s_h(s_h+\beta)}}}{\sqrt{s_h(s_h+\beta)}}\right\} = u(\tau_h - m')e^{-(\beta\tau_h/2)} I_o\left(\frac{\beta}{2}\sqrt{\tau_h^2 - m'^2}\right) \tag{50}$$

where $m' \geq 0$ and I_o is a modified Bessel function. Integrating with respect to m' from m to ∞ gives

$$L^{-1}\left\{\frac{e^{-m\sqrt{s_h(s_h+\beta)}}}{s_h(s_h+\beta)}\right\} = u(\tau_h - m)e^{-(\beta\tau_h/2)} \int_m^{\tau_h} I_o\left(\frac{\beta}{2}\sqrt{\tau_h^2 - m'^2}\right)dm' \tag{51}$$

Multiplying the Laplace transform by s_h and differentiating the inverse transform with respect to τ_h gives

8. AMS 55, Handbook of Mathematical Functions, National Bureau of Standards, 1964, eqn. 29.3.91.

$$L^{-1}\left\{\frac{e^{-m\sqrt{s_h(s_h+\beta)}}}{s_h+\beta}\right\} = u(\tau_h - m)$$

$$\cdot \frac{\partial}{\partial \tau_h}\left\{e^{-(\beta\tau_h/2)} \int_m^{\tau_h} I_o\left(\frac{\beta}{2}\sqrt{\tau_h^2 - m'^2}\right) dm'\right\} \tag{52}$$

Substituting $m = 2n - 1$ and multiplying by $2(-1)^n$ we have ξ'_n for $n \geq 1$.

Note that for $\tau_h < 1$ we have $\xi'(\pi/2) = \xi'_o$. The first part of the waveform is then a step rise followed by an exponential decay. Writing ξ'_o as

$$\xi'_o = e^{-(t^*/t_o)} u(t^*) \tag{53}$$

the time constant of the decay is

$$t_o = \frac{t_h}{\beta} = \frac{L'}{R'} = \frac{L'}{2\Lambda} \tag{54}$$

At $\tau_h = 1$ the first reflection term appears. The step discontinuity at $\tau_h = 1$ is found from the initial value theorem of the Laplace transform as

$$\xi'_1(1+) = \lim_{s_h \to \infty} s_h \left\{\frac{-2}{s_h+\beta} e^{s_h} e^{-\sqrt{s_h(s_h+\beta)}}\right\}$$

$$= \lim_{s_h \to \infty} \left\{\frac{-2s_h}{s_h+\beta} e^{s_h\left(1-\sqrt{1+\frac{\beta}{s_h}}\right)}\right\}$$

$$= -2\, e^{-(\beta/2)} \tag{55}$$

Thus we have the ratio of the step discontinuity at $\tau_h = 1$ to the initial discontinuity at $\tau_h = 0$ as

$$\frac{\xi_1'(1+)}{\xi_o'(0+)} = -2\, e^{-(\beta/2)} = -2\, e^{-(R_o/Z_\infty)} \qquad (56)$$

where R_o (equation 28) is the resistance along one arm of the antenna and Z_∞ is some mean pulse impedance ascribed to the antenna, neglecting the resistive loading.

If, after the initial rise, one wants no more step discontinuities in the radiated waveform, then equation 56 can be used for some quality factor for the waveform. To minimize the magnitude of this discontinuity R_o/Z_∞ can be made larger. However increasing R_o/Z_∞ for a fixed h decreases the time constant of the pulse decay (equation 54) thereby decreasing the "pulse width." If one wants a large pulse width then some compromise is appropriate. Of course, one should recognize the limited accuracy of the transmission-line model from which the above results are obtained. When applied to a real antenna equation 56 is only very approximate.

