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# The Circular Flush-Plate Dipole in a Conducting Plane and Located in Non-Conducting Media

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## Abstract

This note considers the response characteristics of a circular flush-plate dipole in non-conducting media. This sensor is a circular plate coplanar with a conducting plane, but isolated from it by a narrow slot. The current through the resistive load across the slot is proportional to a component of the displacement current density at low frequencies. The response characteristics of this sensor are calculated using cylindrical vector eigenfunction expansions. These calculations include the dependence of the sensor response on both frequency and the direction of wave incidence.

## Foreword

The calculations in this note have a form similar to those in a previous note concerning a spherical dipole. For convenience the figures are grouped together after the summary and before the appendices. Appendix E was written by Mr. Joe Martinez of Dikewood and we would like to thank him for the numerical calculations and thank him and Sgt. Richard T. Clark of AFWL for the graphs.

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## I. Introduction

Of the various types of electromagnetic sensors there is one type which responds to a component of the total current density or, if the medium conductivity is zero, a component of the displacement current density, $\vec{D}$. This sensor is characterized by a geometric parameter, its equivalent area.[1] A previous note[2] has considered the frequency response characteristics of one such device, a slotted hollow sphere. The purpose of this note is to consider another geometry of this type of sensor.

In this note we consider a sensor which might be termed a circular flush-plate dipole. This sensor is a part of a conducting plane and we analyze its performance using vector eigenfunction expansions in cylindrical coordinates. The analysis is similar to that for the spherical dipole in reference 2.

The sensor geometry is shown in figure 1. It is basically a circular disk of radius $\Psi_1$ centered in a hole of radius $\Psi_2$ in a conducting ground plane. Both disk and ground plane are assumed infinitely thin and perfectly conducting. As shown in figure 1 we have cartesian $(x, y, z)$, cylindrical $(\Psi, \phi, z)$, and spherical $(r, \theta, \phi)$ coordinates related as

$$x = \Psi \cos(\phi) , \qquad y = \Psi \sin(\phi)$$

$$z = r \cos(\theta) , \qquad \Psi = r \sin(\theta) \tag{1}$$

The disk is defined by $z = 0$, $0 \le \Psi < \Psi_1$ and the conducting plane by $z = 0$, $\Psi > \Psi_2$; the slot is defined by $\Psi_1 < \Psi < \Psi_2$. The nominal slot center is given by $\Psi = a$ which we define by

$$\frac{\Psi_1}{a} \equiv e^{-\frac{b}{a}} , \qquad \frac{\Psi_2}{a} \equiv e^{\frac{b}{a}} , \qquad \Psi_1\Psi_2 = a^2 \tag{2}$$

so that a is the geometric mean of $\Psi_1$ and $\Psi_2$. b is a lengh assumed small compared to a, so that for $b \ll a$ the slot width is approximately 2b. Note that as $b \to 0$

$$\Psi_1 = a - b + O(b^2) , \qquad \Psi_2 = a + b + O(b^2) \tag{3}$$

1. Capt Carl E. Baum, Sensor and Simulation Note 38, Parameters for Some Electrically-Small Electromagnetic Sensors, March 1967.

2. Capt Carl E. Baum, Sensor and Simulation Note 91, The Single-Gap Hollow Spherical Dipole in Non-Conducting Media, July 1969.

The reason for this choice of the gap center will appear later as a convenient definition in the admittance calculations. We also have

$$\sinh\left(\frac{b}{a}\right) = \frac{\Psi_2 - \Psi_1}{2a} \tag{4}$$

The sensor gap, or slot in the conducting plane, is assumed resistively loaded uniformly in $\phi$ to preserve symmetry about the z axis. In practice this might be several cable inputs uniformly spaced around the gap and bringing the signals with equal delays to one common point. Such cables would be located in positions where they would not significantly affect the sensor response; they are not considered in this note.

In this note we consider the response of such a dipole to a plane wave incident on one side of the conducting plane (say $z > 0$); this includes the reflection of the plane wave by the conducting plane. Both sides of the conducting plane are assumed to be semi-infinite uniform isotropic and homogeneous media, and for the numerical results the media on both sides are taken as having the same parameters with zero conductivity. The sensor has an equivalent area of $\pi a^2$ which at low frequencies relates the vertical component of $\vec{D}$ or the total current density (including the reflection from the $z = 0$ plane) to the current from the sensor.

Note that we take the media on both sides of the $z = 0$ plane as identical and infinite in extent. Practically this may typically mean that if the $z = 0$ plane is a conducting ground plane on the earth's surface, then there is a hollowed-out volume below the sensor which is sufficiently large that the volume boundaries negligibly distort the fields near the sensor. Also the volume is assumed lossy enough that energy radiated from the sensor toward negative z is absorbed and not reflected back into the sensor. There are other geometries for the lower medium that one might consider such as a coaxial TEM transmission line[3,4] or a hemispherical cavity with lossy walls.[5] In this note, however, we take the lower medium as semi-infinite and consider the

3. H. Levine and C. H. Papas, Theory of the Circular Diffraction Antenna, J. Appl. Phys., vol. 22, no. 1, Jan. 1951, pp. 29-43.

4. G. I. Cohn and G. T. Flesher, Theoretical Radiation Pattern and Impedance of a Flush-Mounted Coaxial Aperture, Proc. Natl. Electronics Conf., vol. 14, 1958, pp. 150-168.

5. J. R. Wait, A Low-Frequency Annular-Slot Antenna, J. Res. N.B.S., vol. 60, no. 1, Jan. 1958, pp. 59-64.

receiving characteristics of the sensor. One should note that with both upper and lower media the same the sensor might not be strictly considered an electric dipole since as one drives it a dipole field is not produced if both positive and negative z are included. However, since we are only considering the sensor from the viewpoint of its response to incident fields on one side of the conducting plane and the sensor can be considered to have an electric dipole moment when one considers the conducting plane and only one side of this plane ($z > 0$) because of its far-field characteristics as a radiating antenna, then we still characterize the sensor as an electric dipole.

In outline this note first considers the expansion of electromagnetic fields into cylindrical vector eigenfunctions. These are then used to expand plane waves in cylindrical coordinates. Then the short circuit current from the sensor is calculated, including the effect of the angle of incidence of an incident plane wave. The sensor admittance is then calculated for $b \ll a$ by assuming a quasi-static electric field distribution in the slot. Finally these results are combined to give the sensor response to an incident plane wave for various resistive loads on the sensor output.

## II. Electromagnetic Fields in Cylindrical Coordinates

Consider a linear, homogeneous, isotropic medium with scalar permittivity $\varepsilon$, permeability $\mu$, and conductivity $\sigma$. We have propagation constants[6]

$$k = \sqrt{-i\omega\mu(\sigma + i\omega\varepsilon)}$$

$$\gamma = \sqrt{s\mu(\sigma + s\varepsilon)} \tag{5}$$

and a wave impedance

$$Z = \sqrt{\frac{s\mu}{\sigma + s\varepsilon}} = \sqrt{\frac{i\omega\mu}{\sigma + i\omega\varepsilon}} \tag{6}$$

where s is the Laplace transform variable which is taken as $i\omega$ for frequency-domain analysis. The radian frequency is $\omega$ and i is the unit imaginary. We include $\sigma$ in the analysis for generality but set it to zero for the numerical results.

With time harmonic fields and $e^{i\omega t}$ suppressed Maxwell's equations are

6. All units are rationalized MKSA.

$$\nabla \times \vec{E} = -i\omega\vec{B}\ , \qquad \nabla \times \vec{H} = \vec{J} + i\omega\vec{D}$$

$$\nabla \cdot \vec{B} = 0\ , \qquad \nabla \cdot \vec{D} = \rho \tag{7}$$

and the constitutive relations plus Ohm's law are

$$\vec{B} = \mu\vec{H}\ , \qquad \vec{D} = \varepsilon\vec{E}\ , \qquad \vec{J} = \sigma\vec{E} \tag{8}$$

Assume no charge ($\rho = 0$) or source currents in the medium of interest away from the boundaries giving vector wave equations as

$$\nabla^2\vec{E} + k^2\vec{E} = \vec{0}\ , \qquad \nabla^2\vec{H} + k^2\vec{H} = \vec{0} \tag{9}$$

Note that both $\vec{E}$ and $\vec{H}$ now have zero divergence away from the boundaries of the medium.

In cylindrical coordinates the solution of the scalar wave equation

$$\nabla^2 T + k^2 T = 0 \tag{10}$$

can be written as a linear combination of functions of the form[7]

$$T^{(\ell)}(n,\zeta_1,{}^{e}_{o}) \equiv F_n^{(\ell)}(k\Psi\zeta_2)\,e^{-ikz\zeta_1}\begin{Bmatrix}\cos(n\phi)\\ \sin(n\phi)\end{Bmatrix} \tag{11}$$

with

$$\zeta_1^2 + \zeta_2^2 = 1 \tag{12}$$

where $F_n^{(\ell)}(k\Psi\zeta_2)$ is one of the cylindrical Bessel functions $J_n(k\Psi\zeta_2)$, $Y_n(k\Psi\zeta_2)$, $H_n^{(1)}(k\Psi\zeta_2)$, $H_n^{(2)}(k\Psi\zeta_2)$ for $\ell = 1, 2, 3, 4$ in that order. The third argument of T is listed as e or o (meaning even or odd) and corresponds to using $\cos(n\phi)$ or $\sin(n\phi)$ respectively. We only consider n as an integer for our cases of interest so that the solutions apply for all $\phi$ around a complete circle. On the other hand $\zeta_2$ can take on any value, including complex values, although we will only be using real values.

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7. J. A. Stratton, Electromagnetic Theory, McGraw Hill, 1941, section 7.2.

Note then that $\zeta_2$ is a double valued function of $\zeta_1$ and we must specify which value we are using for a given calculation.

Similar to Stratton (reference 7, section 7.3) we define three sets of solutions to the vector wave equation (as in equations 9) as

$$\vec{L}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) \equiv \frac{1}{k}\nabla T^{(\ell)}(n,\zeta_1,{}^{e}_{o})$$

$$\vec{M}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) \equiv \frac{1}{k}\nabla \times \left[T^{(\ell)}(n,\zeta_1,{}^{e}_{o})\vec{e}_z\right] \tag{13}$$

$$\vec{N}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) \equiv \frac{1}{k}\nabla \times \vec{M}^{(\ell)}(n,\zeta_1,{}^{e}_{o})$$

where $\vec{e}_z$ is a unit vector in the z direction (and similarly for other unit vectors). These vector wave functions have some further relationships as

$$\vec{M}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \frac{1}{k}\left[\nabla T^{(\ell)}(n,\zeta_1,{}^{e}_{o})\right]\times\vec{e}_z = \vec{L}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) \times \vec{e}_z$$

$$= \frac{1}{k}\nabla \times \vec{N}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) \tag{14}$$

$$\vec{N}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \frac{1}{k^2}\nabla \times \nabla \times \left[T^{(\ell)}(n,\zeta_1,{}^{e}_{o})\vec{e}_z\right]$$

$$= \frac{\partial}{\partial(kz)}\vec{L}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) + T^{(\ell)}(n,\zeta_1,{}^{e}_{o})\vec{e}_z$$

All three of these sets of vector eigenfunctions satisfy the vector wave equation as written in equations 9. However the $\vec{N}$ and $\vec{M}$ functions have zero divergence while for the $\vec{L}$ functions we have

$$\nabla \cdot \vec{L}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \frac{1}{k}\nabla^2 T^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = -kT^{(\ell)}(n,\zeta_1,{}^{e}_{o}) \tag{15}$$

Thus in this note we use only the $\vec{N}$ and $\vec{M}$ functions for the field expansions; the $\vec{L}$ functions are included in cases of source currents and charges present in the medium.

Writing out these vector eigenfunctions we have for the vector components of the $\vec{L}$ functions

$$L_{\Psi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \frac{\partial}{\partial(k\Psi)} T^{(\ell)}(n,\zeta_1,{}^{e}_{o})$$

$$L_{\phi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \frac{1}{k\Psi}\frac{\partial}{\partial\phi} T^{(\ell)}(n,\zeta_1,{}^{e}_{o}) \qquad (16)$$

$$L_{z}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \frac{\partial}{\partial(kz)} T^{(\ell)}(n,\zeta_1,{}^{e}_{o})$$

which can be expanded as

$$L_{\Psi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \zeta_2 F_n^{(\ell)'}(k\Psi\zeta_2)e^{-ikz\zeta_1}\begin{Bmatrix}\cos(n\phi)\\ \sin(n\phi)\end{Bmatrix}$$

$$L_{\phi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \frac{F_n^{(\ell)}(k\Psi\zeta_2)}{k\Psi} e^{-ikz\zeta_1}\, n\begin{Bmatrix}-\sin(n\phi)\\ \cos(n\phi)\end{Bmatrix} \qquad (17)$$

$$L_{z}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = -i\zeta_1 F_n^{(\ell)}(k\Psi\zeta_2)e^{-ikz\zeta_1}\begin{Bmatrix}\cos(n\phi)\\ \sin(n\phi)\end{Bmatrix}$$

where a prime used with a Bessel function indicates differentiation with respect to the argument of the Bessel function being considered. For the $\vec{M}$ functions we have the components

$$M_{\Psi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \frac{1}{k\Psi}\frac{\partial}{\partial\phi} T^{(\ell)}(n,\zeta_1,{}^{e}_{o})$$

$$M_{\phi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = -\frac{\partial}{\partial(k\Psi)} T^{(\ell)}(n,\zeta_1,{}^{e}_{o}) \qquad (18)$$

$$M_{z}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = 0$$

which can be expanded as

$$M_{\Psi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \frac{F_n^{(\ell)}(k\Psi\zeta_2)}{k\Psi}\, e^{-ikz\zeta_1}\, n\begin{Bmatrix}-\sin(n\phi)\\ \cos(n\phi)\end{Bmatrix}$$

$$M_{\phi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = -\zeta_2 F_n^{(\ell)'}(k\Psi\zeta_2)\, e^{-ikz\zeta_1}\begin{Bmatrix}\cos(n\phi)\\ \sin(n\phi)\end{Bmatrix} \tag{19}$$

$$M_{z}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = 0 \ .$$

For the $\vec{N}$ functions we have the components

$$N_{\Psi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \frac{\partial^2}{\partial(k\Psi)\partial(kz)}\, T^{(\ell)}(n,\zeta_1,{}^{e}_{o})$$

$$N_{\phi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \frac{1}{k\Psi}\,\frac{\partial^2}{\partial(kz)\partial\phi}\, T^{(\ell)}(n,\zeta_1,{}^{e}_{o}) \tag{20}$$

$$N_{z}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \left[1 + \frac{\partial^2}{\partial(kz)^2}\right] T^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \zeta_2^2 T^{(\ell)}(n,\zeta_1,{}^{e}_{o})$$

which can be expanded as

$$N_{\Psi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = -i\zeta_1\zeta_2 F_n^{(\ell)'}(k\Psi\zeta_2)\, e^{-ikz\zeta_1}\begin{Bmatrix}\cos(n\phi)\\ \sin(n\phi)\end{Bmatrix}$$

$$N_{\phi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = -i\zeta_1\,\frac{F_n^{(\ell)}(k\Psi\zeta_2)}{k\Psi}\, e^{-ikz\zeta_1}\, n\begin{Bmatrix}-\sin(n\phi)\\ \cos(n\phi)\end{Bmatrix} \tag{21}$$

$$N_{z}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \zeta_2^2 F_n^{(\ell)}(k\Psi\zeta_2)\, e^{-ikz\zeta_1}\begin{Bmatrix}\cos(n\phi)\\ \sin(n\phi)\end{Bmatrix}$$

Next we have some orthogonality relationships for these vector eigenfunctions on the circle $0 \leq \phi \leq 2\pi$. If the n index or the ${}^{e}_{o}$ index differs between any two of these vector eigenfunctions (including two of the same kind) then these two functions are orthogonal. Likewise any combination of $\vec{L}$ with $\vec{M}$ or $\vec{M}$ with $\vec{N}$ is orthogonal. For n and ${}^{e}_{o}$ the same we have the relations

$$\int_0^{2\pi} \vec{L}^{(\ell)}(n,\zeta_1,{}^e_o) \cdot \vec{L}^{(\ell')}(n,\zeta_1',{}^e_o)d\phi$$

$$= [1 \pm \delta_{n,o}]\pi\, e^{-ikz(\zeta_1+\zeta_1')}\Bigg\{\zeta_2\zeta_2' F_n^{(\ell)\,\prime}(k\Psi\zeta_2)F_n^{(\ell')\,\prime}(k\Psi\zeta_2)$$

$$+ \left[\frac{n^2}{(k\Psi)^2} - \zeta_1\zeta_1'\right]F_n^{(\ell)}(k\Psi\zeta_2)F_n^{(\ell')}(k\Psi\zeta_2')\Bigg\} \qquad (22)$$

$$\int_0^{2\pi} \vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) \cdot \vec{M}^{(\ell')}(n,\zeta_1',{}^e_o)d\phi$$

$$= [1 \pm \delta_{n,o}]\pi\, e^{-ikz(\zeta_1+\zeta_1')}\Bigg\{\zeta_2\zeta_2' F_n^{(\ell)\,\prime}(k\Psi\zeta_2)F_n^{(\ell')\,\prime}(k\Psi\zeta_2')$$

$$+ \frac{n^2}{(k\Psi)^2} F_n^{(\ell)}(k\Psi\zeta_2)F_n^{(\ell')}(k\Psi\zeta_2')\Bigg\} \qquad (23)$$

$$\int_0^{2\pi} \vec{N}^{(\ell)}(n,\zeta_1,{}^e_o) \cdot \vec{N}^{(\ell')}(n,\zeta_1',{}^e_o)d\phi$$

$$= [1 \pm \delta_{n,o}]\pi\, e^{-ikz(\zeta_1+\zeta_1')}\Bigg\{-\zeta_1\zeta_1'\zeta_2\zeta_2' F_n^{(\ell)\,\prime}(k\Psi\zeta_2)F_n^{(\ell')\,\prime}(k\Psi\zeta_2')$$

$$+ \left[\zeta_2^2\zeta_2'^2 - \frac{n^2}{(k\Psi)^2}\zeta_1\zeta_1'\right]F_n^{(\ell)}(k\Psi\zeta_2)F_n^{(\ell')}(k\Psi\zeta_2')\Bigg\} \qquad (24)$$

$$\int_0^{2\pi} \vec{L}^{(\ell)}(n,\zeta_1,{}^e_o) \cdot \vec{N}^{(\ell')}(n,\zeta_1',{}^e_o)d\phi$$

$$= [1 \pm \delta_{n,o}](-i)\pi e^{-ikz(\zeta_1+\zeta_1')}\Bigg\{\zeta_2\zeta_1'\zeta_2' F_n^{(\ell)\,\prime}(k\Psi\zeta_2)F_n^{(\ell')\,\prime}(k\Psi\zeta_2')$$

$$+ \left[\zeta_1 \zeta_2'^2 + \frac{n^2}{(k\Psi)^2} \zeta_1'\right] F_n^{(\ell)}(k\Psi\zeta_2) F_n^{(\ell')}(k\Psi\zeta_2')\Bigg\} \tag{25}$$

where $\ell'$ and $\zeta_1'$ are used in the second function in place of $\ell$ and $\zeta_1$ respectively and where

$$\zeta_1'^2 + \zeta_2'^2 = 1 \tag{26}$$

The Kronecker delta function is defined by

$$\delta_{n_1,n_2} \equiv \begin{cases} 1 \text{ for } n_1 = n_2 \\ 0 \text{ for } n_1 \neq n_2 \end{cases} \tag{27}$$

