change V to -V in eqns. 4, 6, 7, 9, 10

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Radiation of an Infinite Cylindrical Antenna With Uniform Resistive Loading

by

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Abstract

The time behavior is obtained of the radiation field of an infinite cylindrical antenna loaded along its length with uniform resistance and excited by a step-function voltage across an circumferential gap of infinitesimal width. It is found that the late time behavior of the radiation field is inversely proportional to the square of time, whereas, in the case of no loading, it varies in inverse proportion to the logarithm of time.

Acknowledgment

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I. Introduction

The present note is a generalization of a previous one[1] in which the radiated field is calculated of an infinite cylindrical, perfectly conducting antenna excited by a step-function voltage across a delta gap. Instead of being perfectly conducting the antenna is now loaded with constant resistance along its length, i.e., the loaded resistance is independent of frequency as well as position along the antenna. There are two reasons for studying this particular problem. The first is a mathematical one in the sense that this problem lends itself to exact analysis within the Maxwell field theory. The second reason is a practical one in that, since any antenna that will be built must be of finite length, reflections from the ends of the antenna will occur, thereby introducing undesirable features in the radiation field. One way to minimize such undesirable features is to damp the current pulse to an insignificant magnitude by the time it reaches the ends. A possible method of achieving this is to load the antenna along its length with resistance.

Nonuniform resistive loading along the antenna will undoubtedly provide us more freedom in shaping the radiation field, but this is a much more difficult problem to analyze and may be taken up for study in the future.

In Section II, the time-harmonic far field is obtained by the saddle-point method. Then, assuming the generator voltage to be a step function in time we calculate the radiation field in Section III by performing an inverse Laplace transform. The time behavior of the radiation field is graphed as well as tabulated for a wide range of resistance values.

II. Time-Harmonic Far Field

The point of departure is the integral equation (20) of Reference 2 for the total current on the surface of an axi-symmetric antenna.[2] In the present case where the antenna is an infinite cylinder of radius a , that equation becomes,* in the cylindrical coordinates (ρ, z, ϕ) ,

$$\frac{1}{2} I(z) + \int_{-\infty}^{\infty} K(z - z')I(z')dz' = \int_{-\infty}^{\infty} Y(z - z')\langle E_z(z')\rangle dz' \tag{1}$$

where

$$K(z) = \frac{a}{8\pi} \frac{\partial}{\partial a} \int_{0}^{2\pi} \frac{e^{ik\sqrt{z^2 + 2a^2 - 2a^2 \cos \phi}}}{\sqrt{z^2 + 2a^2 - 2a^2 \cos \phi}} d\phi \tag{2}$$

$$Y(z) = \frac{ika^2}{2Z_o} \int_{0}^{2\pi} \frac{e^{ik\sqrt{z^2 + 2a^2 - 2a^2 \cos \phi}}}{\sqrt{z^2 + 2a^2 - 2a^2 \cos \phi}} \cos \phi \, d\phi \tag{3}$$

$$I(z) = a \int_{0}^{2\pi} H_{\phi}(a,z,\phi)d\phi$$

$$\langle E_z(z)\rangle = \frac{1}{2\pi} \int_{0}^{2\pi} E_z(a,z,\phi)d\phi$$

and Z_o is the free-space wave impedance and k is the wave number.

When the cylindrical antenna is excited by a voltage V across a circumferential gap of infinitesimal width and is loaded along its length by R ohms per meter, we write

*The interested reader may refer to the Appendix for a detailed derivation.

$$E_z(a,z,\phi) = V\delta(z) + RI(z) \quad . \tag{4}$$

The physical meaning of R will be discussed at the end of Section III.

Integration of this equation across the infinitesimal gap gives V , since the integral of RI will go to zero as the gap's width tends to zero. This may be seen from the well-known fact that I(z) varies as $\ln k|z|$ near the delta gap.