IV. Special Case of Nonuniform Resistive Loading

Now go on to let Λ vary as an even function of z'. Make a change of variable in the one-dimensional wave equation for $\tilde{I}$ (equations 15 and 16) by substituting $\tilde{I} = \tilde{F}e^{-\gamma_o \zeta}$; this gives

$$\frac{\partial^2 \tilde{F}}{\partial \zeta^2} - 2\gamma_o \frac{\partial \tilde{F}}{\partial \zeta} - sC'Z'\tilde{F} = 0 \qquad (57)$$

Next try a solution of the form $\tilde{F} = f(\zeta)\tilde{F}_o$ so that $\tilde{F}$ is split into the product of a function of ζ and a function of s; this gives

$$\frac{\partial^2 f}{\partial \zeta^2} - 2\gamma_o \frac{\partial f}{\partial \zeta} - sC'Z'f = 0 \qquad (58)$$

Solving for Z' we have

$$Z' = \frac{1}{sC'}\,\frac{1}{f}\,\frac{\partial^2 f}{\partial \zeta^2} - \frac{2}{cC'}\,\frac{1}{f}\,\frac{\partial f}{\partial \zeta} \qquad (59)$$

This solution has the form

$$Z' = \frac{1}{sC''} + R' \qquad (60)$$

where we have

$$C'' = C'f\left(\frac{\partial^2 f}{\partial \zeta^2}\right)^{-1}$$

$$R' = -\frac{2}{cC'}\frac{1}{f}\frac{\partial f}{\partial \zeta} = -2Z_\infty \frac{1}{f}\frac{\partial f}{\partial \zeta} \tag{61}$$

For this type of solution Z' is the series combination of a resistance per unit length R' and a capacitance-length product C'' (in farad meters). Both R' and C'' can be functions of ζ, depending on the form chosen for $f(\zeta)$. Of course if we want $R' \geq 0$ and $C'' \geq 0$ for $0 \leq \zeta \leq h$ so that R' and C'' can be realized with passive elements, then the form we can choose for $f(\zeta)$ is limited somewhat.

Choosing some particular form of $f(\zeta)$ we have one of the two independent solutions of the second order differential equation (equation 58) and we have a particular form of Z' consistent with this solution. However, to avoid the need for the other independent solution we constrain $f(h) = 0$ so that $\tilde{I}$ is zero at $\zeta = h$ and the other solution is not needed to match the boundary condition there. Using the transmission-line model of the antenna we have only an "outward" propagating current wave which attenuates to zero at the ends of the antenna ($z' = \pm h$).

Consider now C''; it represents a distributed capacitance in series along the antenna. At s = 0 this presents an infinite series impedance along the antenna and at late times prevents the voltage on the equivalent transmission line from being uniform with ζ. At low frequencies the charge is not uniformly distributed along the transmission line (i.e., the charge stored in C', the distributed antenna capacitance). The charge near the ends ($z' = \pm h$) of the antenna is then reduced thereby decreasing the electric dipole moment of the antenna. This in turn reduces the low-frequency radiation of the antenna. Thus for the remainder of the discussion we only consider the case in which we constrain that $1/C'' = 0$ making $Z' = R'$.

In order to make the capacitive term in Z' vanish we set

$$\frac{\partial^2 f}{\partial \zeta^2} \equiv 0 \tag{62}$$

Then with the requirement that $f(h) = 0$ and normalizing $f(\zeta)$ such that $f(0) = 1$ we have

$$f(\zeta) = 1 - \frac{\zeta}{h} \tag{63}$$

This implies

$$Z'(\zeta) = R'(\zeta) = \frac{2Z_\infty}{h}\left[1 - \frac{\zeta}{h}\right]^{-1} \tag{64}$$

In terms of the resistance per unit length on the antenna we have (for $-h < z' < h$)

$$\Lambda(z') = \frac{R'(|z'|)}{2} = \frac{Z_\infty}{h}\left[1 - \frac{|z'|}{h}\right]^{-1} \tag{65}$$

This special form of resistance per unit length has been considered by Wu, King, and Shen.[9,10]

With this special form of $\Lambda(z')$ the transmission-line model gives us a current distribution as

$$\tilde{I}(\zeta) = \left[1 - \frac{\zeta}{h}\right] e^{-\gamma_o \zeta}\, \tilde{I}(0) \tag{66}$$