With expressions for the $\vec{N}$ and $\vec{M}$ functions we can expand an electric field with zero divergence in the form

$$\vec{E} = E_o \sum_{n=o}^{\infty} \int \left\{ \alpha_n(\zeta_1) \vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) + \beta_n(\zeta_1) \vec{N}^{(\ell)}(n,\zeta_1,{}^e_o) \right\} \left\{ \begin{matrix} d\zeta_1 \\ d\zeta_2 \end{matrix} \right\} \tag{28}$$

where $E_o$ is some convenient constant with dimensions of volts per meter. Note that the dimensionless constants $\alpha_n$ and $\beta_n$ are both in general functions of $\zeta_1$ (or equivalently $\zeta_2$) which can take on complex values; so we may wish to integrate over some range of $\zeta_1$ or $\zeta_2$ in the complex plane; this is just indicated in equation 28 by an indefinite integral over $\zeta_1$ or $\zeta_2$. Alternatively we may only need particular discrete $\zeta_1$ and thus no integral of this type. Note that we can also sum over $\ell$ and over even and odd functions, but in the calculations in this note we will not need such summations. Compare equations 7 and 8 relating $\vec{E}$ and $\vec{H}$ to the relations between the $\vec{M}$ and $\vec{N}$ functions as in equations 13 and 14, and note that we can find $\vec{H}$ from $\vec{E}$ by replacing

$$\begin{aligned} \vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) &\to \frac{i}{Z} \vec{N}^{(\ell)}(n,\zeta_1,{}^e_o) \\ \vec{N}^{(\ell)}(n,\zeta_1,{}^e_o) &\to \frac{i}{Z} \vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) \end{aligned} \tag{29}$$

giving an $\vec{H}$ corresponding to equation 28 as

$$\vec{H} = i\,\frac{E_o}{Z} \sum_{n=0}^{\infty} \int \left\{ \alpha_n(\zeta_1)\vec{N}^{(\ell)}(n,\zeta_1,{}^e_o) + \beta_n(\zeta_1)\vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) \right\} \left\{ \begin{matrix} d\zeta_1 \\ d\zeta_2 \end{matrix} \right\} \quad (30)$$

Similarly an expansion for $\vec{H}$ can be converted to one for $\vec{E}$ by substituting

$$\vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) \rightarrow -iZ\vec{N}^{(\ell)}(n,\zeta_1,{}^e_o)$$

$$\vec{N}^{(\ell)}(n,\zeta_1,{}^e_o) \rightarrow -iZ\vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) \quad (31)$$

While we have considered these vector eigenfunctions from the point of view of expanding divergenceless electric and magnetic fields they can also be used (including the $\vec{L}$ functions) for appropriate fields without zero divergence. These functions can also be used for other quantities satisfying the vector and scalar wave equations, such as vector and scalar potentials and Hertz vectors.

## III. Vector Plane Waves in Cylindrical Coordinates

Having the general forms of the electromagnetic field expansions in cylindrical coordinates we go on to consider plane waves of the form

$$\vec{F} \equiv F_o\vec{u}\; e^{-i\vec{k}\cdot\vec{r}} \quad (32)$$

where $\vec{u}$ is some unit vector independent of the coordinates and the propagation vector is

$$\vec{k} \equiv k\;\vec{e}_1 \quad (33)$$

where $\vec{e}_1$ is the direction of propagation of the wave. This plane wave is shown in figure 2A at some position $\vec{r}$ with a fixed polarization for purposes of illustration. The three vectors $\vec{E}$, $\vec{H}$, and $\vec{e}_1$ are mutually perpendicular and related by

$$\vec{E} = Z\vec{H} \times \vec{e}_1 \quad (34)$$

Starting with $\vec{e}_1$ we define two more unit vectors $\vec{e}_2$ and $\vec{e}_3$ so that all 3 are mutually orthogonal. Referring to figure 2B the direction of $\vec{e}_1$ is described by $\theta_1$ and $\phi_1$ in a spherical coordinate system with respect to the cartesian unit vectors ($\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$). Let $\vec{e}_2$ be parallel to the same plane as both $\vec{e}_z$ and $\vec{e}_1$. Then $\vec{e}_3$ is parallel to the x, y plane. Since $\vec{e}_2$ and $\vec{e}_3$ are mutually orthogonal and both orthogonal to $\vec{e}_1$, and since $\vec{E}$ and $\vec{H}$ are both orthogonal to $\vec{e}_1$, then $\vec{e}_2$ and $\vec{e}_3$ can be used in some linear combination to describe the directions of $\vec{E}$ and $\vec{H}$. Thus we consider plane waves as in equation 32 for which $\vec{u}$ is taken as alternately $\vec{e}_2$ and $\vec{e}_3$.

These unit vectors form a right handed system with the relations

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3 , \qquad \vec{e}_2 \times \vec{e}_3 = \vec{e}_1 , \qquad \vec{e}_3 \times \vec{e}_1 = \vec{e}_2 \tag{35}$$

Note that $\vec{e}_2$ is chosen such that for $0 \leq \theta_1 \leq \pi/2$ the polar angle of $\vec{e}_2$ (as in figure 2B) is $\pi/2 - \theta_1$. In terms of the cartesian unit vectors we have

$$\begin{aligned} \vec{e}_1 &= \sin(\theta_1)\cos(\phi_1)\vec{e}_x + \sin(\theta_1)\sin(\phi_1)\vec{e}_y + \cos(\theta_1)\vec{e}_z \\ \vec{e}_2 &= -\cos(\theta_1)\cos(\phi_1)\vec{e}_x - \cos(\theta_1)\sin(\phi_1)\vec{e}_y + \sin(\theta_1)\vec{e}_z \\ \vec{e}_3 &= \sin(\phi_1)\vec{e}_x - \cos(\phi_1)\vec{e}_y \end{aligned} \tag{36}$$

Cartesian and cylindrical coordinates are related by

$$x = \Psi \cos(\phi) , \qquad y = \Psi \sin(\phi) \tag{37}$$

The cartesian and cylindrical unit vectors are related by

$$\begin{aligned} \vec{e}_x &= \cos(\phi)\vec{e}_\Psi - \sin(\phi)\vec{e}_\phi \\ \vec{e}_y &= \sin(\phi)\vec{e}_\Psi + \cos(\phi)\vec{e}_\phi \end{aligned} \tag{38}$$

or

$$\begin{aligned} \vec{e}_\Psi &= \cos(\phi)\vec{e}_x + \sin(\phi)\vec{e}_y \\ \vec{e}_\phi &= -\sin(\phi)\vec{e}_x + \cos(\phi)\vec{e}_y \end{aligned} \tag{39}$$

where $\vec{e}_z$ is common to both cartesian and cylindrical systems. Substitute for the cartesian unit vectors in equations 36 from equations 38 and use some trigonometric identities to give

$$\vec{e}_1 = \sin(\theta_1)\cos(\phi-\phi_1)\vec{e}_\Psi - \sin(\theta_1)\sin(\phi-\phi_1)\vec{e}_\phi + \cos(\theta_1)\vec{e}_z$$

$$\vec{e}_2 = -\cos(\theta_1)\cos(\phi-\phi_1)\vec{e}_\Psi + \cos(\theta_1)\sin(\phi-\phi_1)\vec{e}_\phi + \sin(\theta_1)\vec{e}_z \tag{40}$$

$$\vec{e}_3 = -\sin(\phi-\phi_1)\vec{e}_\Psi - \cos(\phi-\phi_1)\vec{e}_\phi$$

For our sensor configuration (figure 1) we are interested in an incident wave of the form

$$\vec{E}_{inc} = E_o\vec{e}_2 e^{-i\vec{k}\cdot\vec{r}} \quad , \qquad \vec{H}_{inc} = \frac{E_o}{Z}\,\vec{e}_3 e^{-i\vec{k}\cdot\vec{r}} \tag{41}$$

Thus we need $\vec{e}_2 e^{-i\vec{k}\cdot\vec{r}}$ and $\vec{e}_3 e^{-i\vec{k}\cdot\vec{r}}$ in terms of the cylindrical vector eigenfunctions discussed in the previous section. Other polarizations of the incident wave could also be considered but due to the symmetry of the sensor about the z axis only the vertical electric field component produces any signal in the sensor and this field component is only associated with the polarization as in equations 41. As a further simplification, since the sensor geometry is independent of $\phi$ we set $\phi_1 = 0$ without loss of generality.

To expand these vector plane waves we first observe

$$\vec{k} \cdot \vec{r} = k\vec{r} \cdot \vec{e}_1 = k[\Psi\vec{e}_\Psi \cdot \vec{e}_1 + z\vec{e}_z \cdot \vec{e}_1]$$

$$= k\Psi \sin(\theta_1) \cos(\phi) + kz \cos(\theta_1) \tag{42}$$

Now set

$$\zeta_1 \equiv \cos(\theta_1) \quad , \qquad \zeta_2 \equiv \sin(\theta_1) \tag{43}$$

and assume that $\zeta_2 \neq 0$. Then we can write

$$e^{-i\vec{k}\cdot\vec{r}} = e^{-ik\Psi\zeta_2\cos(\phi)}\; e^{-ikz\zeta_1} \tag{44}$$

and

$$\vec{e}_2 = -\zeta_1 \cos(\phi)\vec{e}_\Psi + \zeta_1 \sin(\phi)\vec{e}_\phi + \zeta_2\vec{e}_z$$

$$\vec{e}_3 = -\sin(\phi)\vec{e}_\Psi - \cos(\phi)\vec{e}_\phi \tag{45}$$

Now we have the expansion[8]

$$e^{-ik\Psi\zeta_2\cos(\phi)} = \sum_{n=o}^{\infty} [2 - \delta_{n,o}](-i)^n J_n(k\Psi\zeta_2) \cos(n\phi) \tag{46}$$

Our task is now simply to identify the above expressions with the vector eigenfunctions developed in section II. The wave function components of interest are

$$N_\Psi^{(1)}(n,\zeta_1,e) = -i\zeta_1\zeta_2 J_n'(k\Psi\zeta_2)e^{-ikz\zeta_1} \cos(n\phi)$$

$$N_\phi^{(1)}(n,\zeta_1,e) = i\zeta_1 \frac{J_n(k\Psi\zeta_2)}{k\Psi} e^{-ikz\zeta_1} n \sin(n\phi) \tag{47}$$

$$N_z^{(1)}(n,\zeta_1,e) = \zeta_2^2 J_n(k\Psi\zeta_2)e^{-ikz\zeta_1} \cos(n\phi)$$

and

$$M_\Psi^{(1)}(n,\zeta_1,e) = -\frac{J_n(k\Psi\zeta_2)}{k\Psi} e^{-ikz\zeta_1} n \sin(n\phi)$$

$$M_\phi^{(1)}(n,\zeta_1,e) = -\zeta_2 J_n'(k\Psi\zeta_2)e^{-ikz\zeta_1} \cos(n\phi) \tag{48}$$

$$M_z^{(1)}(n,\zeta_1,e) = 0$$

8. Abramowitz and Stegun, ed., Handbook of Mathematical Functions, AMS 55, National Bureau of Standards, 1964, eqns. 9.1.44 and 9.1.45.

Note that by our choice of $\zeta_1$ in equation 43 then the z dependence of the plane waves as in equation 45 matches that of the vector eigenfunctions. This leaves the $\Psi$ and $\phi$ dependence. Now $\vec{e}_3$ has no z component so that $\vec{H}_{inc}$ (equations 41) has no z component and can only be expanded with $\vec{M}$ functions. This implies (from equations 31) that $\vec{E}_{inc}$ can only be expanded in $\vec{N}$ functions. For the electric field expansion we then identify the z component of $\vec{e}_2$ in equations 45 plus the expansion in equation 46 with $N_z$ in equations 47 to give

$$e_{2_z} e^{-i\vec{k}\cdot\vec{r}} = \zeta_2 \sum_{n=o}^{\infty} \left\{[2 - \delta_{n,o}](-i)^n J_n(k\Psi\zeta_2)\right\} e^{-ikz\zeta_1} \cos(n\phi)$$

$$= \frac{1}{\zeta_2} \sum_{n=o}^{\infty} [2 - \delta_{n,o}](-i)^n N_z^{(1)}(n,\zeta_1,e) \tag{49}$$

Since the $\vec{M}$ functions cannot be included in $\vec{e}_2 e^{-i\vec{k}\cdot\vec{r}}$ the only expansion choice available for the electric field as in equations 41 is

$$\vec{E}_{inc} = E_o \vec{e}_2 e^{-i\vec{k}\cdot\vec{r}}$$

$$\vec{e}_2 e^{-i\vec{k}\cdot\vec{r}} = \frac{1}{\zeta_2} \sum_{n=o}^{\infty} [2 - \delta_{n,o}](-i)^n \vec{N}^{(1)}(n,\zeta_1,e) \tag{50}$$

Then making the transformation from the electric field to the associated magnetic field as indicated in equations 29 gives the magnetic field in equations 41 as

$$\vec{H}_{inc} = \frac{E_o}{Z} \vec{e}_3 e^{-i\vec{k}\cdot\vec{r}}$$

$$\vec{e}_3 e^{-i\vec{k}\cdot\vec{r}} = \frac{1}{\zeta_2} \sum_{n=o}^{\infty} [2 - \delta_{n,o}](-i)^n i\vec{M}^{(1)}(n,\zeta_1,e) \tag{51}$$

As a check one can expand the $\Psi$ and $\phi$ components of the right side of equations 50 and by use of trigonometric identities and recurrence relations for the Bessel functions manipulate the series into the forms given by equations 44 through 46. We do not record these manipulations here but note that they do check equations 50. Equations 51 follow by a simple transformation (as in equations 29) from equations 50.

For the results of equations 50 and 51 we have set $\phi_1 = 0$. However the results can be easily generalized to include $\phi_1 \neq 0$ by making a coordinate rotation by replacing $\phi$ everywhere by $\phi - \phi_1$. (Note the combination $\phi - \phi_1$ throughout equations 40.) Alternatively both even and odd $\vec{N}$ and $\vec{M}$ functions could be used in the expansions.

The expansion of plane waves of the form $\vec{e}_1 e^{-i\vec{k}\cdot\vec{r}}$ would be another generalization of the present results, allowing any $\vec{u}$ in equation 32 as a linear combination of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$. This form of plane wave would apply to the case of source currents and charges in the medium and $\vec{L}$ functions would be included in the expansion.

## IV. Short Circuit Current

Having expanded plane waves in cylindrical coordinates we go on to consider the short circuit current from the sensor. For these calculations we short out the slot in the sensor at all $\phi$ so that the problem becomes one of considering the surface current on the perfectly conducting x, y plane as shown in figure 3. The incident plane wave has its propagation vector $\vec{k}$ pointing at an angle $\theta_1$ with respect to the z direction; the reflected plane wave has its propagation vector $\vec{k}_{re}$ pointing at an angle $\pi - \theta_1$ with respect to the z direction. For convenience we define another angle

$$\theta_1' \equiv \pi - \theta_1 \tag{52}$$

so that for our incident wave from equations 43

$$\begin{aligned} \zeta_1 &= \cos(\theta_1) = -\cos(\theta_1') \\ \zeta_2 &= \sin(\theta_1) = \sin(\theta_1') \end{aligned} \tag{53}$$

Then $\theta_1'$ is the angle of the incident and reflected wave propagation with respect to the negative x axis as shown in figure 3. Note that we only consider $\pi/2 \leq \theta_1 \leq \pi$ or $0 \leq \theta_1' \leq \pi/2$ for these calculations.