Substituting (4) into (1) we get

$$\frac{1}{2} I(z) + \int_{-\infty}^{\infty} K(z - z')I(z')dz' - R \int_{-\infty}^{\infty} Y(z - z')I(z')dz' = VY(z) \quad . \tag{5}$$

To solve this integral equation we employ Fourier transforms. Defining

$$\overline{I}(\zeta) = \int_{-\infty}^{\infty} I(z)e^{-i\zeta z}\, dz$$

and similarly for $\overline{Y}(\zeta)$ and $\overline{K}(\zeta)$ we have, from (5),

$$\overline{I}(\zeta) = 2V \frac{\overline{Y}(\zeta)}{1 + 2\overline{K}(\zeta) - 2R\overline{Y}(\zeta)} \quad . \tag{6}$$

Since[3]

$$\overline{Y}(\zeta) = - \frac{\pi^2 ka^2}{Z_o} H_1^{(1)}(\lambda a)J_1(\lambda a) \quad ,$$

and[4]

$$\overline{K}(\zeta) = \frac{a}{4}\frac{\partial}{\partial a}\left[\pi i\ H_o^{(1)}(\lambda a) J_o(\lambda a)\right]$$

$$= -\frac{\pi i}{4}\lambda a\left[J_o(\lambda a)H_1^{(1)}(\lambda a) + J_1(\lambda a)H_o^{(1)}(\lambda a)\right]$$

$$= -\frac{1}{2} - \frac{\pi i}{2}\lambda a J_1(\lambda a)H_o^{(1)}(\lambda a) \qquad , \text{(Wronskians)}$$

where $\lambda = \sqrt{k^2 - \zeta^2}$, the inverse Fourier transform of (6) is then given by

$$I(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{I}(\zeta)e^{i\zeta z}\, d\zeta$$

$$= -ika^2\frac{V}{Z_o}\int_{-\infty}^{\infty}\frac{H_1^{(1)}(\lambda a)}{\lambda a H_o^{(1)}(\lambda a) + i\beta ka H_1^{(1)}(\lambda a)}\, e^{i\zeta z}\, d\zeta \qquad , \qquad (7)$$

where $\beta = 2\pi aR/Z_o$. The path of integration in (7) is along the real axis in the complex ζ-plane with upward indentation at $\zeta = -k$ and downward indentation at $\zeta = k$.

To obtain the fields off the surface of the cylindrical antenna we regard (7) as a boundary condition for H_ϕ which, due to the symmetry of the problem, is the only component of the magnetic field and is independent of the azimuthal coordinate ϕ . The equation that $H_\phi(\rho,z)$ satisfies is

$$\left(\frac{\partial^2}{\partial\rho 2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2} + k^2\right)H_\phi(\rho,z) = 0 \tag{8}$$

which is directly derivable from Maxwell's equations. The solution of (8) that satisfies the radiation condition at infinity and is equal to $I(2\pi a)^{-1}$ at $\rho = a$, I being given by (7), is easily seen to be

$$H_\phi(\rho,z) = -\frac{ikV}{2\pi Z_o}\int_{-\infty}^{\infty}\frac{H_1^{(1)}(\lambda\rho)}{\lambda H_o^{(1)}(\lambda a) + i\beta k H_1^{(1)}(\lambda a)}\, e^{i\zeta z}\, d\zeta \quad , \tag{9}$$

from which we obtain

$$E_z(\rho,z) = -\frac{1}{i\omega\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)$$

$$= \frac{V}{2\pi}\int_{-\infty}^{\infty}\frac{\lambda H_o^{(1)}(\lambda\rho)}{\lambda H_o^{(1)}(\lambda a) + ik\beta H_1^{(1)}(\lambda a)}\, e^{i\zeta z}\, d\zeta \quad . \tag{10}$$

In the far zone where $\theta \neq 0$, (r,θ,ϕ) being the spherical coordinates, one may use the saddle-point method to evaluate (10). Thus[5]

$$E_\theta = -\frac{Vi}{\pi}\frac{e^{ikr}}{\rho}\frac{1}{H_o^{(1)}(ka\sin\theta) + i\beta\csc\theta\, H_1^{(1)}(ka\sin\theta)} \quad ,$$

which becomes, in terms of $p = -i\omega$,

$$E_\theta = \frac{V}{2\rho}\frac{e^{-pr/c}}{K_o(p\frac{a}{c}\sin\theta) + \beta\csc\theta\, K_1(p\frac{a}{c}\sin\theta)} \quad , \tag{11}$$

where K_o and K_1 are modified Bessel functions.