The voltage along the transmission line is given from equations 14 by

$$\tilde{V}(\zeta) = -\frac{1}{sC'}\frac{\partial \tilde{I}(\zeta)}{\partial \zeta} = -\frac{1}{sC'}\left[-\gamma_o\left(1 - \frac{\zeta}{h}\right) - \frac{1}{h}\right] e^{-\gamma_o \zeta}\, \tilde{I}(0) \tag{67}$$

so that we have

$$\tilde{V}(0) = \frac{1}{sC'h}\,[\gamma_o h + 1]\,\tilde{I}(0) \tag{68}$$

In the transmission line model the antenna impedance is then

9. T. T. Wu and R. W. P. King, The Cylindrical Antenna with Nonreflecting Resistive Loading, IEEE Trans. on Antennas and Propagation, AP-13, May 1965, pp. 369-373.

10. L. C. Shen and R. W. P. King, The Cylindrical Antenna with Nonreflecting Resistive Loading, IEEE Trans. on Antennas and Propagation, AP-13, November 1965, p. 998.

$$Z_a \equiv \frac{\tilde{V}(0)}{\tilde{I}(0)} = \frac{1}{sC'h}\,[\gamma_o h + 1]$$

$$= \frac{1}{sC_a} + Z_\infty = Z_\infty\left[\frac{1}{s_h} + 1\right] \tag{69}$$

This impedance is the series combination of a capacitance and a resistance. The impedance of our ideal capacitive generator is just

$$Z_g = \frac{1}{sC_g} \tag{70}$$

$\tilde{I}(0)$ can now be found as

$$\tilde{I}(0) = \frac{V_o}{s}\,[Z_g + Z_a]^{-1} = \frac{V_o}{s}\left[\frac{1}{sC_g} + \frac{1}{sC_a} + Z_\infty\right]^{-1}$$

$$= \frac{V_o}{s}\left[\frac{\alpha}{sC_a} + Z_\infty\right]^{-1} = \frac{V_o}{sZ_\infty}\left[\frac{\alpha}{s_h} + 1\right]^{-1}$$

$$= \frac{V_o}{Z_\infty}\,\frac{t_h}{s_h + \alpha} \tag{71}$$

Using the transmission-line model the current on the antenna is then

$$\tilde{I}(z') = \frac{V_o}{Z_\infty}\,\frac{t_h}{s_h + \alpha}\left[1 - \frac{|z'|}{h}\right] e^{-s_h\frac{|z'|}{h}} \tag{72}$$

From equation 21 the normalized waveform can be calculated as

$$\tilde{\xi}'(\theta) = \sin(\theta)\,\frac{\mu_o f_g}{2Z_\infty t_h}\,\frac{s_h}{s_h+\alpha}\int_{-h}^{h}\left[1 - \frac{|z'|}{h}\right]$$

$$\cdot\; e^{-s_h\frac{|z'|}{h}}\; e^{s_h\frac{z'}{h}\cos(\theta)}\,dz'$$

$$= \frac{\sin(\theta)}{2} \frac{s_h}{s_h+\alpha} \frac{1}{h} \int_{-h}^{h} \left[1 - \frac{|z'|}{h}\right] e^{s_h\left(-\frac{|z'|}{h}+\frac{z'}{h}\cos(\theta)\right)} dz' \tag{73}$$

Substituting

$$\nu \equiv \frac{z'}{h} \tag{74}$$

we have

$$\tilde{\xi}'(\theta) = \frac{\sin(\theta)}{2} \frac{s_h}{s_h+\alpha} \int_{-1}^{1} (1 - |\nu|) e^{s_h(-|\nu|+\nu\cos(\theta))} d\nu$$

$$= \frac{\sin(\theta)}{2} \frac{s_h}{s_h+\alpha} \int_{0}^{1} (1 - \nu) \Big[e^{-s_h(1-\cos(\theta))\nu}$$

$$+ e^{-s_h(1+\cos(\theta))\nu}\Big] d\nu \tag{75}$$

This last integral is composed of two integrals of the form

$$\int_0^1 (1 - \nu) e^{-b\nu}\, d\nu = e^{-b\nu}\left(-\frac{1}{b} + \frac{\nu}{b} + \frac{1}{b^2}\right)\Bigg|_0^1$$