The expansion of the incident wave is given in equations 50 and 51. To obtain the reflected wave we merely replace $\theta_1$ by $\pi - \theta_1$ which replaces $\zeta_1$ by $-\zeta_1$ giving the reflected wave as

$$\vec{E}_{re} = E_o \frac{1}{\zeta_2} \sum_{n=o}^{\infty} [2 - \delta_{n,o}] (-i)^n \vec{N}^{(1)} (n, -\zeta_1, e)$$

$$\vec{H}_{re} = \frac{E_o}{Z} \frac{1}{\zeta_2} \sum_{n=o}^{\infty} [2 - \delta_{n,o}] (-i)^n i\vec{M}^{(1)} (n, -\zeta_1, e) \tag{54}$$

this gives a total field distribution as

$$\vec{E} = \vec{E}_{inc} + \vec{E}_{re} = E_o \frac{1}{\zeta_2} \sum_{n=o}^{\infty} [2 - \delta_{n,o}] (-i)^n \cdot \left[\vec{N}^{(1)} (n, \zeta_1, e) + \vec{N}^{(1)} (n, -\zeta_1, e)\right]$$

$$\vec{H} = \vec{H}_{inc} + \vec{H}_{re} = \frac{E_o}{Z} \frac{1}{\zeta_2} \sum_{n=o}^{\infty} [2 - \delta_{n,o}] (-i)^n \cdot i\left[\vec{M}^{(1)} (n, \zeta_1, e) + \vec{M}^{(1)} (n, -\zeta_1, e)\right] \tag{55}$$

Since the sensor is part of the conducting plane which of course reflects an incident wave we use the fields in equations 55 as our definition of the fields to which the sensor must respond; response functions will be normalized in terms of these fields.

Note in equation 47 that $N_\psi$ and $N_\phi$ are odd in $\zeta_1$ so that tangential E is zero on the x, y plane as required. At $\vec{r} = \vec{0}$, which is at the center of the sensor, the only E field component is $E_z$ which from equations 47 and 55 is

$$\vec{E}\Big|_{\vec{r}=\vec{o}} = 2E_o\zeta_2\vec{e}_z = 2E_o \sin(\theta_1)\vec{e}_z = 2E_o \sin(\theta_1')\vec{e}_z \tag{56}$$

The total current density in the medium at $\vec{r} = \vec{0}$ (just above the x, y plane) is

$$\vec{J}_t\Big|_{\vec{r}=\vec{o}} = (\sigma + i\omega\varepsilon)\vec{E}\Big|_{\vec{r}=\vec{o}} = 2(\sigma + i\omega\varepsilon)E_o \sin(\theta_1)\vec{e}_z \tag{57}$$

This will be used in defining the low-frequency sensitivity of the sensor.

The surface current density on the x, y plane is

$$\vec{J}_s = J_{s_\Psi}\vec{e}_\Psi + J_{s_\phi}\vec{e}_\phi \tag{58}$$

where

$$J_{s_\Psi} = -H_\phi\Big|_{z=o+} = 2\,\frac{E_o}{Z}\sum_{n=o}^{\infty}\,[2 - \delta_{n,o}]\,(-i)^{n-1}J_n'(k\Psi\zeta_2)\cos(n\phi)$$

(59)

$$J_{s_\phi} = H_\Psi\Big|_{z=o+} = -2\,\frac{E_o}{Z}\sum_{n=o}^{\infty}\,[2 - \delta_{n,o}]\,(-i)^{n-1}\,\frac{J_n(k\Psi\zeta_2)}{k\Psi\zeta_2}\,n\,\sin(n\phi)$$

The total current crossing $\Psi = a$ (outward) on the $z = 0$ plane is just

$$I = a\int_o^{2\pi} J_{s_\Psi}\Big|_{\Psi=a} d\phi = 4\pi a\,\frac{E_o}{Z}\,iJ_o'(ka\zeta_2)$$

$$= -i4\pi a\,\frac{E_o}{Z}\,J_1(ka\sin(\theta_1)) \tag{60}$$

As $ka \to 0$ we have

$$I = -i2\pi a\,\frac{E_o}{Z}\,ka\sin(\theta_1) + O((ka)^3)$$

$$= -2\pi a^2(\sigma + i\omega\varepsilon)E_o\sin(\theta_1) + O((ka)^3)$$

$$= -A_{eq}\left\{2(\sigma + i\omega\varepsilon)E_o\sin(\theta_1)\right\} + O((ka)^3) \tag{61}$$

where the equivalent area of the sensor is

$$A_{eq} = \pi a^2 \tag{62}$$

or as a vector

$$\vec{A}_{eq} = \pi a^2 \vec{e}_z \tag{63}$$

so that we can write as $ka \to 0$

$$I = -(\sigma + i\omega\varepsilon)\vec{E}\Big|_{\vec{r}=\vec{o}} \cdot \vec{A}_{eq} + O((ka)^3) \tag{64}$$

Thus the short circuit current from the sensor is proportional to the total current density (equation 57) at low frequencies. For the case that $\sigma = 0$, which we use for the graphs, the total current density is just the displacement current density.

Note that the sensor slot is for $\Psi_1 < \Psi < \Psi_2$ and a is the geometric mean of $\Psi_1$ and $\Psi_2$. The above results for the short circuit current then assume that the slot is narrow compared to a. This implies that the distance b which characterizes the slot in equations 2 through 4 is restricted by $b \ll a$.

For convenience we define a short circuit transfer function as

$$T(\theta_1') \equiv [-2A_{eq}(\sigma + i\omega\varepsilon)E_o \sin(\theta_1)]^{-1} I$$

$$= \frac{2}{ka \sin(\theta_1)} J_1(ka \sin(\theta_1)) = \frac{2}{ka \sin(\theta_1')} J_1(ka \sin(\theta_1')) \tag{65}$$

so that as $ka \to 0$ we have $T \to 1$. The deviation of T from 1 as ka increases then shows for various $\theta_1$ the departure of the short circuit current from its ideal dependence.

We also define a special short circuit transfer function as

$$T_1 \equiv T(\pi/2) = \frac{2}{ka} J_1(ka) \tag{66}$$

This transfer function applies to the case of a plane wave propagating parallel to the x, y plane and is a special case of interest. It is used for some of the later response function calculations when we want to consider only one value of $\theta_1'$.

The short circuit current transfer function is plotted in figure 4 as a function of $ka \sin(\theta_1)$ with $\sigma = 0$. For convenience the phase is plotted as $\arg(T) - ka \sin(\theta_1')$ which corresponds to multiplying T by $e^{-ika\sin(\theta_1')}$. The reason for this is that in the time domain the wave reaches the sensor slot before it reaches the reference point for the fields, i.e. the center of the sensor or $\vec{r} = \vec{0}$. This phase shift references the phase to the first arrival of the fields at some position around the sensor gap. Note that the magnitude of T starts rolling off for $ka\sin(\theta_1') > 1$ and has zeros corresponding to the positive zeros of $J_1$ (not including $ka\sin(\theta_1') = 0$); the first such zero occurring at $ka\sin(\theta_1') \simeq 3.83$. Also note that the phase jumps by $\pi$ at each of the zeros of T. As a convention we add $\pi$ to the phase on going through a zero in the direction of increasing $ka\sin(\theta_1')$. This makes the phase oscillate around some finite negative value for large $ka\sin(\theta_1')$. We keep this definition of phase for cases which later use T or $T_1$ as part of various response functions for the sensor.

## V. Admittances

Now consider the admittances when the sensor is driven at the gap, the annular slot in the x, y plane described by $\Psi_1 < \Psi < \Psi_2$. As shown in figure 5 there is a voltage $V_{gap}$ uniformly distributed around the gap. Associated with $V_{gap}$ there are 3 surface current densities parallel to $\vec{e}_\psi$. These are $J_{s_c}$, $J_{s_u}$, and $J_{s_\ell}$ and are associated respectively with currents into cables or other transmission lines loading the gap, with fields above the conducting plane (z > 0), and fields below the conducting plane (z < 0). Taking the conventions for these surface current densities as indicated in figure 5 we define three admittances as

$$Y_c \equiv 2\pi a \frac{J_c}{V_{gap}} , \quad Y_u \equiv 2\pi a \frac{J_{s_u}}{V_{gap}} , \quad Y_\ell \equiv 2\pi a \frac{J_{s_\ell}}{V_{gap}} \tag{67}$$

Note that for our case of interest $Y_u = Y_\ell$ because of the identical geometries above and below the x, y plane. Thus we define a single normalized admittance as

$$y_a \equiv ZY_u = ZY_\ell \tag{68}$$

The total normalized admittance associated with the fields both above and below the x, y plane is just $2y_a$. If in some other problem the geometry of the region below the ground plane were changed, then another admittance besides $y_a$ would be needed. Also define a normalized cable conductance as

$$y_c \equiv \frac{1}{r_c} \equiv \frac{Z}{Z_c} \tag{69}$$

where $Z_c$ is the net cable impedance (resistive) loading the gap. Since we use $\sigma = 0$ in the numerical results then $r_c > 0$ for these calculations and we can specify $r_c$ parametrically.

A. Boundary Conditions at Annular Slot

As in reference 2 and a few other previous notes we find the sensor admittance by specifying a quasi-static electric field distribution in the gap. Again the gap width $\Psi_2 - \Psi_1$ is assumed small compared to the characteristic sensor dimension a. This field distribution is written as

$$E_\Psi \Big|_{z=o} = \frac{1}{b} V_{gap} f_E \tag{70}$$

where $f_E$ is a normalized distribution function subject to the condition

$$\frac{1}{b} \int_o^\infty f_E d\Psi = \frac{1}{b} \int_{\Psi_1}^{\Psi_2} f_E d\Psi = 1 \tag{71}$$

Actually $f_E$ is non zero only in the range $\Psi_1 < \Psi < \Psi_2$. For later use we introduce the normalized cylindrical radius

$$\nu \equiv \frac{\Psi}{a} , \qquad d\nu = \frac{1}{a} d\Psi \tag{72}$$

and another variable $\xi$ for use in defining positions in the sensor gap as

$$\xi \equiv \frac{a}{b} \ln(\nu) = \frac{a}{b} \ln\left(\frac{\Psi}{a}\right) , \qquad \nu = e^{\frac{b}{a}\xi}$$

$$\frac{d\nu}{\nu} = \frac{b}{a} d\xi , \qquad d\nu = \frac{b}{a} e^{\frac{b}{a}\xi} d\xi \tag{73}$$

Back in equations 2 we defined the annular slot in terms of $\Psi$. In terms of $\nu$ we define

$$\nu_1 \equiv \frac{\Psi_1}{a} = e^{-\frac{b}{a}}, \qquad \nu_2 \equiv \frac{\Psi_2}{a} = e^{\frac{b}{a}} \tag{74}$$

Note now that in terms of $\nu$ the slot is defined by $\nu_1 < \nu < \nu_2$; in terms of $\xi$ it is defined by $-1 < \xi < 1$. In terms of these new variables the normalization condition on the electric field distribution function becomes

$$\frac{a}{b} \int_{\nu_1}^{\nu_2} f_E d\nu = 1$$

$$\int_{-1}^{1} f_E e^{\frac{b}{a}\xi} \, d\xi = 1 \tag{75}$$

For this note we choose $f_E$ as

$$f_E \equiv \begin{cases} \frac{1}{\pi}[1 - \xi^2]^{-1/2} \; e^{-\frac{b}{a}\xi} & \text{for } |\xi| < 1 \\ 0 & \text{for } |\xi| > 1 \end{cases} \tag{76}$$

Substituting this definition in the second of equations 75 one finds that the normalization condition is met. In terms of $\nu$ the distribution function is

$$f_E = \begin{cases} \frac{1}{\pi}\left\{1 - \left[\frac{a}{b} \ln(\nu)\right]^2\right\}^{-1/2} \frac{1}{\nu} & \text{for } \nu_1 < \nu < \nu_2 \\ 0 & \text{for } \nu < \nu_1 \text{ and for } \nu > \nu_2 \end{cases} \tag{77}$$

Now as $b/a \to 0$ we have

$$\nu_1 = 1 + \frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right), \qquad \nu_2 = 1 - \frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right) \tag{78}$$

and for $\nu_1 < \nu < \nu_2$ so that $\nu - 1 = O(b/a)$ we have

$$\begin{aligned} f_E &= \frac{1}{\pi}\left\{1 - \left[\frac{a}{b}(\nu - 1) + O\left(\frac{b}{a}\right)\right]^2\right\}^{-1/2} \frac{1}{\nu} \\ &= \frac{1}{\pi}\left\{1 - \left[\frac{a}{b}(\nu - 1)\right]^2 + O\left(\frac{b}{a}\right)\right\}^{-1/2} \frac{1}{\nu} \end{aligned} \tag{79}$$

This field distribution has the proper form of singularity at the edges of the annular slot, assuming perfectly conducting edges of zero thickness. The factor of $1/\nu$ makes the field decrease slightly going from the inside to the outside edge of the slot which is not inappropriate for such a geometry. While there is a simpler form for $f_E$ in terms of $\nu$ which still meets these conditions (by dropping the $1/\nu$ and $O(b/a)$ terms in equation 79), we choose the form in equations 76 and 77 because it is helpful in the solution of a certain integral over $f_E$ to be encountered later. In addition this form goes to the simpler form as $b/a \to 0$, as do the slot edges in equations 78.

### B. Hankel Transforms

Later in this section we use Hankel transforms. For later reference we have the Hankel transform pair as[9]

$$\begin{aligned} F(u_2) &= \int_0^\infty f(u_1)(u_1u_2)^{1/2} J_\eta(u_1u_2)\,du_1 \\ f(u_1) &= \int_0^\infty F(u_2)(u_1u_2)^{1/2} J_\eta(u_1u_2)\,du_2 \end{aligned} \tag{80}$$

where $\eta$ is the order of the Hankel transform. Substituting the first of equations 80 into the second gives the identity

$$f(u_1) = \int_0^\infty \int_0^\infty f(u_3)(u_2u_3)^{1/2}(u_1u_2)^{1/2} J_\eta(u_2u_3) J_\eta(u_1u_2)\,du_3\,du_2 \tag{81}$$

9. Magnus, Oberhettinger, and Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, 3rd ed., Springer-Verlag, New York, 1966, p. 397.

Define

$$g(u_1) = u_1^{-1/2} f(u_1) \ , \qquad G(u_2) = u_2^{-1/2} F(u_2) \tag{82}$$

giving another form to equations 80 as

$$G(u_2) = \int_0^\infty g(u_1) u_1 J_\eta(u_1 u_2) du_1$$

$$g(u_1) = \int_0^\infty G(u_2) u_2 J_\eta(u_1 u_2) du_2 \tag{83}$$

and equation 81 takes the form

$$g(u_1) = \int_0^\infty \int_0^\infty g(u_3) u_2 u_3 J_\eta(u_2 u_3) J_\eta(u_1 u_2) du_3 du_2 \tag{84}$$

These Hankel transform relations have an important application in that suitably well behaved functions, say $f(u_1)$ or $g(u_1)$, can be represented as integrals of other functions (their transforms) times Bessel functions of the first kind with the integration limits 0 and $\infty$. Knowing that functions can be so represented will allow us to find an explicit expression for $y_a$ in terms of $f_E$ using Hankel transforms.

C. Formulation of the Integral Expression for the Admittance Associated with One Side of the Sensor

In solving for the sensor admittance we drive the slot uniformly in $\phi$ with only a $\Psi$ component of the electric field. By symmetry the only non zero field components are $E_\psi$, $E_z$, $H_\phi$ and they are all independent of $\phi$. We only consider the fields above the z = 0 plane in calculating $y_a$. The components of the vector wave functions of interest are

$$N_\psi^{(1)}(0,\zeta_1,e) = -i\zeta_1\zeta_2 J_0'(k\Psi\zeta_2) e^{-ikz\zeta_1}$$

$$= i\zeta_1\zeta_2 J_1(k\Psi\zeta_2) e^{-ikz\zeta_1}$$

$$N_\phi^{(1)}(0,\zeta_1,e) = 0$$

$$N_z^{(1)}(0,\zeta_1,e) = \zeta_2^2 J_o(k\Psi\zeta_2)e^{-ikz\zeta_1} \tag{85}$$

and

$$M_\Psi^{(1)}(0,\zeta_1,e) = 0$$

$$M_\phi^{(1)}(0,\zeta_1,e) = -\zeta_2 J_o'(k\Psi\zeta_2)e^{-ikz\zeta_1}$$

$$= \zeta_2 J_1(k\Psi\zeta_2)e^{-ikz\zeta_1} \tag{86}$$

$$M_z^{(1)}(0,\zeta_1,e) = 0$$

Note that $\zeta_1$ and $\zeta_2$ have definitions for these admittance calculations which are not the same as for the short circuit current and plane wave calculations in previous sections.

For $z > 0$ the fields are expanded (as in equations 28 and 30) in the form

$$\vec{E} = E_1\int_o^\infty A_1(\zeta_2)\vec{N}^{(1)}(0,\zeta_1,e)d\zeta_2$$

$$\vec{H} = i\,\frac{E_1}{Z}\int_o^\infty A_1(\zeta_2)\vec{M}^{(1)}(0,\zeta_1,e)d\zeta_2 \tag{87}$$

Then $E_\Psi$ on the $z = 0$ plane is given by

$$E_\Psi\Big|_{z=o} = iE_1\int_o^\infty A_1(\zeta_2)\zeta_1\zeta_2 J_1(k\Psi\zeta_2)d\zeta_2 \tag{88}$$

which is to match the $E_\psi$ on the z = 0 plane as specified by the field distribution discussed in section V A. Looking at equation 88 and comparing it to the Hankel transform relations in section V B, note that $E_\psi$ on z = 0 is set up as the Hankel transform (of order one) of $A_1(\zeta_2)$ (with a few other terms) where $E_\psi$ is considered as a function of $k\Psi$. We then know that we can have such an $A_1(\zeta_2)$ because it is given through the Hankel transform of $E_\psi$. This also shows why the limits of zero and ∞ are chosen on the integrals in equations 87. $E_1$ is just a convenient constant with units volts/meter.

Note that $\zeta_1$ and $\zeta_2$ are related by equation 12 and since we are interested in a range for $\zeta_2$ of $0 < \zeta_2 < \infty$ then we have to define which branch of the square root to use for $\zeta_1$. We define this as

$$\zeta_1 \equiv \begin{cases} \sqrt{1-\zeta_2^2} & \text{for } 0 \leq \zeta_2 \leq 1 \\ -i\sqrt{\zeta_2^2 - 1} & \text{for } 1 \leq \zeta_2 \end{cases} \tag{89}$$

where the square roots here have their standard positive definition. This choice assures that for all $\zeta_2$ being considered the vector wave functions for z > 0 represent either outward propagating or attenuating waves. Note that we are only considering $\zeta_2$ on the positive real axis in equation 89 but it can be considered a more general complex number provided the definition of $\zeta_1$ is made in a manner consistent with this one.