III. Radiation Field for a Step Voltage

In the case where the voltage of the slice generator is a step function in time, i.e., $v(t) = v_o U(t)$, equation (11) becomes

$$\frac{\rho E_\theta}{v_o} = \frac{1}{2p} \frac{e^{-pr/c}}{K_o(p \frac{a}{c} \sin\theta) + \beta \csc\theta \, K_1(p \frac{a}{c} \sin\theta)} \quad , \qquad (12)$$

and its inverse Laplace transform is

$$\frac{\rho E_\theta(r,\theta,t)}{v_o} = \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{(t - r/c)p}}{K_o(p \frac{a}{c} \sin\theta) + \beta_\theta K_1(p \frac{a}{c} \sin\theta)} \frac{dp}{p}$$

$$= \frac{1}{4\pi i} \int_C \frac{e^{q_\theta \zeta}}{K_o(\zeta) + \beta_\theta K_1(\zeta)} \frac{d\zeta}{\zeta} \qquad (13)$$

where $q_\theta = a^{-1}(ct - r)\csc\theta$, $\beta_\theta = \beta \csc\theta$, and the path C is shown in Fig. 2 of Ref. 1.

For $\beta_\theta > 0$ (passive resistance) the function, $K_o(\zeta) + \beta_\theta K_1(\zeta)$, has no zeros for which $|\arg \zeta| < \pi$. Thus, following the same procedure as in Sec. II of Ref. 1 we have

$$\frac{\rho E_\theta}{v_o} = 0 \quad , \qquad \text{if } ct < r - a\sin\theta \qquad (14a)$$

$$= \frac{1}{2} \int_0^\infty \frac{I_o(x) + \beta_\theta I_1(x)}{[K_o(x) - \beta_\theta K_1(x)]^2 + \pi^2[I_o(x) + \beta_\theta I_1(x)]^2} e^{-xq_\theta} \frac{dx}{x} \quad , \qquad (14b)$$

$$\text{if } ct > r - a\sin\theta \quad .$$

In terms of the normalized time T_θ defined by

$$T_\theta = q_\theta + 1 = \frac{ct - (r - a \sin \theta)}{a \sin \theta} \quad ,$$

equations (14) become

$$\frac{\rho E_\theta}{v_o} = 0 \quad , \qquad \text{if} \quad T_\theta < 0 \tag{15a}$$

$$= \int_o^\infty f(x, \beta_\theta) e^{-xT_\theta} dx \quad , \qquad \text{if} \quad T_\theta > 0 \tag{15b}$$

where

$$f(x,\beta_\theta) = \frac{I_o(x) + \beta_\theta I_1(x)}{[K_o(x) - \beta_\theta K_1(x)]^2 + \pi^2[I_o(x) + \beta_\theta I_1(x)]^2} \frac{e^x}{2x} \quad . \tag{15c}$$

Equation (15b) was evaluated numerically for a wide range of β_θ values, and the results are presented in Table I and Table II and also in figures 1 and 2.

Early time behavior of $\rho E_\theta / v_o$

The technique described in Ref. 2 can be applied directly to the integral (15b) for $T_\theta << 1$. The result is

$$\frac{\rho E_\theta}{v_o} \sim \frac{1}{\pi\sqrt{2}} \frac{1}{(1 + \beta_\theta)\sqrt{T_\theta}} \quad , \qquad \text{as} \quad T_\theta \to 0 \tag{16}$$

which is plotted in broken lines in figure 1.