$$= \frac{1}{b^2} e^{-b} - \frac{1}{b^2} + \frac{1}{b} \tag{76}$$

The normalized radiated waveform is then

$$\tilde{\xi}'(\theta) = \frac{\sin(\theta)}{2(s_h+\alpha)} \left\{ \frac{1}{1-\cos(\theta)} \left[\frac{e^{-s_h(1-\cos(\theta))} - 1}{s_h(1-\cos(\theta))} + 1 \right] \right.$$

$$\left. + \frac{1}{1+\cos(\theta)} \left[\frac{e^{-s_h(1+\cos(\theta))} - 1}{s_h(1+\cos(\theta))} + 1 \right] \right\}$$

$$= \frac{2\pi f_g}{t_h} \frac{r\tilde{E}_{f_\theta}}{V_o} e^{\gamma_o r} \tag{77}$$

Taking the inverse transform gives the time-domain results as

$$\xi'(\theta) = \frac{\sin(\theta)}{2}\left\{\left[\frac{e^{-\alpha\tau_h}}{1-\cos(\theta)} - \frac{1-e^{-\alpha\tau_h}}{\alpha(1-\cos(\theta))^2}\right]u(\tau_h)\right.$$

$$+ \frac{1}{\alpha}\frac{1-e^{-\alpha[\tau_h-(1-\cos(\theta))]}}{(1-\cos(\theta))^2} u(\tau_h-[1-\cos(\theta)])$$

$$+ \left[\frac{e^{-\alpha\tau_h}}{1+\cos(\theta)} - \frac{1-e^{-\alpha\tau_h}}{\alpha(1+\cos(\theta))^2}\right]u(\tau_h)$$

$$\left. + \frac{1}{\alpha}\frac{1-e^{-\alpha[\tau_h-(1+\cos(\theta))]}}{(1+\cos(\theta))^2} u(\tau_h-[1+\cos(\theta)])\right\}$$

$$= 2\pi f_g \frac{rE_{f_\theta}}{V_o} \tag{78}$$

Note the interesting result that after $\tau_h = 0$ there are no step discontinuities in $\xi'(\theta)$. There are, however, discontinuities in the slope of $\xi'(\theta)$ at $\tau = 1 - \cos(\theta)$ and at $\tau = 1 + \cos(\theta)$. In this respect the special resistance distribution in this section gives a significantly better waveform than that associated with a uniform resistance distribution.

The low frequency asymptotic form of $\tilde{\xi}'(\theta)$ is given by equation 27. For high frequencies we have as $s_h \to \infty$ the asymptotic form

$$\tilde{\xi}'(\theta) \approx \frac{\sin(\theta)}{2s_h}\left[\frac{1}{1-\cos(\theta)} + \frac{1}{1+\cos(\theta)}\right] = \frac{1}{s_h \sin(\theta)} \tag{79}$$

Note then that for low frequencies the normalized waveform is proportional to $\sin(\theta)$ while for high frequencies it is proportional to $1/\sin(\theta)$. The initial step discontinuity in $\xi'(\theta, \tau_h)$ is given by

$$\xi'(\theta, 0+) = \lim_{s_h \to \infty} s_h \tilde{\xi}'(\theta) = \frac{1}{\sin(\theta)} \tag{80}$$

Consider the case that $C_g >> C_a$ so that we can set $\alpha = 1$. For this case $\tilde{\xi}'(\theta)$ is plotted in figure 3 as a function of ωt_h for a few values of θ. Similarly $\xi'(\theta)$ is plotted in figure 4 as a function of τ_h for a few values of θ. Note that as θ decreases from $\pi/2$ to 0 the low frequency content of $\tilde{\xi}'(\theta)$ decreases; correspondingly the rate of decay of $\xi'(\theta)$ from its peak value increases and the pulse "width" decreases. Also as θ decreases from $\pi/2$ to 0 the high frequency content of $\tilde{\xi}'(\theta)$ increases; correspondingly the initial amplitude of $\xi'(\theta)$ increases.