The surface current density on the z = 0 plane (associated with the fields for z > 0) has only a $\Psi$ component given by

$$J_{s_\Psi} = -H_\phi\Big|_{z=0+} = -i\,\frac{E_1}{Z}\int_0^\infty A_1(\zeta_2)\zeta_2 J_1(k\Psi\zeta_2)d\zeta_2 \tag{90}$$

With this result we find the surface current density driven from the slot by setting $\Psi = a$ and using the convention in figure 5 to give

$$J_{s_u} = -J_{s_\Psi}\Big|_{\Psi=a} = i\,\frac{E_1}{Z}\int_0^\infty A_1(\zeta_2)\zeta_2 J_1(ka\zeta_2)d\zeta_2 \tag{91}$$

We evaluate this surface current density at $\Psi = a$ for convenience under the assumption of a narrow slot (b << a) so that for wavelengths large compared to b the surface current density as in equation 90 is approximately uniform across the gap. The normalized admittance associated with the upper half space is then

$$y_a = 2\pi a Z \frac{J_{s_u}}{V_{gap}} = i2\pi a \frac{E_1}{V_{gap}} \int_0^\infty A_1(\zeta_2)\zeta_2 J_1(ka\zeta_2)d\zeta_2 \qquad (92)$$

Now we need an expression for $A_1(\zeta_2)$. Equating the expressions for the tangential electric field on the z = 0 plane from equations 70 and 88 gives

$$f_E = ib \frac{E_1}{V_{gap}} \int_0^\infty A_1(\zeta_2)\zeta_1\zeta_2 J_1(k\Psi\zeta_2)d\zeta_2 \qquad (93)$$

This is a Hankel transform relation for $A_1(\zeta_2)$ in terms of $f_E$. For convenience define

$$A_2(\zeta_2) \equiv ib \frac{E_1}{V_{gap}} A_1(\zeta_2)\zeta_1 \qquad (94)$$

so that equations 93 and 92 become

$$f_E = \int_0^\infty A_2(\zeta_2)\zeta_2 J_1(k\Psi\zeta_2)d\zeta_2 \qquad (95)$$

$$y_a = 2\pi \frac{a}{b} \int_0^\infty A_2(\zeta_2) \frac{\zeta_2}{\zeta_1} J_1(ka\zeta_2)d\zeta_2 \qquad (96)$$

This equation pair contains our solution.

Now take equation 95, change $\zeta_2$ to $\zeta_2'$, multiply both sides by $k\Psi J_1(k\Psi\zeta_2)$, and integrate over $k\Psi$ from 0 to $\infty$ to give

$$\int_0^\infty f_E k\Psi J_1(k\Psi\zeta_2)d(k\Psi) = \int_0^\infty \int_0^\infty A_2(\zeta_2')k\Psi\zeta_2' J_1(k\Psi\zeta_2')J_1(\zeta_2 k\Psi)d\zeta_2' d(k\Psi)$$

$$= A_2(\zeta_2) \qquad (97)$$

where we have used the Hankel transform result of equation 84. Now change from $k\Psi$ as an integration variable to $\nu(= \Psi/a)$ from equation 72 giving

$$A_2(\zeta_2) = (ka)^2 \int_o^\infty f_E \nu J_1(ka\nu\zeta_2) d\nu \tag{98}$$

This is an explicit solution for $A_2(\zeta_2)$ in terms of an integral over $f_E$ which is given as a function of $\nu$ in equation 77. Substituting this result for $A_2$ into equation 96 gives an explicit result for the normalized admittance as

$$y_a = 2\pi \frac{a}{b} (ka)^2 \int_o^\infty \int_o^\infty f_E \frac{\zeta_2}{\zeta_1} \nu J_1(ka\nu\zeta_2) J_1(ka\zeta_2) d\zeta_2 d\nu \tag{99}$$

For convenience we write this last result as

$$y_a = 2\pi \frac{a}{b} (ka)^2 \int_o^\infty f_E \nu \Gamma_1 d\nu \tag{100}$$

where

$$\Gamma_1 \equiv \int_o^\infty \frac{\zeta_2}{\zeta_1} J_1(ka\nu\zeta_2) J_1(ka\zeta_2) d\zeta_2 \tag{101}$$

We go on to consider some manipulations of $\Gamma_1$.

### D. Manipulation of $\Gamma_1$

This integral $\Gamma_1$ is considered in reference 3 and we follow a similar derivation to manipulate it into a more convenient form. Starting with a form of the well known addition theorem for cylindrical Bessel functions (ref. 8 eqns. 9.1.79 and 9.1.5) we have as a special case

$$J_o(p\zeta_2) = \sum_{n=-\infty}^{\infty} J_n(ka\nu\zeta_2) J_n(ka\zeta_2) \cos(n\beta)$$

$$= \sum_{n=o}^{\infty} [2 - \delta_{n,o}] J_n(ka\nu\zeta_2) J_n(ka\zeta_2) \cos(n\beta) \tag{102}$$

where

$$p \equiv ka\ q \tag{103}$$

and

$$q \equiv [1 + \nu^2 - 2\nu \cos(\beta)]^{1/2} \tag{104}$$

The square root is defined as positive for $\nu$ real. Multiply both sides of equation 102 by $\cos(\beta)$ and integrate over $0 \le \beta \le 2\pi$ giving

$$J_1(ka\nu\zeta_2)J_1(ka\zeta_2) = \frac{1}{2\pi}\int_0^{2\pi} J_o(p\zeta_2)\ \cos(\beta)d\beta \tag{105}$$

As a special case of a result of Watson[10] we have another integral as

$$\int_0^{\infty} J_o(p\zeta_2)\ \frac{\zeta_2}{\zeta_1}\, d\zeta_2 = i\ \frac{e^{-ip}}{p} \tag{106}$$

where the definition of $\zeta_1$ is as in equation 89. Then we can evaluate $\Gamma_1$ using equations 105 and 106 as

$$\Gamma_1 = \int_0^{\infty} \frac{\zeta_2}{\zeta_1}\, J_1(ka\nu\zeta_2)J_1(ka\zeta_2)d\zeta_2$$

$$= \frac{1}{2\pi}\int_0^{\infty} \frac{\zeta_2}{\zeta_1}\int_0^{2\pi} J_o(p\zeta_2)\ \cos(\beta)d\beta d\zeta_2$$

$$= \frac{i}{2\pi}\int_0^{2\pi} \frac{e^{-ip}}{p}\cos(\beta)d\beta$$

10. G. N. Watson, A Treatise on the Theory of Bessel Functions, 2nd ed., Cambridge, 1966, p. 416, eqn. 13.47(4).

$$= \frac{i}{\pi} \int_0^{\pi} \frac{e^{-ip}}{p} \cos(\beta) d\beta \tag{107}$$

This result can also be derived from an expansion of $(1/p)e^{-ip}$ found in another reference.[11]

There is another result for $\Gamma_1$ in which $\Gamma_1$ is expanded in an infinite series of spherical Bessel functions. However this expansion is not directly used for our present results and it is presented in appendix A for completeness.

For convenience we define an integral related to $\Gamma_1$ as

$$\Gamma_2 \equiv -i2\pi ka\Gamma_1 = ka\int_0^{2\pi} \frac{e^{-ip}}{p} \cos(\beta) d\beta$$

$$= \int_0^{2\pi} \frac{e^{-ikaq}}{q} \cos(\beta) d\beta = 2\int_0^{\pi} \frac{e^{-ikaq}}{q} \cos(\beta) d\beta \tag{108}$$

where q is defined in equation 104. The normalized admittance now has the form

$$y_a = i\,\frac{a}{b}\,ka\int_0^{\infty} f_E \nu \Gamma_2 d\nu \tag{109}$$

Next we go on to consider $\Gamma_2$.

E. Manipulation of $\Gamma_2$

Now expand the exponential in equation 108 in a power series (an absolutely convergent one) as

$$e^{-ikaq} = \sum_{n=0}^{\infty} \frac{(-ika)^n}{n!} q^n \tag{110}$$

Substituting this series in equation 108 and interchanging summation and integration gives

11. Reference 9, p. 487.

$$\Gamma_2 = \sum_{n=o}^{\infty} \frac{(-ika)^n}{n!} \Lambda_n \tag{111}$$

where

$$\Lambda_n \equiv \int_o^{2\pi} q^{n-1} \cos(\beta) d\beta$$

$$= \int_o^{2\pi} [1 + \nu^2 - 2\nu \cos(\beta)]^{\frac{n-1}{2}} \cos(\beta) d\beta \tag{112}$$

For convenience define

$$\eta \equiv \frac{1 - n}{2}$$

$$Q \equiv q^2 = 1 + \nu^2 - 2\nu \cos(\beta) \tag{113}$$

giving another form for $\Lambda_n$ as

$$\Lambda_n = \int_o^{2\pi} Q^{-\eta} \cos(\beta) d\beta \tag{114}$$

Now expand $Q^{-\eta}$ for $\eta \neq 0, -1, -2, \cdots$ in the form[12,13,14]

$$Q^{-\eta} = \begin{cases} \sum_{\lambda=o}^{\infty} C_\lambda^\eta(\cos(\beta))\nu^\lambda & \text{for } |\nu| < 1 \\ \nu^{-2\eta} \sum_{\lambda=o}^{\infty} C_\lambda^\eta(\cos(\beta))\nu^{-\lambda} & \text{for } |\nu| > 1 \end{cases} \tag{115}$$

12. Reference 9, pp. 218-224.

13. A. Erdelyi, ed., Higher Transcendental Functions, vol. 2, McGraw Hill, 1953, pp. 174-178.

14. A. Erdelyi, ed., Higher Transcendental Functions, vol. 1, McGraw Hill, 1953, pp. 175-177.

where the functions $C_\lambda^\eta$ are called Gegenbauer or ultraspherical polynomials. For $\eta = 0$ we have simply

$$Q_o = 1 \tag{116}$$

For $\eta = 0$ the ultraspherical polynomials need a special definition to make them not identically zero and this definition is inconsistent with equation 115. Note also that the range of $\lambda$ is often restricted to $\eta > -1/2$ in order to make the weight functions integrable. We allow $\eta \leq -1/2$ since we do not need the integrability of the weight functions for our present purposes. We also have the useful representation of the ultraspherical polynomials for $\eta \neq 0$ as

$$C_\lambda^\eta\ (\cos(\beta)) = \sum_{\tau=o}^{\lambda} \frac{(\eta)_\tau (\eta)_{\lambda-\tau}}{\tau!\,(\lambda - \tau)!} \cos((\lambda - 2\tau)\beta) \tag{117}$$

where the Pochhammer symbol is given by

$$(\eta)_o = 1$$

$$(\eta)_\tau = \eta(\eta + 1)(\eta + 2) \cdots (\eta + \tau - 1) \quad \text{for } \tau = 1, 2, \cdots \tag{118}$$

$$= \frac{\Gamma(\eta + \tau)}{\Gamma(\eta)}$$

Note that $\eta$ can even be a negative integer if the $\Gamma$ functions are not used. The expansion in equation 117 is derived in reference 14 using the expansion in equation 115 as the definition of the ultraspherical polynomials. Provided the Pochhammer symbols are used in the derivation instead of $\Gamma$ functionsthen $\eta$ can be a negative integer and the derivation still applies.

Now $\Lambda_n$ for $n \neq 1$ (or $\eta \neq 0$) can be rewritten as

$$\Lambda_n = \begin{cases} \displaystyle\sum_{\lambda=o}^{\infty} \nu^\lambda \Xi_{n,\lambda} & \text{for } |\nu| < 1 \\ \displaystyle\sum_{\lambda=o}^{\infty} \nu^{-\lambda+n-1}\ \Xi_{n,\lambda} & \text{for } |\nu| > 1 \end{cases} \tag{119}$$

where

$$\Xi_{n,\lambda} = \int_0^{2\pi} C_\lambda^\eta(\cos(\beta))\ \cos(\beta)d\beta \tag{120}$$

while for n = 1 (or $\eta \neq 0$) we have

$$\Lambda_1 = 0 \tag{121}$$

Substituting the expansion of $C_\lambda^\eta$ as in equation 117 into the integral in equation 120 the only contribution comes from the terms for which $\lambda - 2\tau = \pm 1$ due to the orthogonality of the trigonometric functions over the interval $0 \leq \beta \leq 2\pi$. Thus we have the result

$$\Xi_{n,\lambda} = \begin{cases} \pi \dfrac{(\eta)_{\frac{\lambda+1}{2}}(\eta)_{\frac{\lambda-1}{2}}}{\left(\frac{\lambda+1}{2}\right)!\left(\frac{\lambda-1}{2}\right)!} + \pi \dfrac{(\eta)_{\frac{\lambda-1}{2}}(\eta)_{\frac{\lambda+1}{2}}}{\left(\frac{\lambda-1}{2}\right)!\left(\frac{\lambda+1}{2}\right)!} & \text{for } \lambda \text{ odd} \\ 0 \quad \text{for } \lambda \text{ even} & \end{cases} \tag{122}$$

Since only odd $\lambda$ contribute let

$$\lambda \equiv 2\ell + 1 \tag{123}$$

and define a new coefficient for integer $\ell$ as

$$B_{n,\ell} \equiv \frac{1}{\pi}\Xi_{n,\lambda} = 2\,\frac{(\eta)_{\ell+1}(\eta)_\ell}{(\ell+1)!\,\ell!} = 2\,\frac{\left(\frac{1-n}{2}\right)_{\ell+1}\left(\frac{1-n}{2}\right)_\ell}{(\ell+1)!\,\ell!} \tag{124}$$

so that we have the result for $n \neq 1$ as

$$\Lambda_n = \begin{cases} \pi \displaystyle\sum_{\ell=0}^{\infty} B_{n,\ell}\ \nu^{2\ell+1} & \text{for } |\nu| < 1 \\ \pi \displaystyle\sum_{\ell=0}^{\infty} B_{n,\ell}\ \nu^{n-2\ell-2} & \text{for } |\nu| > 1 \end{cases} \tag{125}$$

If equations 121, 124, and 125 are substituted into equation 111 then $\Gamma_2$ will be expanded as a doubly infinite series in ka and $\nu$.

F. Expansion of $y_a$

With the last results for $\Gamma_2$ we substitute them into equation 109 to obtain

$$y_a = ika \sum_{n=0}^{\infty} \frac{(-ika)^n}{n!} \Omega_n \tag{126}$$

where we have defined

$$\Omega_n \equiv \frac{a}{b} \int_0^\infty f_E \nu \Lambda_n d\nu \tag{127}$$

and where $\Lambda_n$ is given by equations 121, 124, and 125.

Now recall the change of variable from equations 73

$$\nu = e^{\frac{b}{a}\xi}, \qquad d\nu = \frac{b}{a} e^{\frac{b}{a}\xi} d\xi \tag{128}$$

and the distribution function $f_E$ for the electric field in the slot $(-1 < \xi < 1)$ from equation 76 as

$$f_E = \begin{cases} \frac{1}{\pi} [1 - \xi^2]^{-1/2} e^{-\frac{b}{a}\xi} & \text{for } |\xi| < 1 \\ 0 \quad \text{for } |\xi| > 1 \end{cases} \tag{129}$$

Substituting these in equation 127 gives for $n \neq 1$

$$\Omega_n = \sum_{\ell=0}^{\infty} \pi B_{n,\ell} \left\{ \frac{a}{b} \int_0^1 f_E \nu^{2\ell+2} d\nu + \frac{a}{b} \int_1^\infty f_E \nu^{n-2\ell-1} d\nu \right\}$$

$$= \sum_{\ell=0}^{\infty} \pi B_{n,\ell} \left\{ \frac{1}{\pi} \int_{-1}^{0} [1-\xi^2]^{-1/2} e^{\frac{b}{a}(2\ell+2)\xi} d\xi \right.$$

$$+ \frac{1}{\pi} \int_0^1 [1-\xi^2]^{-1/2} e^{-\frac{b}{a}(2\ell+1-n)} d\xi \Big\}$$

$$= \sum_{\ell=0}^{\infty} B_{n,\ell} \Big\{ \int_0^1 [1-\xi^2]^{-1/2} e^{-\frac{b}{a}(2\ell+2)\xi} d\xi$$

$$+ \int_0^1 [1-\xi^2]^{-1/2} e^{-\frac{b}{a}(2\ell+1-n)} d\xi \Big\} \tag{130}$$

Define a function as

$$X(\alpha) \equiv \int_0^1 [1 - \xi^2]^{-1/2} e^{\alpha\xi} d\xi \tag{131}$$

This function can also be written as[15]

$$X(\alpha) = \frac{\pi}{2}[I_o(\alpha) + L_o(\alpha)] \tag{132}$$

where $I_o$ is a modified Bessel function and $L_o$ is a modified Struve function, both of zero order. The series and asymptotic expansions of $X(\alpha)$ are treated in appendix B. With this function we can write for $n \neq 1$

$$\Omega_n = \sum_{\ell=0}^{\infty} B_{n,\ell} \Big\{ X\Big(- \frac{b}{a}(2\ell + 2)\Big) + X\Big(- \frac{b}{a}(2\ell + 1 - n)\Big) \Big\} \tag{133}$$

while for $n = 1$ equation 121 gives

$$\Omega_1 = 0 \tag{134}$$

15. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, 1965, p. 322, eqn. 3.387(5).

Summarizing we have a solution for the normalized admittance associated with one side of the z = 0 plane as

$$y_a = ika \sum_{n=o}^{\infty} \frac{(-ika)^n}{n!} \Omega_n$$

$$\Omega_n = \begin{cases} \displaystyle\sum_{\ell=o}^{\infty} B_{n,\ell}\left\{X\left(-\frac{b}{a}(2\ell+2)\right) + X\left(-\frac{b}{a}(2\ell+1-n)\right)\right\} & \text{for } n \neq 1 \\ 0 & \text{for } n = 1 \end{cases} \tag{135}$$

$$B_{n,\ell} = 2\,\frac{\left(\frac{1-n}{2}\right)_{\ell+1}\left(\frac{1-n}{2}\right)_{\ell}}{(\ell+1)!\,\ell!}$$

The behavior of these series for large $\ell$ and n are treated in appendices C and D respectively.