Late time behavior of $\rho E_\theta/v_o$

An examination of (15c) shows that so long as $\beta_\theta \neq 0$, $f(x,\beta_\theta)$ and all its derivatives with respect to x exist at $x = 0$. Thus, integrating (15b) by parts one can easily develop the following asymptotic series:

$$\int_0^\infty f(x,\beta_\theta)e^{-xT_\theta}\,dx \sim \sum_{n=0}^{\infty} \frac{f^n(0,\beta_\theta)}{T_\theta^{n+1}} \quad , \qquad \text{for} \quad T_\theta >> 1 \quad .$$

Keeping the first two terms in the series we have

$$\frac{\rho E_\theta}{v_o} \sim \frac{1}{2}\left(\frac{1}{\beta_\theta^2 T_\theta^2} + \frac{1}{\beta_\theta T_\theta^3}\right) \quad , \qquad \text{for} \quad T_\theta >> 1 \quad . \tag{17}$$

This equation is plotted in broken lines in figure 2.

In Tables I and II, the radiation field is tabulated for a wide range of β_θ values and for $0.2 \leq T_\theta \leq 1000$. If the radiation field is desired for $T_\theta < 0.2$ and $T_\theta > 1000$ (or $T_\theta \geq 10^5$ when β_θ is of the order 10^{-2}), it can be calculated from the asymptotic forms (16) and (17), respectively. If the radius of the antenna is about one meter, the value of T_θ equal to 1000 roughly corresponds to 3 microseconds after the arrival of the leading edge of the pulse at a distant observation point.

At this point it is perhaps pertinent to say a few words about the physical meaning of R $(= Z_o\beta_\theta \sin\theta/2\pi a)$ introduced in equation (4). According to (4)

$$R = \frac{\langle E_z\rangle}{I} = \frac{\frac{1}{2\pi}\int_0^{2\pi} E_z(a,z,\phi)\,d\phi}{a\int_0^{2\pi} H_\phi(a,z,\phi)\,d\phi} = \frac{E_z(a,z)}{2\pi a\, H_\phi(a,z)} \tag{18}$$

where the last step follows from the symmetry of the present problem. Thus, R is defined as the ratio of the averaged longitudinal surface electric field to the total (conduction and displacment) current flowing through the cross-sectional area of the antenna. R is sometimes referred to as the "internal" impedance in contradistinction to the "surface" impedance defined by E_z/H_ϕ . If one integrates the time-average Poynting vector over the antenna surface, he will find that the total time-average ohmic loss per unit length along the antenna is exactly given by $R|I|^2/2$. Hence, $\Delta z \cdot R$ can be appropriately interpreted as the total resistance between two cross sections of Δz apart. Of course, Δz should be smaller than all relevant wavelengths so that $\Delta z E_z$ can be meaningfully defined as the voltage drop.

Table I. Values of $\frac{\rho E_\theta}{v_o} \times 10^2$.