Now consider the special case that $\theta = \pi/2$ so that the observer is located on the x, y plane, a plane of symmetry. The results simplify significantly. In the Laplace-transform domain we have

$$\tilde{\xi}'\left(\frac{\pi}{2}\right) = \frac{1}{s_h + \alpha}\left\{1 - \frac{1}{s_h} + \frac{e^{-s_h}}{s_h}\right\} \tag{81}$$

In the time domain we have

$$\tilde{\xi}'\left(\frac{\pi}{2}\right) = \frac{1}{\alpha}\left\{\left[(1+\alpha)e^{-\alpha\tau_h}-1\right]u(\tau_h)+\left[1-e^{-\alpha(\tau_h-1)}\right]u(\tau_h-1)\right\} \tag{82}$$

In figure 5A we have $|\tilde{\xi}'(\pi/2)|$ plotted as a function of ωt_h for several values of α. Note that as α decreases toward 1 the low frequency content increases. $\xi'(\pi/2)$ is plotted in figure 5B as a function of τ_h for several values of α. Note that as α decreases toward 1 the rate of decay of the waveform (after the initial peak) decreases and the pulse "width" increases. Note also for $0 < \tau_h < 1$ that $\xi'(\pi/2)$ is a monotonically decreasing function of τ_h, while for $1 < \tau_h$ it is a monotonically increasing function of τ_h, increasing from the minimum at $\tau_h = 1$ approaching 0 as $\tau_h \to \infty$. This waveform has one zero crossing which occurs for some τ_h satisfying $0 < \tau_h < 1$.

If we set both $\theta = \pi/2$ and $\alpha = 1$ then the time-domain normalized waveform has the form

$$\xi'\left(\frac{\pi}{2}\right) = \left[2e^{-\tau_h} - 1\right]u(\tau_h) + \left[1 - e^{-(\tau_h-1)}\right]u(\tau_h - 1) \tag{83}$$