The normalized admittance is plotted in figure 6 as a function of ka with $\sigma = 0$ for several values of b/a. Remember for small b/a the slot width is very nearly 2b. For convenience this normalized admittance is plotted in the form $y_a/ka$. As $ka \to 0$ $y_a/ka$ tends to a constant, i.e. $i\Omega_o$; the numerically determined coefficients are listed in table 1.

| $\frac{b}{a}$ | $\frac{y_a}{ika}$ |
|---|---|
| .001 | 15.36 |
| .01 | 10.76 |
| .1 | 6.16 |

Table 1. Asymptotic form of $y_{int}$ for small ka

For small ka this admittance represents a capacitance.

G. Behavior of $y_a$ for Small ka

From equation 135 note that as $ka \to 0$ we have, since these series are convergent and $\Omega_1 = 0$,

$$\frac{y_a}{ika} = \Omega_o + O((ka)^2) \qquad , \tag{136}$$

The term $\Omega_o$ is given from equations 127 through 129 as

$$\Omega_o = \frac{1}{\pi} \int_{-1}^{1} [1 - \xi^2]^{-1/2} \Lambda_o d\xi \tag{137}$$

where from equation 112

$$\Lambda_o = 2\int_0^{\pi} [1 + \nu^2 - 2\nu \cos(\beta)]^{-1/2} \cos(\beta) d\beta \tag{138}$$

Now let

$$\beta \equiv \pi - 2\beta' , \qquad \cos(\beta) = 2 \sin^2(\beta') - 1 \tag{139}$$

giving

$$\Lambda_o = 4\int_0^{\pi/2} \frac{2 \sin^2(\beta') - 1}{[(1 + \nu)^2 - 4\nu \sin^2(\beta')]^{1/2}} d\beta'$$

$$= \frac{4}{1 + \nu} \frac{2}{m} \left\{ \left[1 - \frac{m}{2}\right] K(m) - E(m) \right\} \tag{140}$$

where K and E are complete elliptic integrals and are functions of the parameter m given by

$$m \equiv 4\nu(1 + \nu)^{-2} = \left[\frac{\nu^{-1/2} + \nu^{1/2}}{2}\right]^{-2} = \left[\cosh\left(\frac{\xi}{2} \frac{b}{a}\right)\right]^{-2} \tag{141}$$

and where the complementary parameter is

$$m_1 \equiv 1 - m = \left[\frac{1 - \nu}{1 + \nu}\right]^2 = \left[\tanh\left(\frac{\xi}{2} \frac{b}{a}\right)\right]^2 \tag{142}$$

For small $m_1$ with $|\xi| < 1$ (and thus small b/a) we have[16]

16. H. B. Dwight, Table of Integrals and Other Mathematical Data, 4th ed., Macmillan, 1965, eqns. 777.3 and 774.2.

$$K(m) = \ln\left(\frac{4}{m_1^{1/2}}\right)[1 + O(m_1)]$$

$$E(m) = 1 + O(m_1) \tag{143}$$

$$m_1 = \left(\frac{\xi}{2}\,\frac{b}{a}\right)^2 + O\left(\left(\xi\,\frac{b}{a}\right)^4\right)$$

so that as $b/a \to 0$ equation 140 becomes

$$\Lambda_o = \frac{4}{1+\nu}\left\{[1+O(m_1)]\ln\left(\frac{4}{m_1^{1/2}}\right) - 2 + O(m_1)\right\}$$

$$= 2\left[1 - \frac{1}{2}\,\frac{b}{a}\xi + O\left(\left(\xi\frac{b}{a}\right)^2\right)\right]\left\{\left[1 + O\left(\left(\xi\frac{b}{a}\right)^2\right)\right]\ln\left(\frac{8}{|\xi|}\,\frac{a}{b}\right) - 2\right\}$$

$$= 2\left\{\ln\left(\frac{8}{|\xi|}\,\frac{a}{b}\right) - 2\right\} - \frac{b}{a}\xi\left\{\ln\left(\frac{8}{|\xi|}\,\frac{a}{b}\right) - 2\right\} + O\left(\left(\xi\frac{b}{a}\right)^2 \ln\left(\frac{a}{\xi b}\right)\right) \tag{144}$$

Next substitute this result into equation 137 noting that terms odd in $\xi$ give no contribution. Thus as $b/a \to 0$ we have[17]

$$\Omega_o = \frac{2}{\pi}\int_{-1}^{1}[1-\xi^2]^{-1/2}\ln\left(\frac{1}{|\xi|}\right)d\xi + \frac{2}{\pi}\left\{\ln\left(8\,\frac{a}{b}\right) - 2\right\}\int_{-1}^{1}[1-\xi^2]^{-1/2}d\xi$$

$$+ O((b/a)^2\ln(a/b))$$

$$= 2\ln(2) + 2\left\{\ln\left(8\,\frac{a}{b}\right) - 2\right\} + O((b/a)^2\ln(a/b))$$

$$= 2\left\{\ln\left(16\,\frac{a}{b}\right) - 2\right\}[1 + O((b/a)^2)] \tag{145}$$

17. Reference 16, eqn. 863.41.

This result can be compared with table 1 to give an indication of the accuracy of this first term in the asymptotic expansion for small b/a. Note that the results rely on the form of $f_E$ chosen in equation 76. In fact for b/a = .1 this result differs by less than one part in $10^3$ from the result determined by summing the series for $\Omega_o$. For small b/a this choice of $f_E$ should be quite accurate since as $b/a \to 0$ $f_E$ goes to the known solution for the case of an infinitely long straight slot in a conducting sheet of zero thickness at $\omega = 0$. For $b/a > 0$, but still small, there may be some small error in $f_E$. The presence of signal cables connecting across the slot will of course further alter the field distribution in the slot.

Now as $\omega \to 0$ the admittance associated with the space above the z = 0 plane has the asymptotic form (from equations 68 and 135)

$$Y_u = i\omega C_u + O((ka)^3) \tag{146}$$

where $\sigma = 0$ is assumed. This capacitance is just

$$C_u = \frac{ika}{i\omega Z}\,\Omega_o = \varepsilon a \Omega_o \tag{147}$$

If the lower half space is the same as the upper half space then the sensor capacitance is just $2C_u$ or $2\varepsilon a\Omega_o$.

## VI. Frequency Response Characteristics

With the short circuit current and admittances calculated in normalized forms we go on to consider some frequency response characteristics of the sensor. First define a response function including only the admittances as

$$R_Y \equiv \frac{Y_c}{Y_c + Y_u + Y_\ell} = \frac{Y_c}{Y_c + 2y_a} = [1 + 2r_c y_a]^{-1} \tag{148}$$

This is plotted as a function of ka in figures 7 and 8 for two values of b/a (.01 and .1) with $\sigma = 0$. For each graph there are several values of $r_c$. As $ka \to 0$ we have $R_Y \to 1$. As $r_c$ is decreased $R_Y$ is maintained as a flat response characteristic out to larger values of ka.

Including the short circuit current transfer function from equation 65 we have the response function

$$R(\theta_1') \equiv T(\theta_1')R_y = T(\theta_1')[1 + 2r_c y_a]^{-1} \tag{149}$$

Another convenient response function uses the short circuit current transfer function for only one value of $\theta_1'$ as in equation 66 with $\theta_1' = \pi/2$. This response function is defined as

$$R_1 \equiv T_1 R_y = T_1[1 + 2r_c y_a]^{-1} \tag{150}$$

Figures 9 and 10 have $R_1$ plotted as a function of ka for two values of b/a (.01 and .1) and several values of $r_c$.

Based on $R_1$ we define an upper frequency response as the minimum positive value of ka for which

$$|R_1| = \frac{1}{\sqrt{2}} \tag{151}$$

This value of ka is plotted as a function of $r_c$ in figure 11 for two values of b/a. Frequency response is increased by increasing b/a and by decreasing $r_c$.

Figures 12 and 13 show R plotted as a function of ka with $\sigma = 0$ for various values of $\theta_1'$ with b/a = .1 for two specific values of $r_c$. These two values are $r_c \simeq .1327$ and $r_c \simeq .2654$ which correspond to $Z_c = 50\ \Omega$ and $Z_c = 100\ \Omega$ respectively if the media both above and below the z = 0 plane are assumed to have the same constitutive parameters as free space so that the wave impedance is

$$Z = Z_o \simeq 376.7\ \Omega \tag{152}$$

Using $R_1$ so as to pick a specific $\theta_1'$ of $\pi/2$, the frequency responses for these two cases as defined by equation 151 are given by ka ≃ .33 and ka ≃ .17 for $Z_c = 50\ \Omega$ and $Z_c = 100\ \Omega$ respectively.

## VII. Summary

In this note we have developed equations and curves for the response of a flush circular plate dipole with a uniformly resistively loaded slot as a function of frequency and the angle of incidence of the incident electromagnetic wave. For low frequencies the response of this sensor is proportional to a component of the displacement current density (or total current density if the medium is conducting).

The present calculations consider the case that both upper and lower media are semi infinite half spaces. An extension to the present calculations would be to put finite boundaries on the lower medium and/or to divide the lower medium into two or more distinct media with different electrical properties.

Note that the sensor slot has been assumed small in this note to allow the use of simple quasi static approximations of the electric field in the sensor slot for wavelengths large compared to the slot width. Perhaps some future notes could consider detailed static solutions for this geometry, even for b/a not small. Such solutions could be used for accurate calculations of equivalent area and capacitance of the sensor.

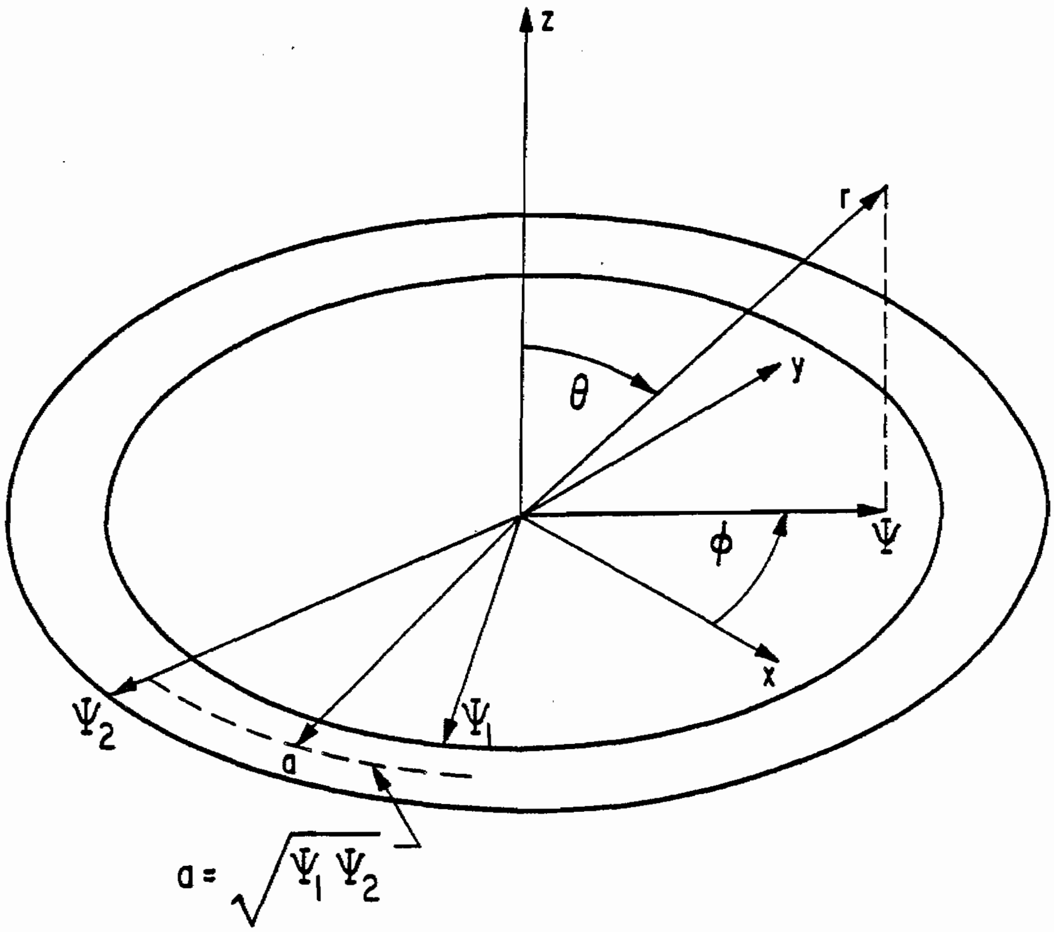
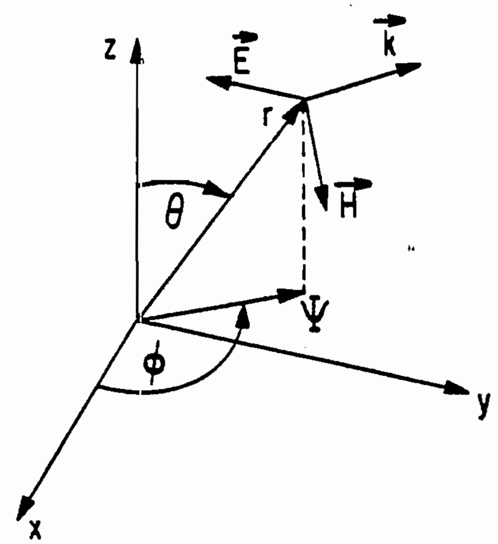
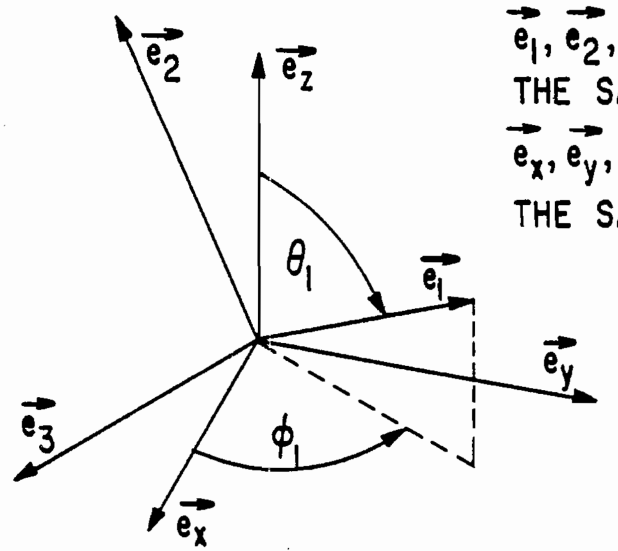