T_θ \ β_θ	0	.02	.03	.04	.05	.06	.07	.08	.09	.10	.20	.40	.80
.2	51.5	51.5	50.9	50.3	49.8	49.2	48.7	48.1	47.6	47.1	42.6	35.7	26.9
.4	38.3	37.8	37.3	36.8	36.3	35.9	35.4	35.0	34.6	34.1	30.5	25.1	18.3
.6	32.6	31.8	31.4	30.9	30.5	30.1	29.7	29.2	28.9	28.5	25.1	20.3	14.4
.8	29.3	28.4	27.9	27.5	27.1	26.6	26.2	25.9	25.5	25.1	21.9	17.4	12.1
1	27.0	26.0	25.6	25.2	24.7	24.3	23.9	23.6	23.2	22.8	19.7	15.4	10.4
2	21.4	20.3	19.9	19.4	19.0	18.6	18.2	17.8	17.4	17.1	14.1	10.1	6.22
4	17.3	16.2	15.7	15.2	14.8	14.3	13.9	13.5	13.1	12.7	9.75	6.26	3.26
6	15.5	14.4	13.8	13.3	12.8	12.3	11.9	11.4	11.0	10.6	7.65	4.46	2.05
8	14.5	13.2	12.6	12.0	11.5	11.0	10.5	10.1	9.66	9.26	6.32	3.39	1.40
10	13.8	12.3	11.7	11.1	10.5	10.0	9.54	9.08	8.66	8.26	5.36	2.67	1.01
14	12.8	11.1	10.4	9.80	9.20	8.65	8.14	7.67	7.23	6.83	4.06	1.79	.583
20	11.8	9.98	9.20	8.49	7.85	7.27	6.74	6.26	5.83	5.43	2.88	1.08	.304
26	11.3	9.16	8.32	7.56	6.89	6.30	5.76	5.29	4.86	4.47	2.14	.715	.181
30	11.0	8.73	7.84	7.07	6.39	5.78	5.25	4.78	4.36	3.98	1.80	.559	.134
40	10.5	7.87	6.92	6.10	5.40	4.79	4.26	3.81	3.41	3.07	1.21	.329	.073
50	10.3	7.22	6.21	5.37	4.66	4.06	3.55	3.12	2.75	2.43	.859	.211	.045
60	10.0	6.69	5.64	4.78	4.07	3.49	3.00	2.60	2.26	1.97	.632	.144	.030
70	9.76	6.24	5.16	4.30	3.60	3.03	2.57	2.20	1.88	1.63	.479	.103	.021
80	9.74	5.84	4.75	3.89	3.21	2.66	2.23	1.88	1.59	1.36	.371	.077	.016
90	9.52	5.50	4.40	3.54	2.87	2.35	1.94	1.62	1.35	1.14	.294	.059	.012
100	9.50	5.22	4.09	3.24	2.59	2.09	1.70	1.40	1.16	.973	.237	.047	.010
1000	7.20	.369	.137	.064	.036	.022	.015	.011	.008	.006	.001	0	0

Table II. Values of $\frac{\rho E_\theta}{v_o} \times 10^3$

T_θ \ β_θ	1	2	4	6	8	10	20	40	60	80	10^2	10^3	10^4
.2	239	154	90.3	63.7	49.3	40.1	20.8	10.6	7.13	5.37	4.30	.433	.043
.4	162	101	57.6	40.2	30.9	25.1	12.9	6.55	4.39	3.30	2.64	.266	.027
.6	126	76.5	42.5	29.4	22.4	18.1	9.26	4.68	3.13	2.35	1.88	.189	.019
.8	104	61.6	33.4	22.9	17.3	14.0	7.07	3.56	2.38	1.78	1.43	.143	.014
1	89.2	51.3	27.2	18.4	13.9	11.1	5.60	2.80	1.87	1.40	1.12	.112	.011
2	51.3	26.2	12.5	8.05	5.90	4.65	2.24	1.10	.726	.543	.433	.043	.004
4	25.4	10.7	4.29	2.55	1.78	1.36	.608	.285	.186	.138	.109	.011	.001
6	15.2	5.55	1.96	1.09	.732	.543	.228	.103	.066	.049	.038	.004	0
8	10.0	3.28	1.05	.557	.362	.263	.104	.045	.029	.021	.017	.002	0
10	7.02	2.11	.630	.321	.204	.146	.055	.023	.015	.011	.008	.001	0
14	3.89	1.04	.283	.138	.085	.059	.021	.008	.005	.004	.003	0	0
20	1.95	.474	.121	.057	.034	.023	.008	.003	.002	.001	.001	0	0
26	1.13	.264	.066	.030	.018	.012	.004	.001	.001	0	0	0	0
30	.832	.191	.047	.022	.013	.009	.003	.001	0	0	0	0	0
40	.446	.101	.025	.011	.007	.004	.001	0	0	0	0	0	0
50	.273	.062	.015	.007	.004	.003	.001	0	0	0	0	0	0
60	.183	.042	.010	.006	.003	.002	0	0	0	0	0	0	0
70	.131	.030	.007	.003	.002	.001	0	0	0	0	0	0	0
80	.098	.023	.005	.002	.001	.001	0	0	0	0	0	0	0
90	.076	.018	.004	.002	.001	0	0	0	0	0	0	0	0
100	.060	.014	.003	.001	.001	0	0	0	0	0	0	0	0
1000	0	0	0	0	0	0	0	0	0	0	0	0	0