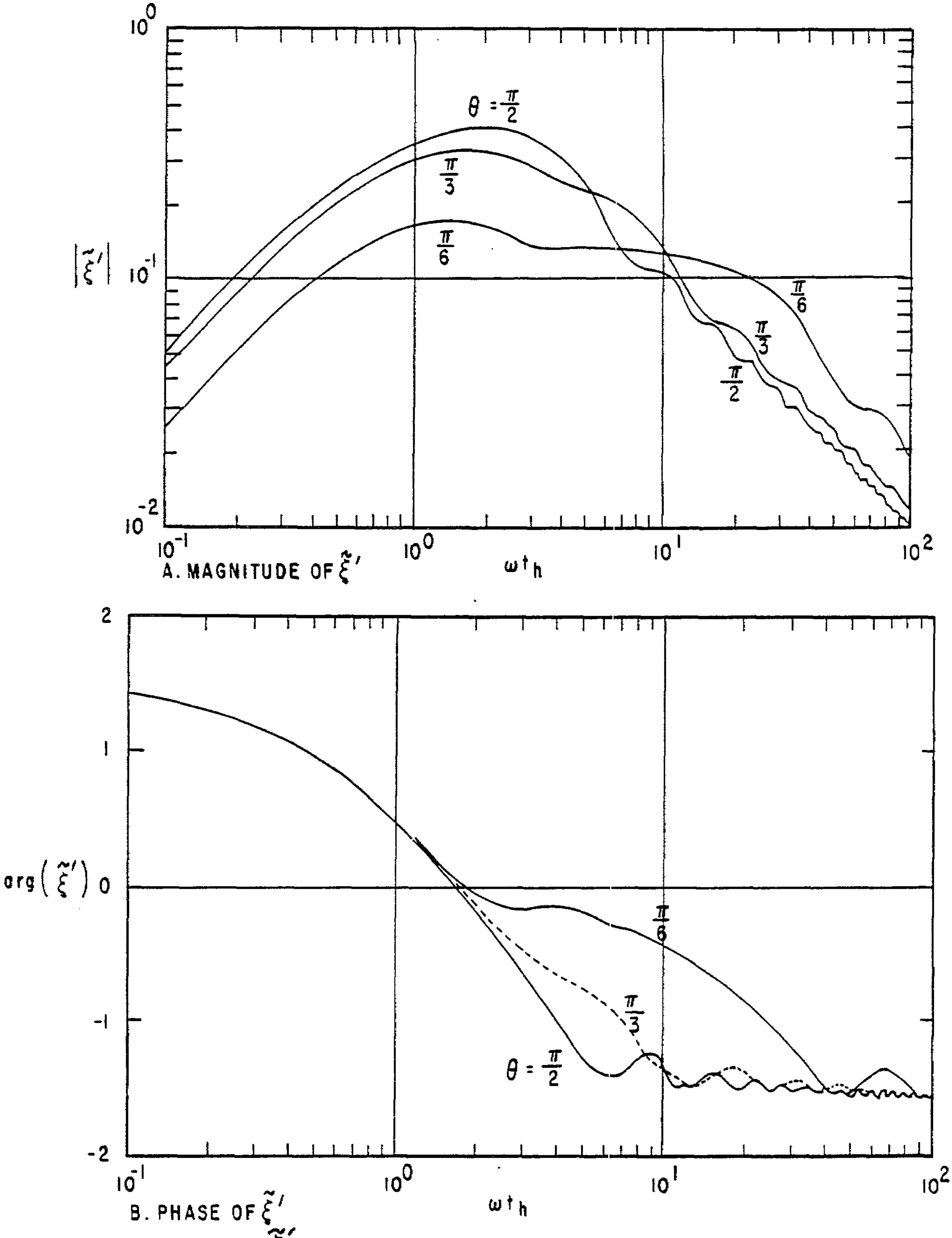


FIGURE 3. $\tilde{\xi}'$ FOR VARIOUS θ WITH $\alpha = 1$