FIGURE I. CIRCULAR FLUSH-PLATE DIPOLE




A. PLANE WAVE WITH FIXED POLARIZATION




B. UNIT VECTORS FOR PLANE WAVES

FIGURE 2. VECTOR PLANE WAVES IN CYLINDRICAL COORDINATES

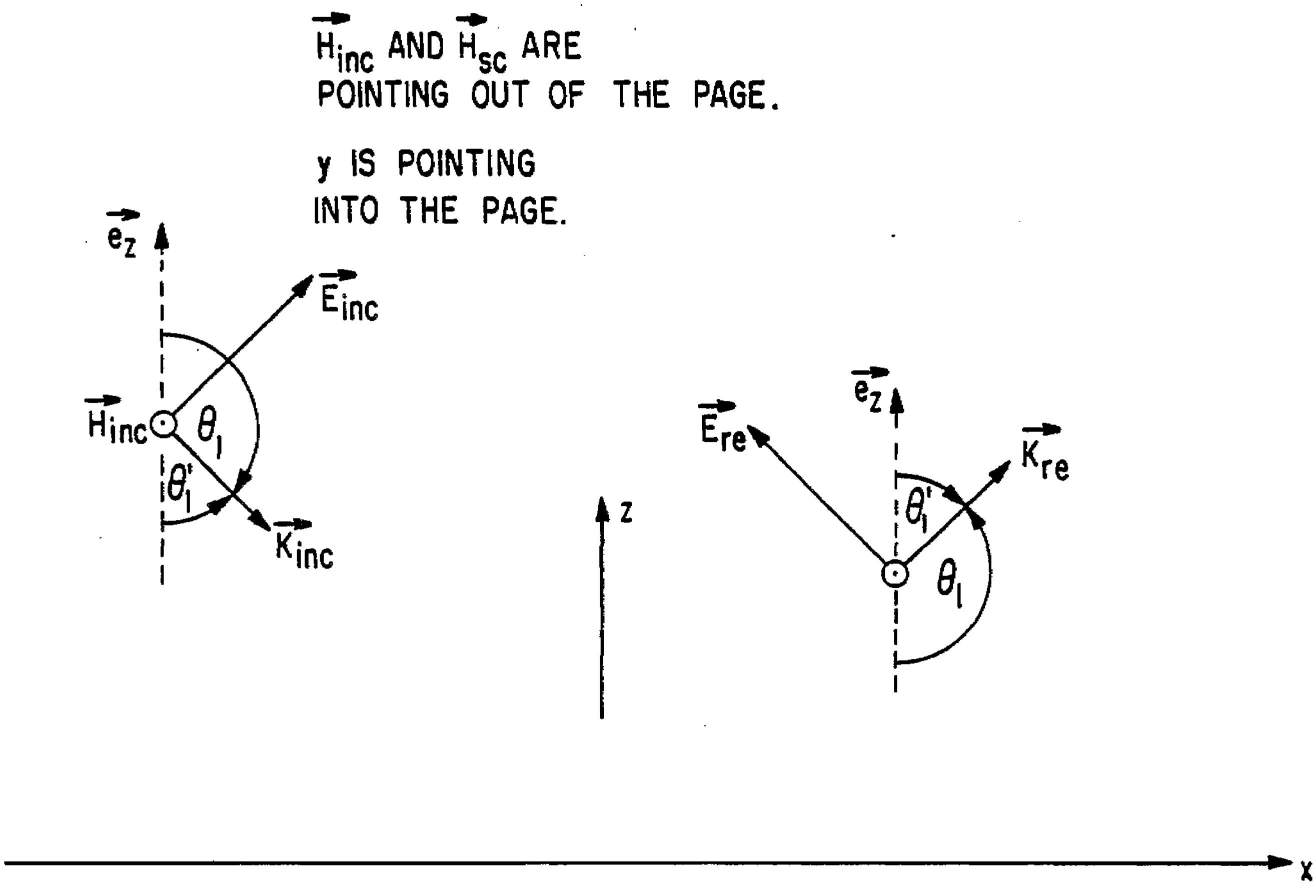



FIGURE 3. INCIDENT AND SCATTERED PLANE WAVES AT CONDUCTING PLANE: CROSS SECTION VIEW

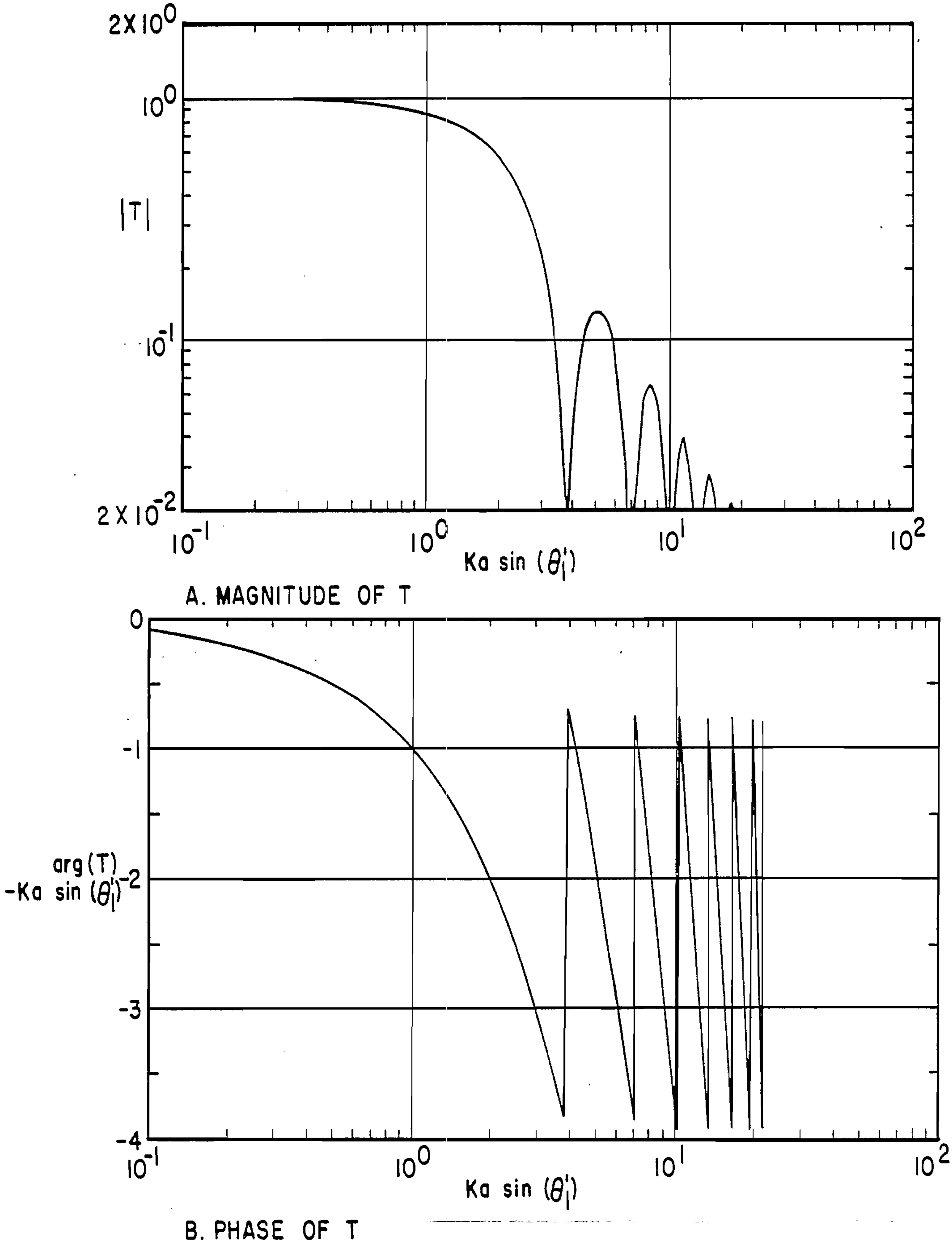



FIGURE 4. SHORT CIRCUIT CURRENT TRANSFER FUNCTION

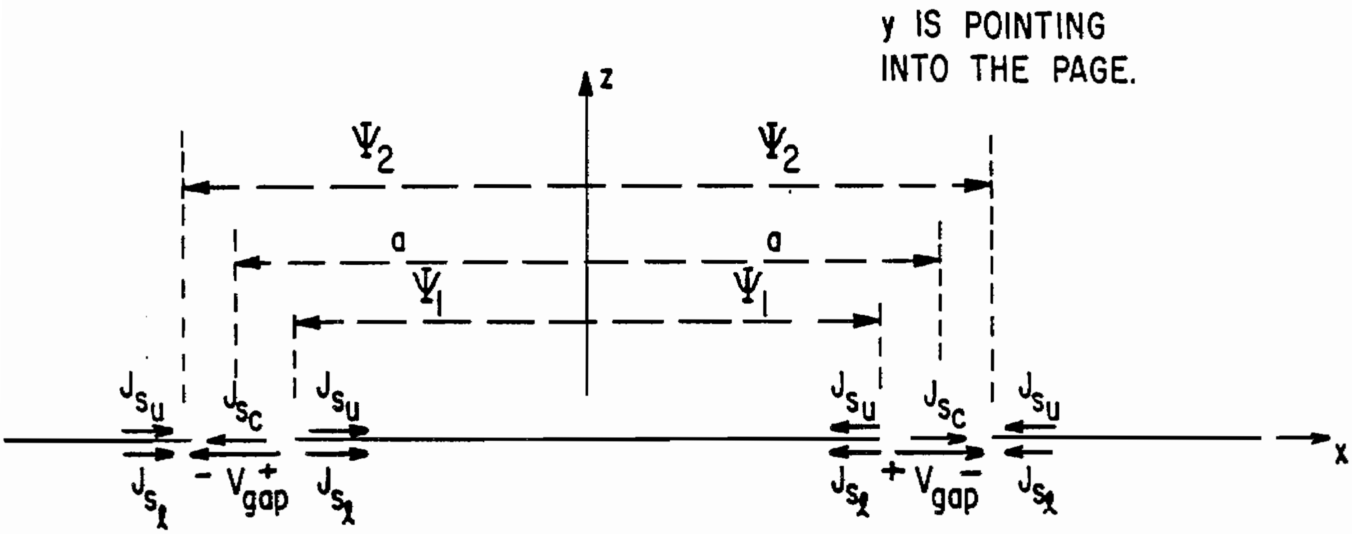



FIGURE 5. CURRENTS ON SENSOR AND CONDUCTING PLANE
FOR ADMITTANCES: CROSS SECTION VIEW

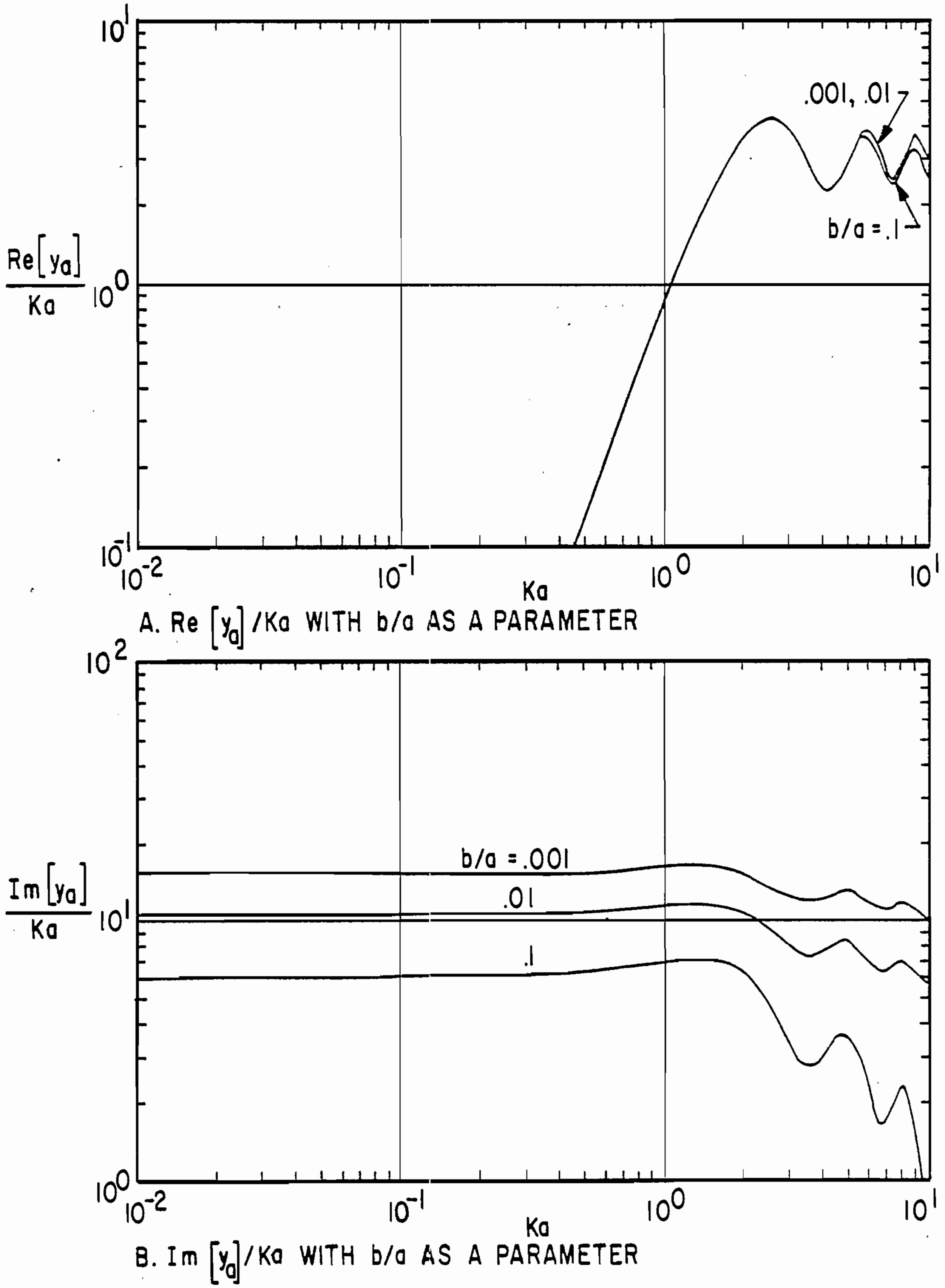
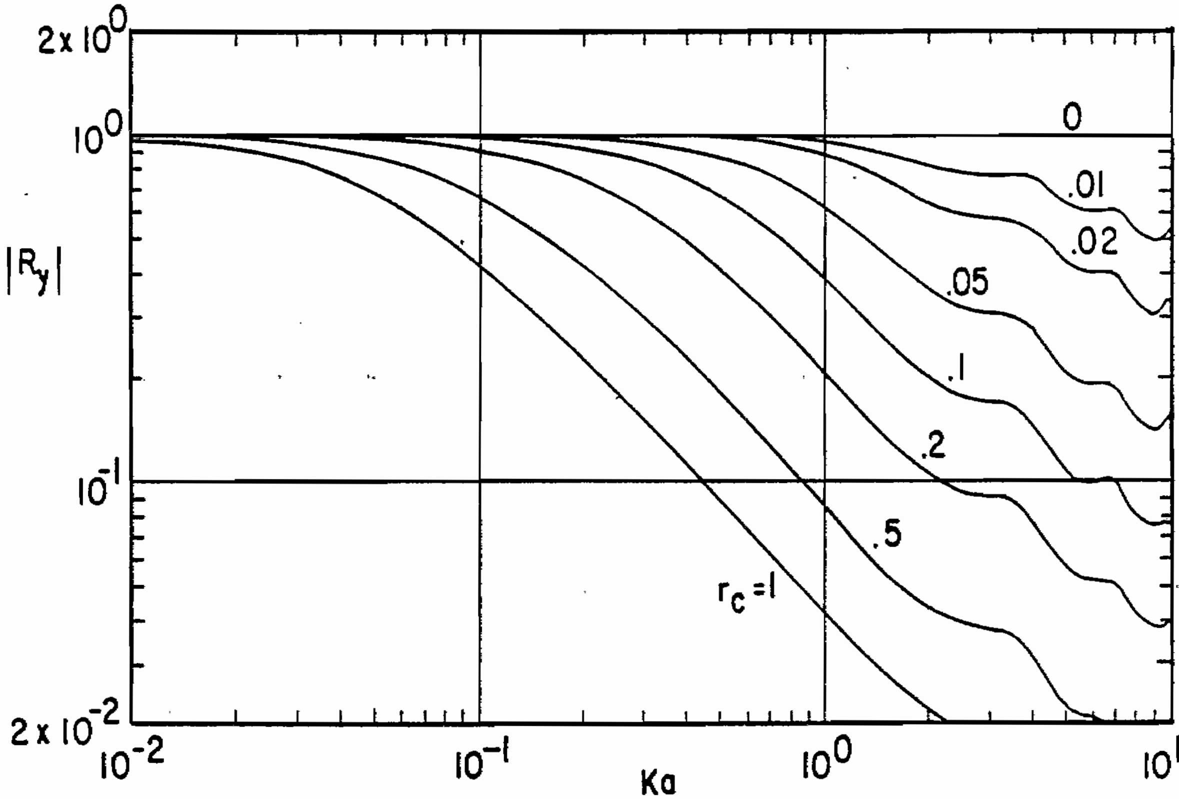
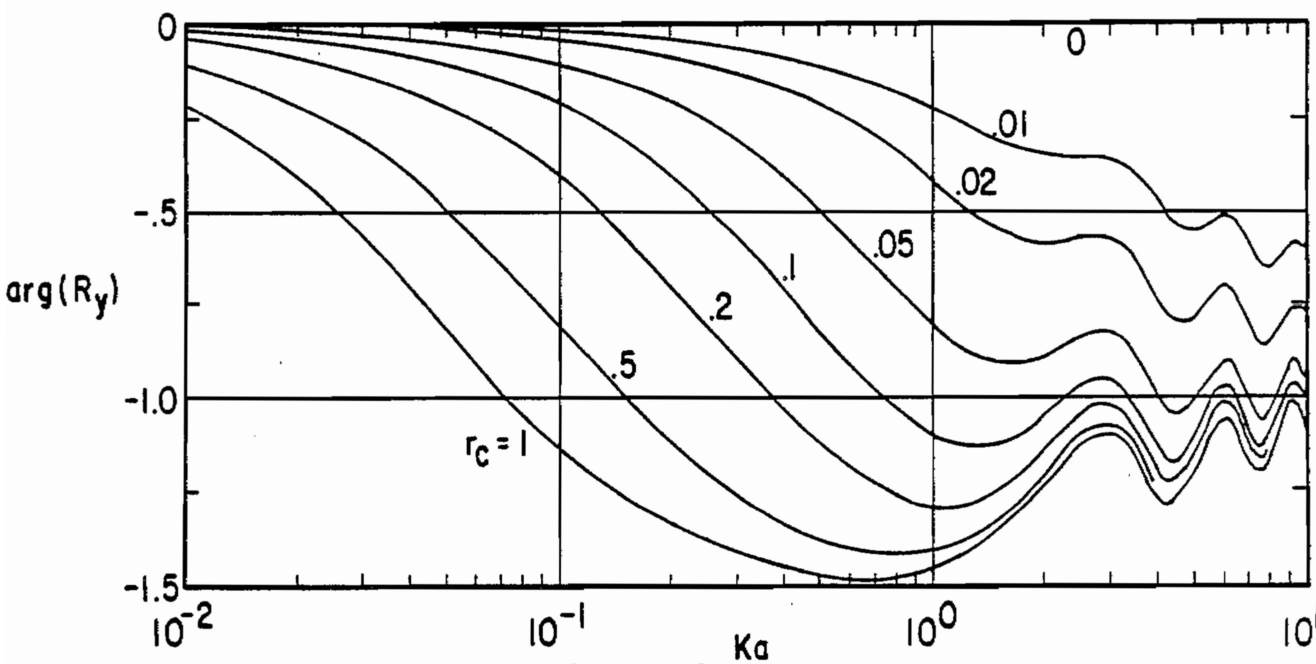