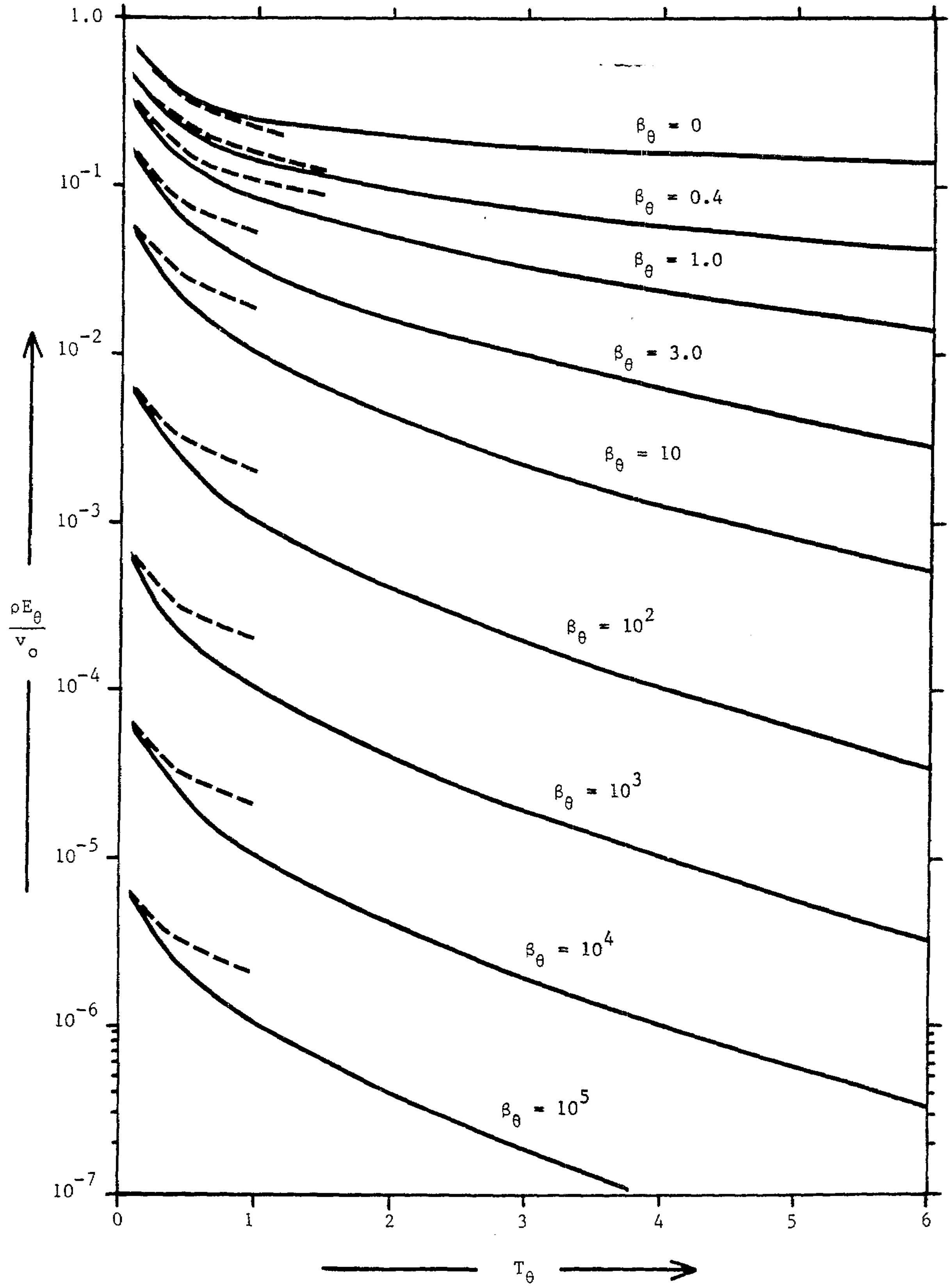

Figure 1. Radiation field for a step-function voltage.