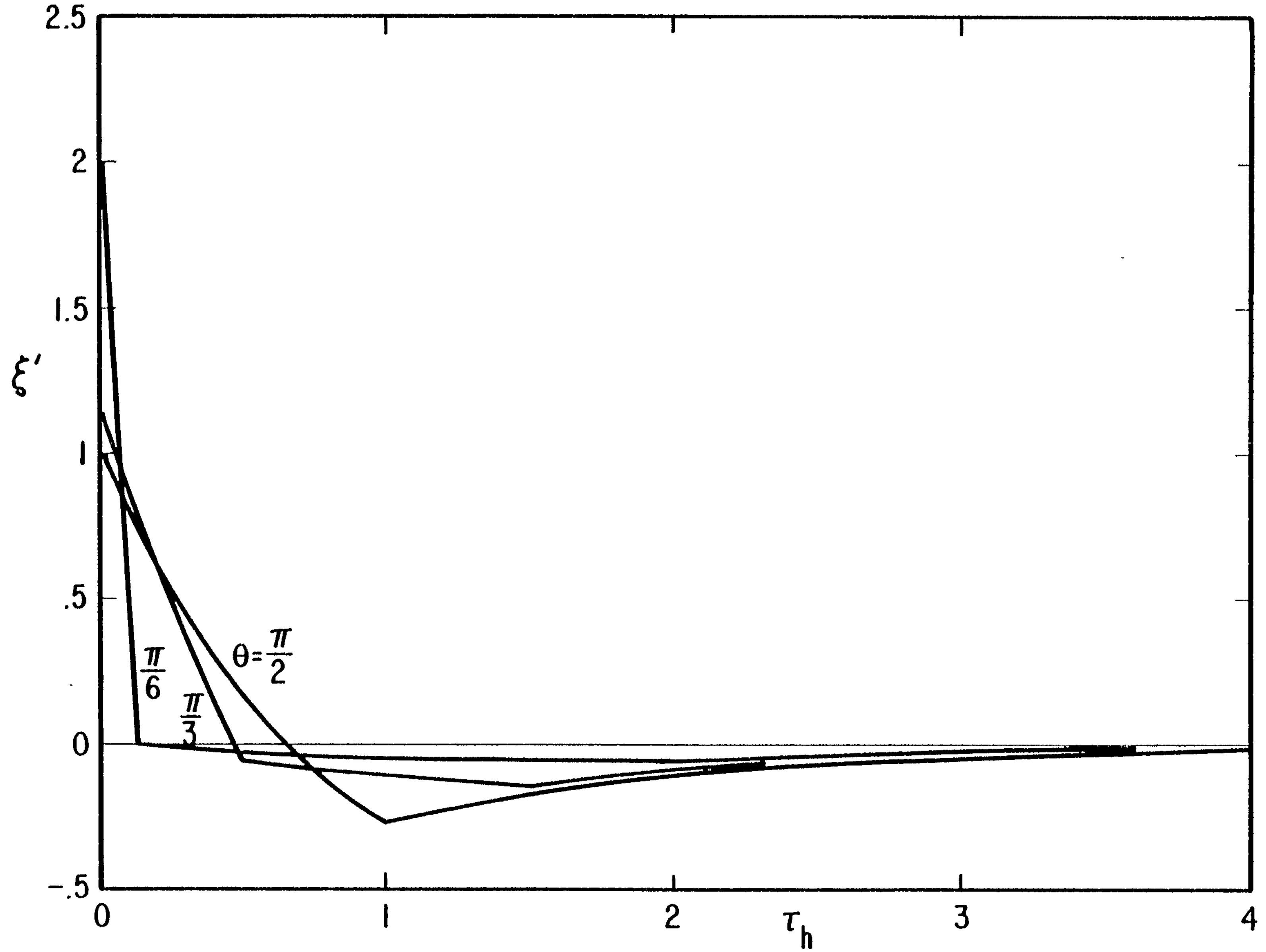
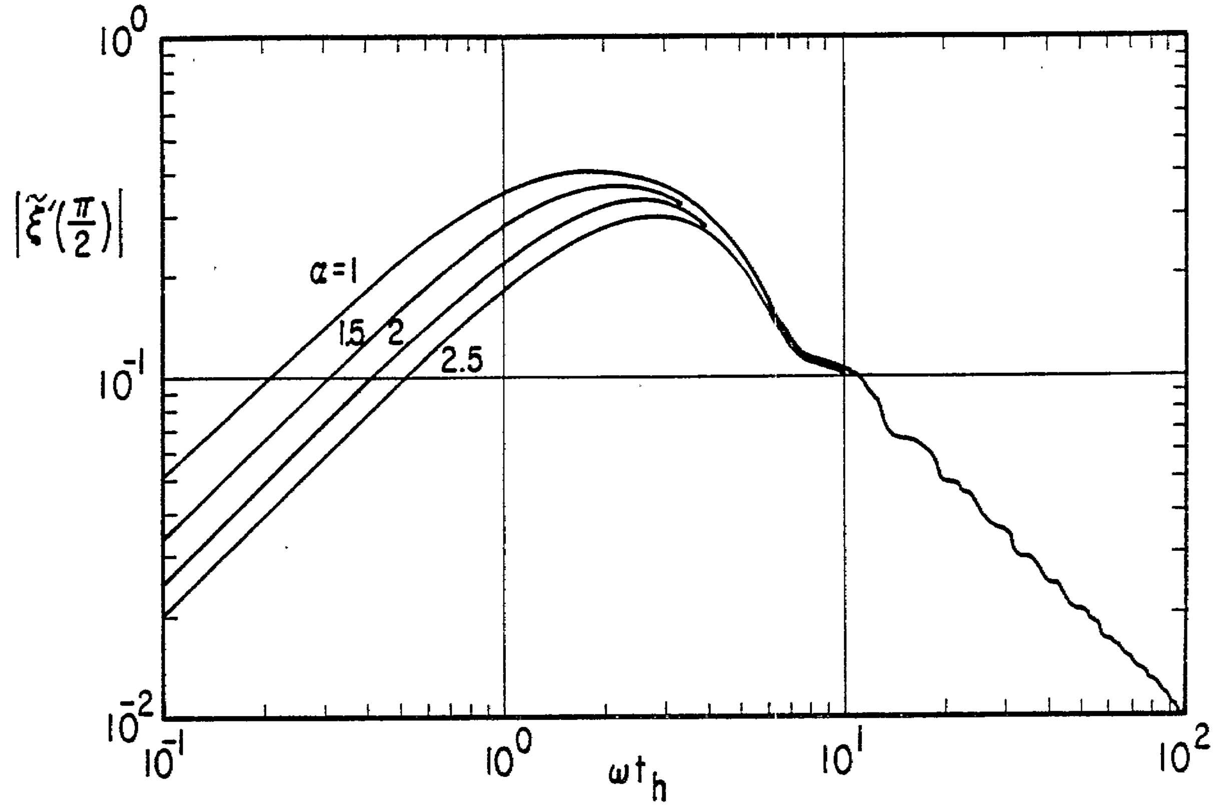
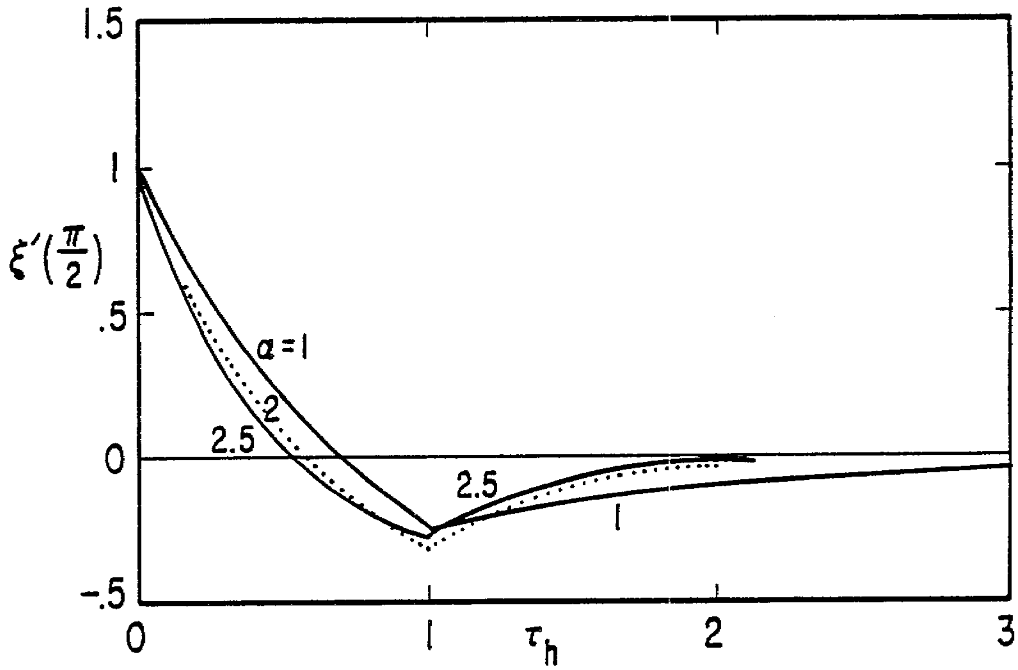