FIGURE 6. NORMALIZED ADMITTANCE FOR ONE SIDE OF SENSOR

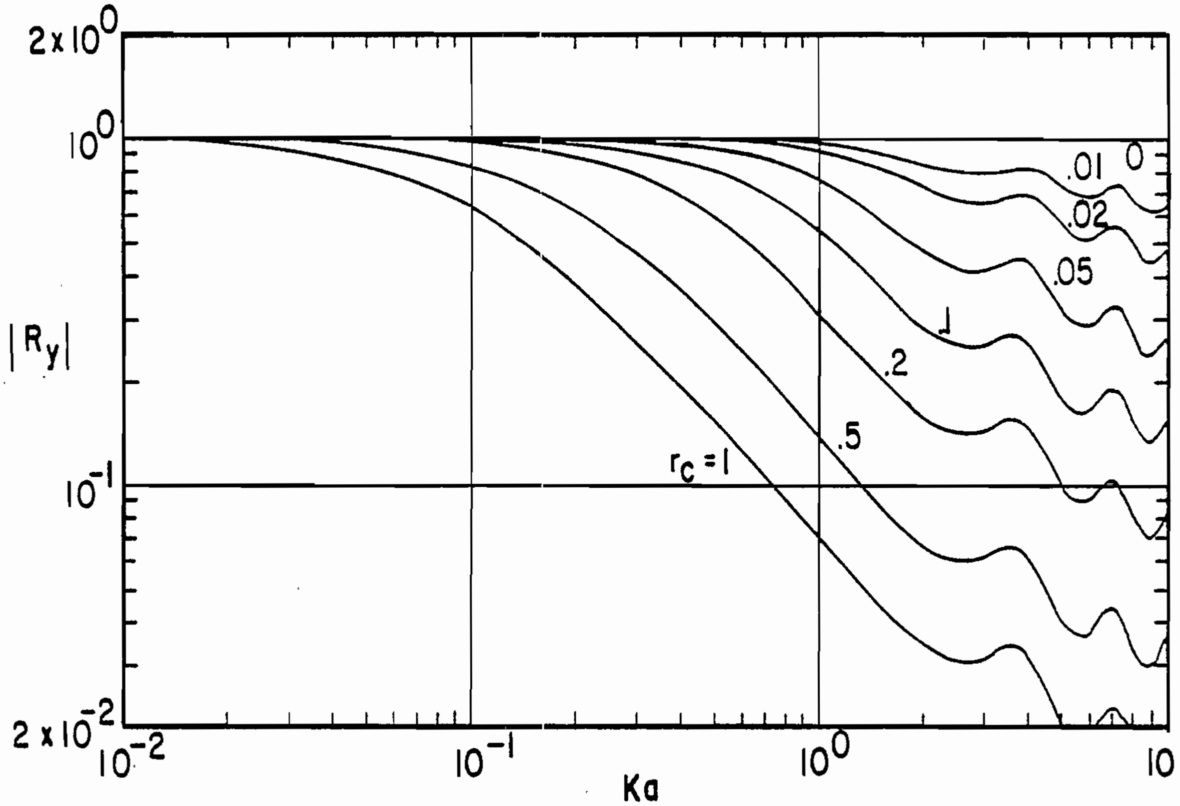



A. MAGNITUDE OF $R_y$ WITH $r_c$ AS A PARAMETER

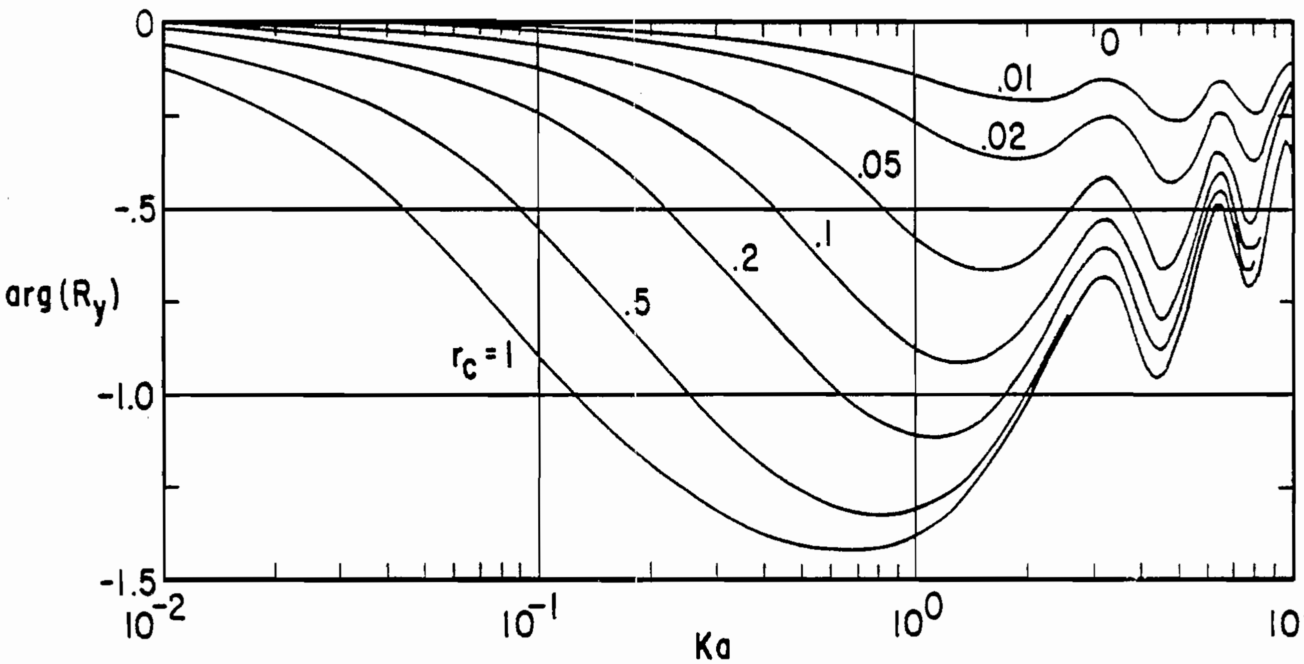



B. PHASE OF $R_y$ WITH $r_c$ AS A PARAMETER

FIGURE 7. EFFECT OF ADMITTANCES ON RESPONSE: $b/a = .01$

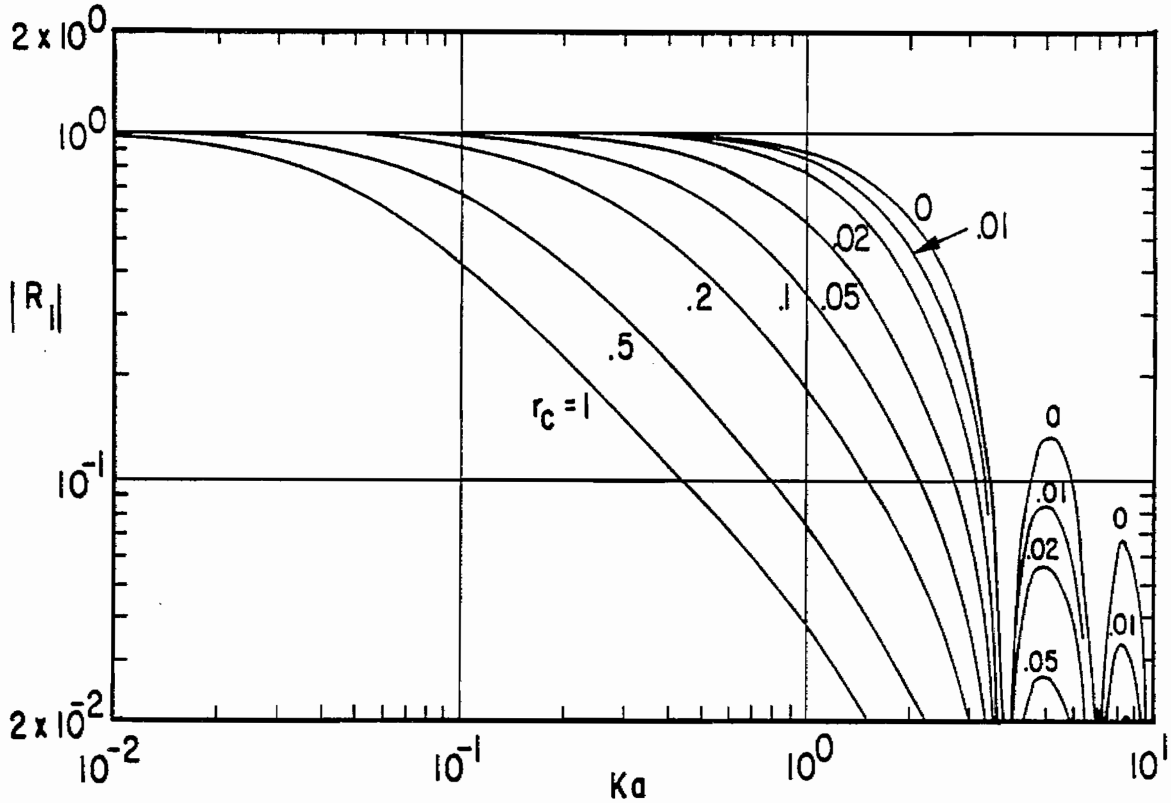



A. MAGNITUDE OF $R_y$ WITH $r_c$ AS A PARAMETER

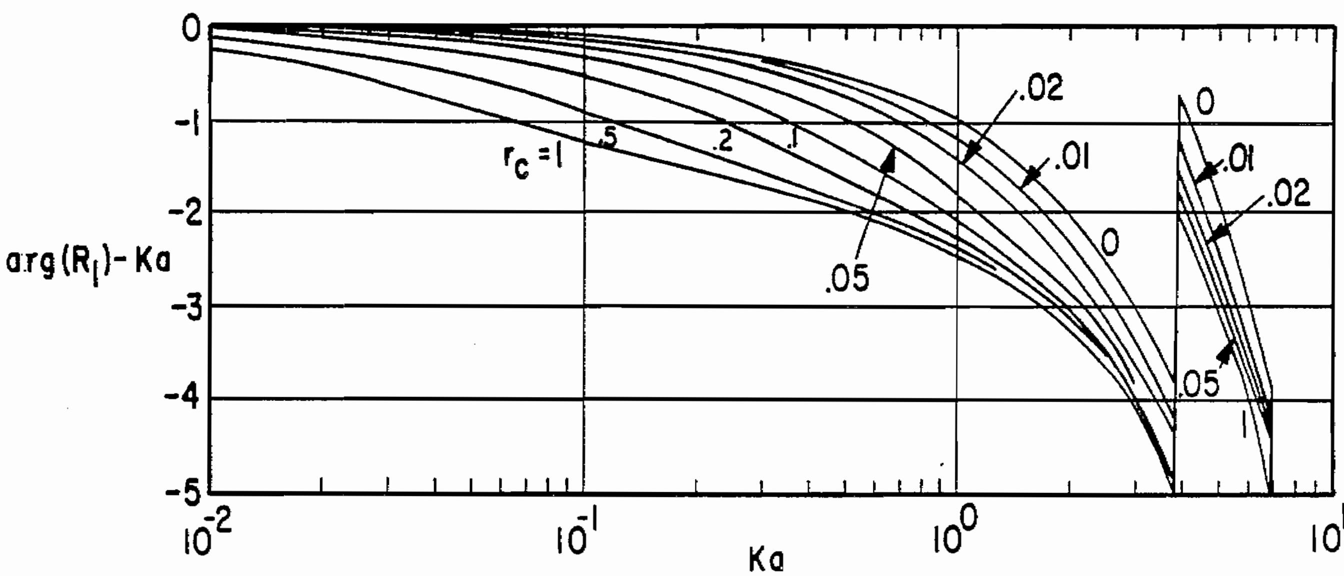



B. PHASE OF $R_y$ WITH $r_c$ AS A PARAMETER

FIGURE 8. EFFECT OF ADMITTANCES ON RESPONSE: b/a = .1

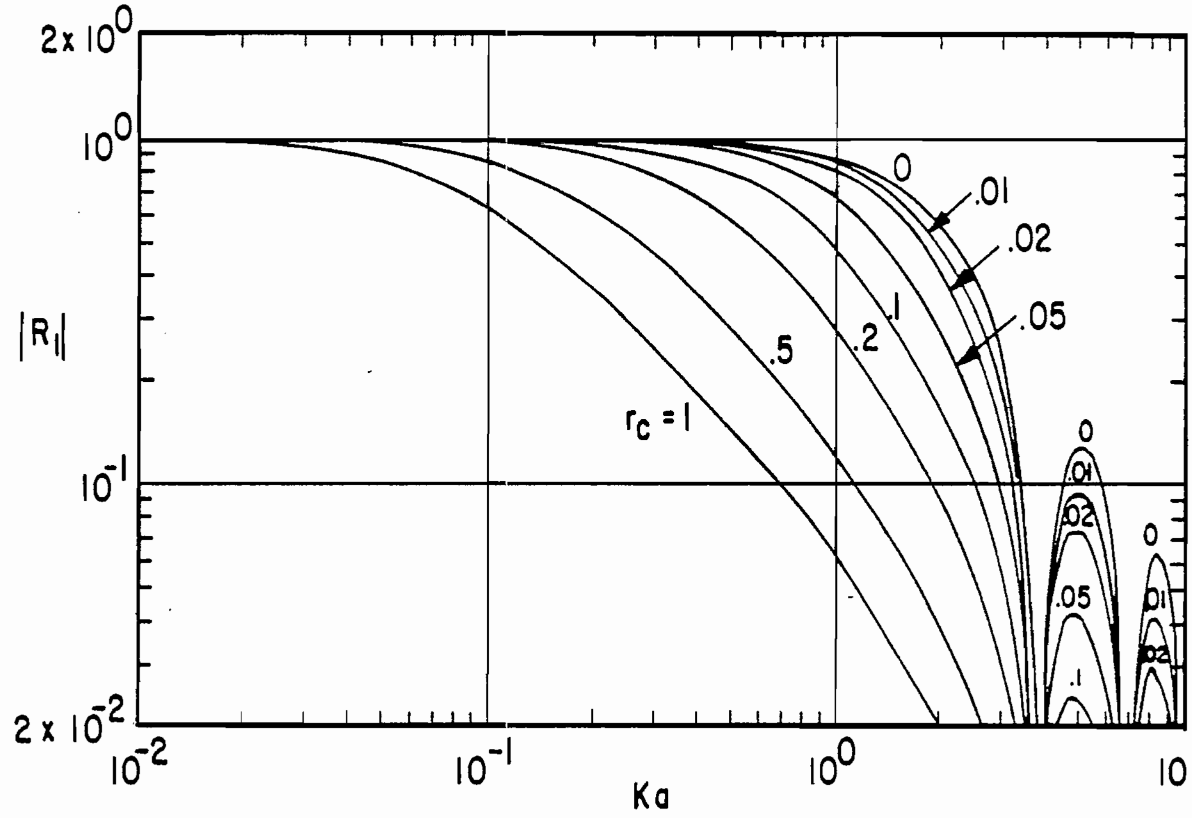



A. MAGNITUDE OF $R_1$ WITH $r_c$ AS A PARAMETER

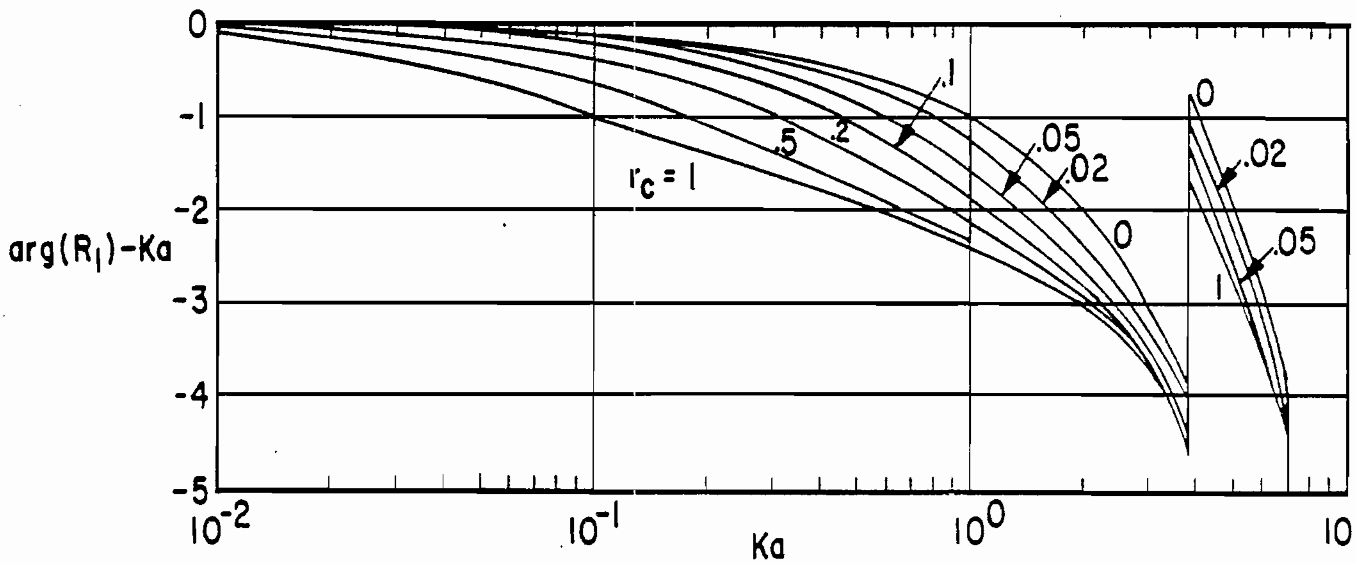



B. PHASE OF $R_1$ WITH $r_c$ AS A PARAMETER

FIGURE 9. RESPONSE CHARACTERISTICS FOR $\theta_1' = \pi/2$ WITH b/a = .01




A. MAGNITUDE OF $R_1$ WITH $r_c$ AS A PARAMETER




B. PHASE OF $R_1$ WITH $r_c$ AS A PARAMETER

FIGURE 10. RESPONSE CHARACTERISTICS FOR $\theta_1' = \pi/2$ WITH $b/a = .1$

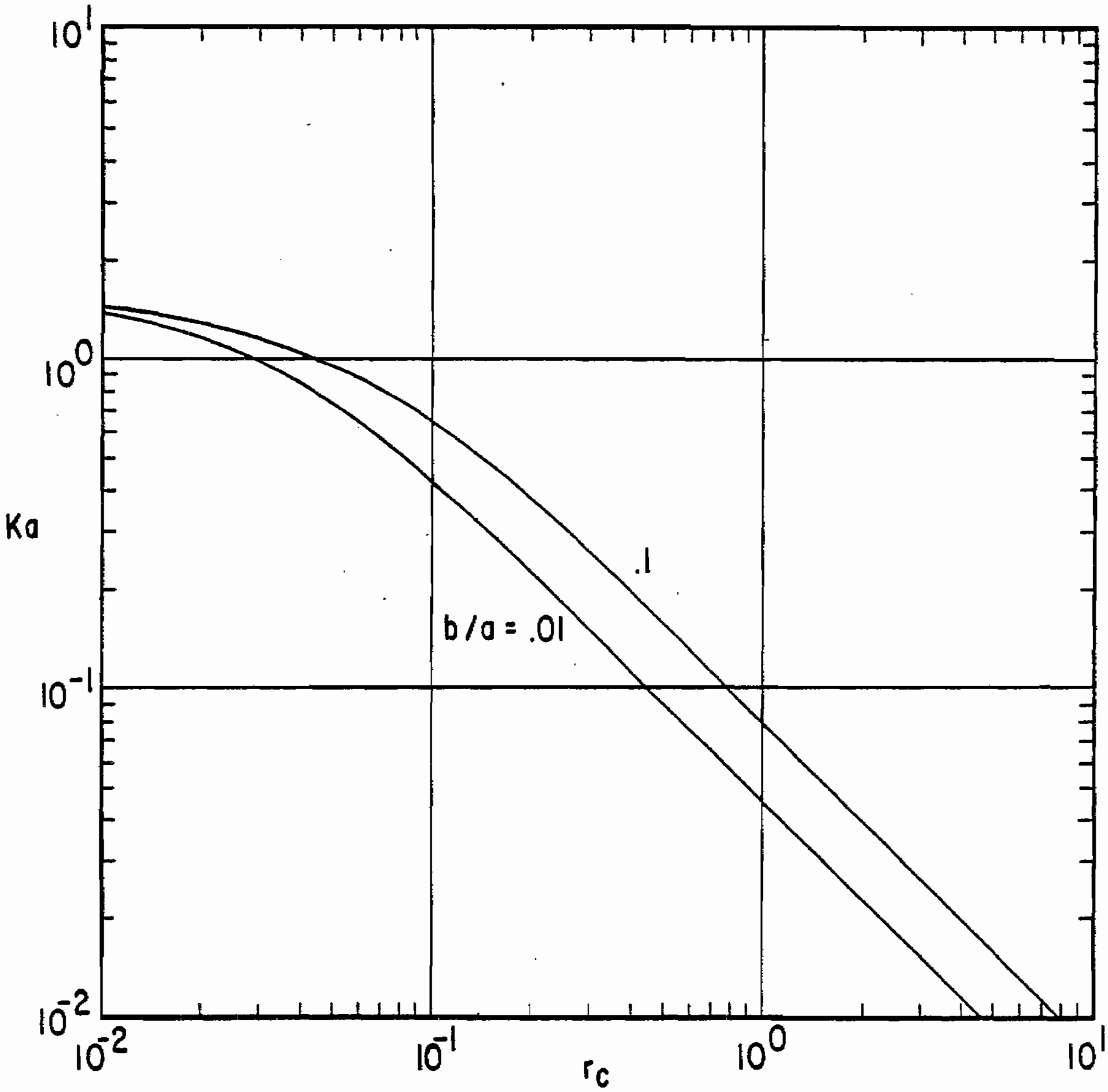
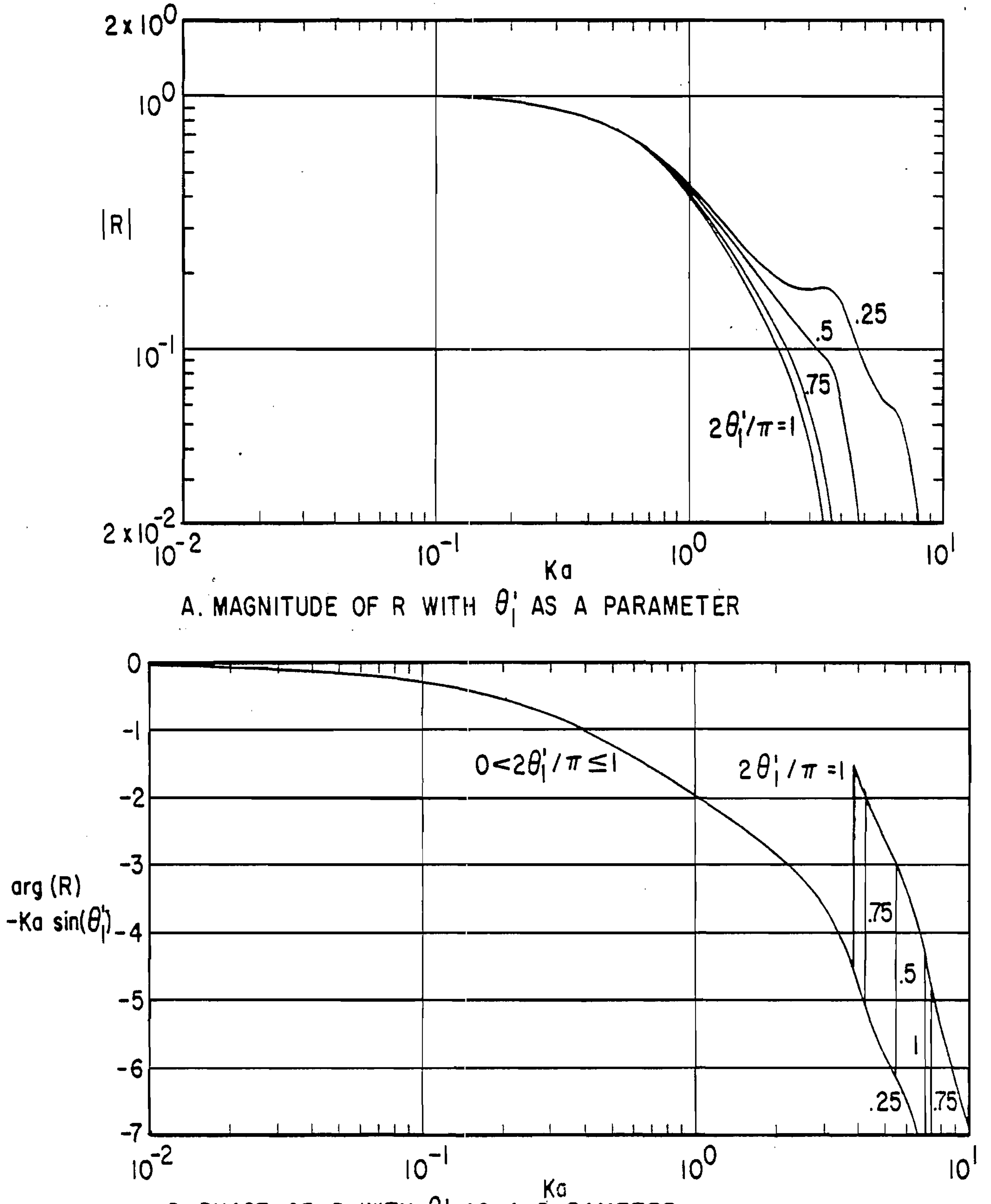



FIGURE 11. DEPENDENCE OF FREQUENCY RESPONSE ON CABLE IMPEDANCE WITH b/a AS A PARAMETER: $|R_i| = 1/\sqrt{2}$

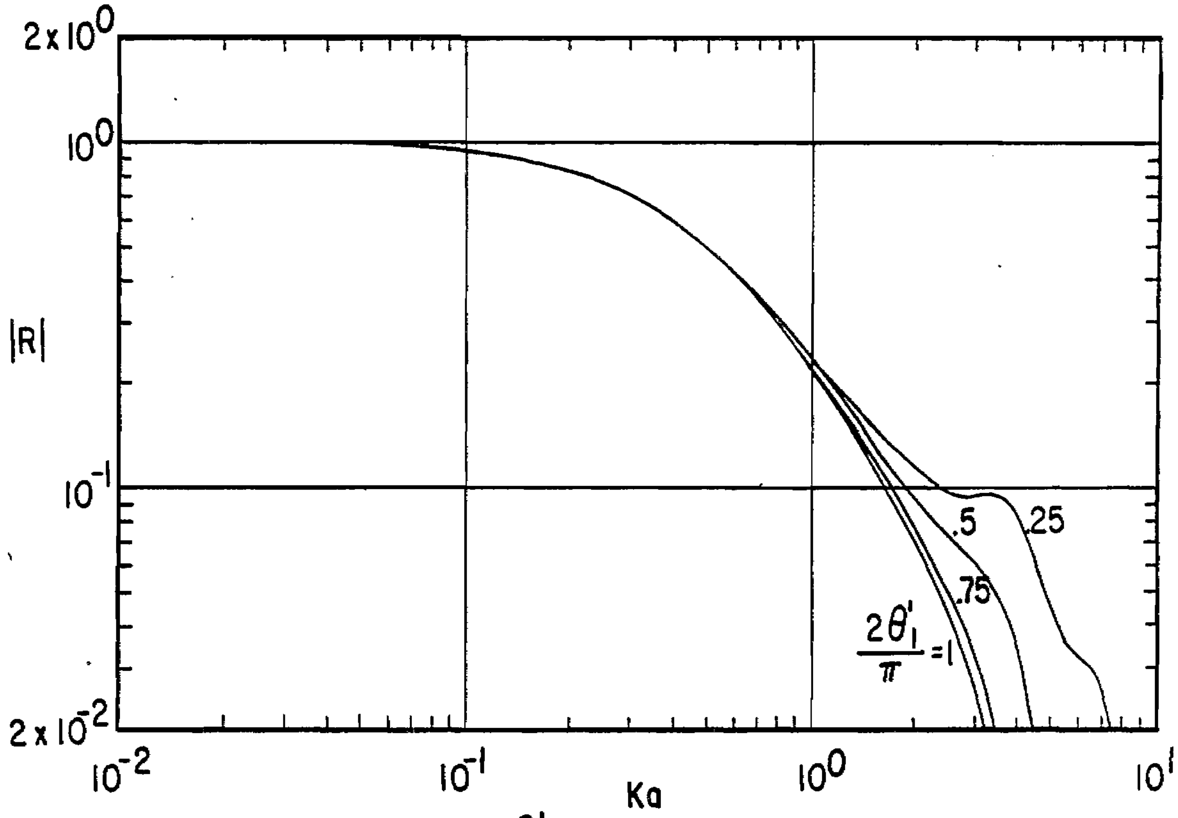



A. MAGNITUDE OF R WITH $\theta_1'$ AS A PARAMETER

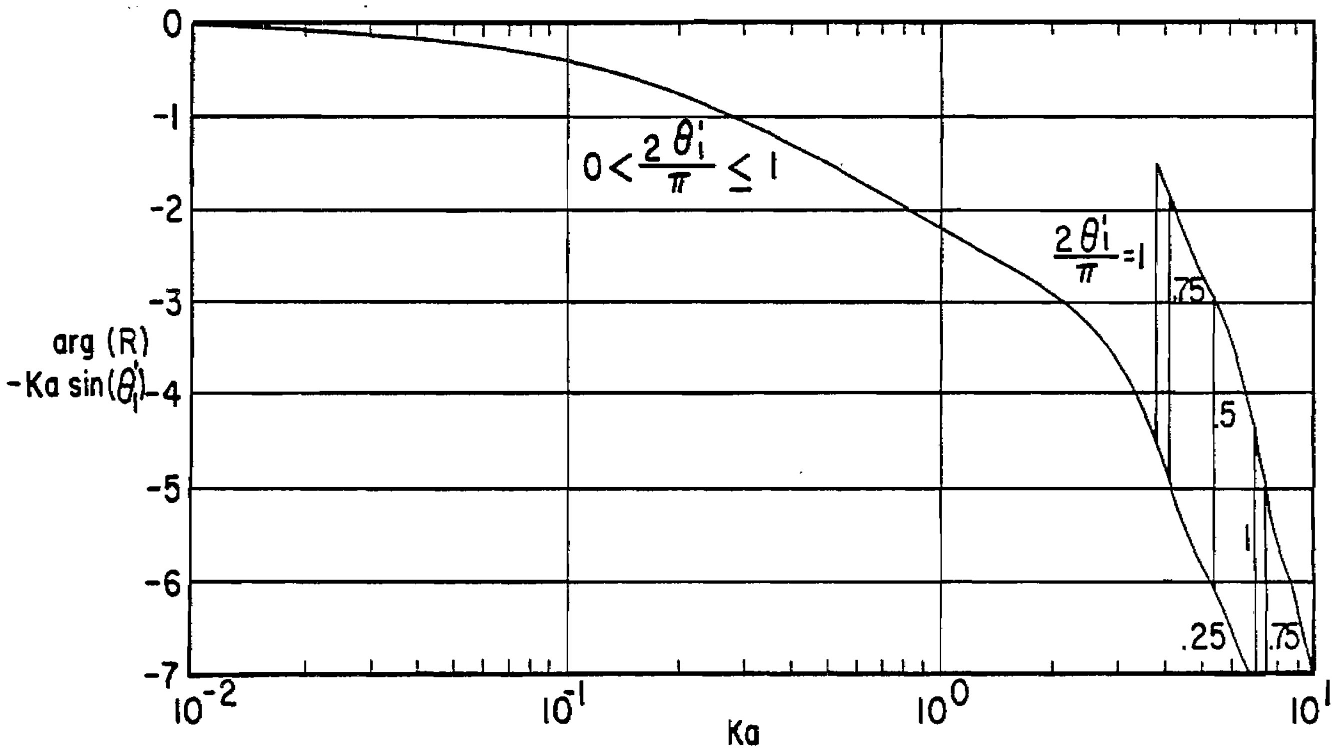


B. PHASE OF R WITH $\theta_1'$ AS A PARAMETER

FIGURE 12. RESPONSE CHARACTERISTICS: $b/a = .1$, $r_c = 50\Omega/Z_0 \simeq .1327$




A. MAGNITUDE OF R WITH $\theta_1'$ AS A PARAMETER




B. PHASE OF R WITH $\theta_1'$ AS A PARAMETER

FIGURE 13. RESPONSE CHARACTERISTICS: $\frac{b}{a} = .1$, $r_c = \frac{100\Omega}{Z_0} \simeq .2654$

## Appendix A: Expansion of $\Gamma_1$ in Spherical Bessel Functions

In equation 107 we have the integral

$$\Gamma_1 = \frac{i}{\pi}\int_0^{\pi} \frac{e^{-ip}}{p}\cos(\beta)d\beta \tag{A1}$$

where from equations 103 and 104

$$p \equiv ka[1 + \nu^2 - 2\nu\cos(\beta)]^{1/2} \tag{A2}$$

and where we have $0 \leq \nu$. Define

$$\nu_- \equiv \min[1, \nu]$$
$$\nu_+ \equiv \max[1, \nu] \tag{A3}$$

giving the relation