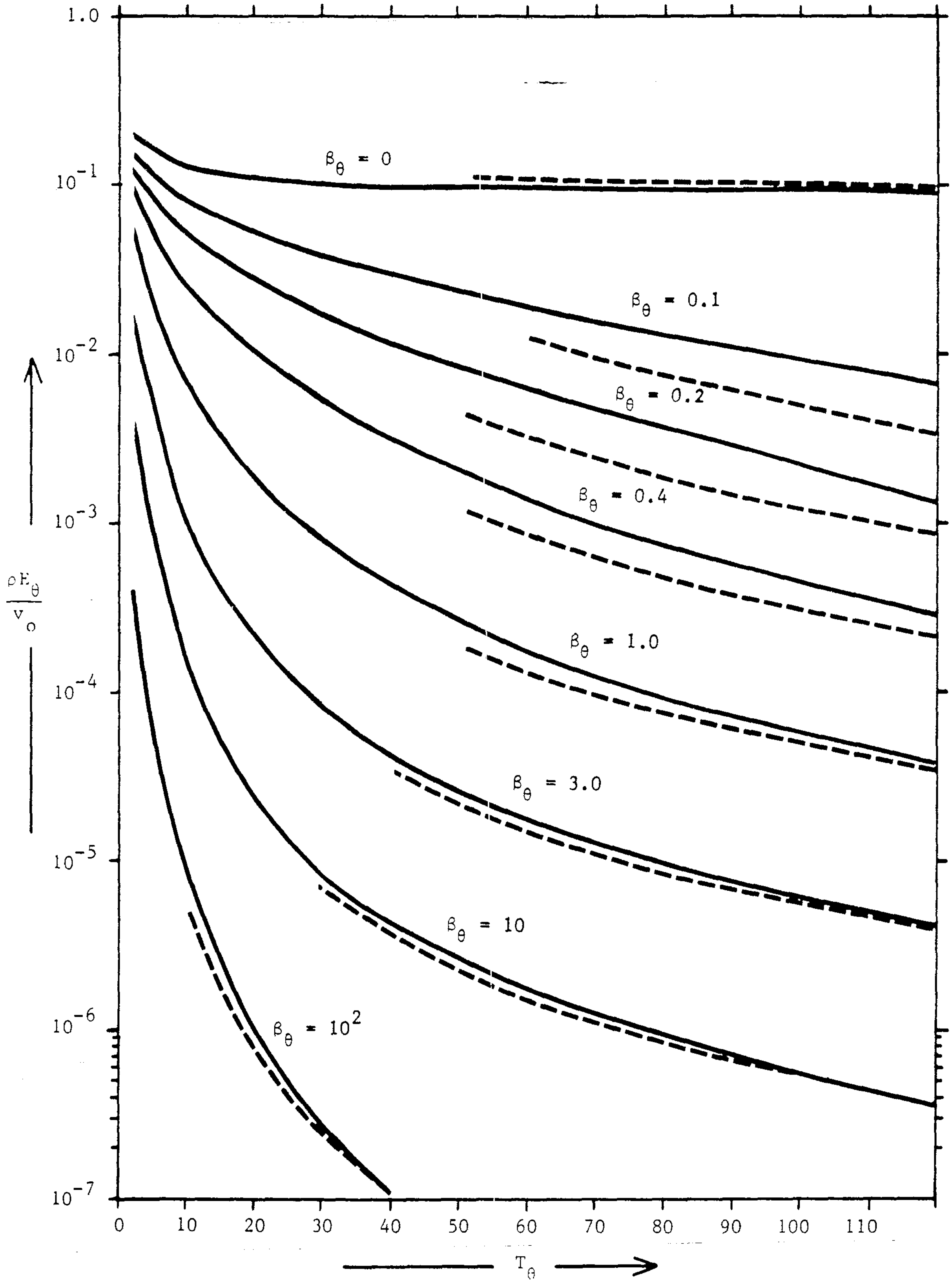

Figure 2. Radiation field for a step-function voltage.