FIGURE 4. ξ' FOR VARIOUS θ WITH $\alpha = 1$



A. FREQUENCY DOMAIN



B. TIME DOMAIN

FIGURE 5. INFLUENCE OF GENERATOR CAPACITANCE ON WAVEFORM

The zero crossing for this special case occurs at $\tau_h = \ln(2) \simeq .69$, while the minimum occurs at $\tau_h = 1$ and has a value of about $-.26$. Equation 83 gives a very simple result for these special values of θ and α.

V. Summary

The transmission-line model for calculating the radiation from a dipole antenna, while rather approximate, can give some interesting results. In this note we have used this model to consider some aspects of the problem of loading the antenna structure with series impedance. Using a uniform resistance per unit length the abrupt changes in the radiated waveform associated with the reflection of the current at the ends of the antenna can be reduced as one increases the resistance. At the same time, however, the pulse decays somewhat faster so that the pulse width is decreased. As discussed in section IV one can use a special resistance distribution which goes to ∞ at the ends of the antenna. The resulting time-domain waveform has no abrupt changes (after the initial rise) due to current reflections. There is, however, a discontinuity in the slope of the waveform, but this is a lower order type of discontinuity. Perhaps there are other forms of impedance loading which can further improve the radiated waveform in some way. Of course, one should recognize that while the transmission-line model introduces a significant simplification into antenna calculations the results are not rigorous in the sense of an exact solution to a given electromagnetic boundary value problem. As such the results can be rather approximate.

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