Appendix

For pedagogical reason we shall give here some steps that lead to equation (1) in the text. Instead of treating the cylindrical structure as a special case of an axially symmetric body we consider, ab initio, an infinite cylindrical structure to which equation (1) applies. From the fact that $\vec{H}$ at an interior point $\vec{r}$ in a source-free region bounded by a regular surface S can be expressed in terms of the values of $\vec{E}$ and $\vec{H}$ on S, we write,[6] with the time factor $e^{-i\omega t}$ suppressed,

$$\vec{H}(\vec{r}) = -\int_S \left\{ i\omega\varepsilon(\vec{n}' \times \vec{E})G - (\vec{n}' \times \vec{H}) \times \nabla' G - (\vec{n}' \cdot \vec{H})\nabla' G \right\} dS' \tag{A-1}$$

where

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi} \frac{e^{ik\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + (z - z')^2}}}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + (z - z')^2}}$$

and $\vec{n}'$ is the inward unit normal to S. In the case under consideration there are no sources at infinity and the surface S is just that enclosing the infinite cylindrical antenna.

Taking the ϕ-component of (A-1) and noting that $\nabla' = -\nabla$ we obtain, after some vector algebra,

$$H_\phi(\rho,z,\phi) = i\omega\varepsilon \int \cos(\phi - \phi')E_z \, G \, dS' - \frac{\partial}{\partial\rho} \int H_\phi \, G \, dS'$$

$$- \frac{1}{\rho} \frac{\partial}{\partial\phi} \int H_\rho \, G \, dS' \tag{A-2}$$

where $dS' = a\,d\phi' dz'$. We now multiply (A-2) by a, the radius of the cylindrical antenna, and then integrate the resulting equation with respect to ϕ from 0 to 2π . Since the last term on the right side of (A-2) integrates to zero, we have

$$a\int_0^{2\pi} H_\phi(\rho,z,\phi)d\phi = 2\pi i\omega\varepsilon a^2 \int_{-\infty}^{\infty} dz' \frac{1}{2\pi}\int_0^{2\pi} E_z(a,z',\phi')d\phi' \int_0^{2\pi} d\psi \cos\psi\, G(\rho,a;z,z';\psi)$$

$$- a\frac{\partial}{\partial\rho}\int_{-\infty}^{\infty} dz' \int_0^{2\pi} aH_\phi(a,z',\phi')d\phi' \int_0^{2\pi} d\psi\, G(\rho,a;z,z';\psi) \qquad \text{(A-3)}$$

where we have made use of

$$\int_0^{2\pi}\int_0^{2\pi} f(\phi')G(\phi-\phi')d\phi'd\phi = \int_0^{2\pi} f(\phi')d\phi' \int_0^{2\pi} G(\psi)d\psi$$

which follows from the fact that $G(\phi)$ is a function of $\cos\phi$.

Noting that

$$\left[\frac{\partial}{\partial\rho} G(\rho,a;z,z';\phi)\right]_{\rho=a} = \frac{1}{2}\frac{\partial}{\partial a} G(a,a;z,z';\phi) \qquad \text{(A-4)}$$

and

$$\lim_{\rho\to a}\int_{-\infty}^{\infty}\int_0^{2\pi} I(z')\frac{\partial}{\partial\rho} G(\rho,a;z,z';\psi)a\,d\psi\, dz'$$

$$= -\frac{1}{2} I(z) + \int_{-\infty}^{\infty}\int_0^{2\pi} I(z')\left[\frac{\partial}{\partial\rho} G(\rho,a;z,z';\psi)\right]_{\rho=a} a\,d\psi\, dz' \quad , \qquad \text{(A-5)}$$

one immediately obtains (1) from (A-3). (A-4) can be verified by straightforward differentiation. The term $-I/2$ on the right side of (A-5) comes from the contribution of the integral over a small area on the surface surrounding the point that the observation point approaches when taking the limit $\rho \to a$.

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