$$\nu_-\nu_+ \equiv \nu \tag{A4}$$

Then we have from an addition theorem for spherical Bessel functions[1A,2A,3A]

$$\frac{e^{-ip}}{p} = -\frac{i}{\pi}\sum_{n=o}^{\infty}(2n + 1)j_n(ka\nu_-)h_n^{(2)}(ka\nu_+)P_n(\cos(\beta)) \tag{A5}$$

where $P_n$ is a Legendre polynomial.

To obtain the expansion for $\Gamma_1$ multiply both sides of equation A5 by $\cos(\beta)$ and integrate over $0 \leq \beta \leq \pi$ giving

$$\Gamma_1 = \frac{1}{\pi}\sum_{n=o}^{\infty}(2n + 1)D_n j_n(ka\nu_-)h_n^{(2)}(ka\nu_+) \tag{A6}$$

---

1A. Reference 9, p. 107.

2A. Morse and Feshbach, Methods of Theoretical Physics, McGraw Hill, 1953, p. 1466.

3A. Reference 8, eqns. 10.1.45 and 10.1.46.

where

$$D_n = \int_0^{\pi} P_n\ (\cos(\beta))\ \cos(\beta)d\beta = \int_{-1}^{1} P_n(\alpha)\ \frac{\alpha}{\sqrt{1 - \alpha^2}}\ d\alpha \qquad (A7)$$

For n even $P_n(\alpha)$ is even in $\alpha$ and the integral is zero. For n odd and given by

$$n = 2n' + 1 \qquad (A8)$$

we have[4A,5A]

$$D_n = \frac{\pi}{4^{2n'+1}} \binom{2n'}{n'} \binom{2n' + 2}{n' + 1} = \frac{\pi}{4^{2n'+1}} \frac{(2n')!(2n' + 2)!}{(n'!)^2((n' + 1)!)^2}$$

$$= \pi \frac{(2n' - 1)!!(2n' + 1)!!}{(2n')!!(2n' + 2)!!} = \pi \frac{(n - 2)!!n!!}{(n - 1)!!(n + 1)!!}$$

$$= \frac{\pi}{n(n + 1)} \left[\frac{n!!}{(n - 1)!!}\right]^2 \qquad (A9)$$

where the double factorial function is defined by

$$(2n)!! \equiv (2n)(2n - 2) \cdots (4)(2) \qquad \text{(even)}$$

$$(2n - 1)!! \equiv (2n - 1)(2n - 3) \cdots (3)(1) \qquad \text{(odd)} \qquad (A10)$$

$$0!! \equiv (-1)!! \equiv 1$$

For $\Gamma_1$ we then have

$$\Gamma_1 = \sum_{n=1}^{\infty,2} \frac{2n + 1}{n(n + 1)} \left[\frac{n!!}{(n - 1)!!}\right]^2 j_n(ka\nu_-)h_n^{(2)}(ka\nu_+) \qquad (A11)$$

4A. Reference 8, eqn. 22.13.7.

5A. A. Erdelyi, ed., Tables of Integral Transforms, vol. 2, McGraw Hill, 1954, p. 276, eqn. 16.2(5).

In another form we have some special values of the Legendre functions (using ref. 8 definitions) as[6A]

$$P_n^{n''}(0) = \begin{cases} 0 & \text{for } n + n'' \text{ odd} \\ (-1)^{\frac{n+n''}{2}} \dfrac{(n + n'' - 1)!!}{(n - n'')!!} & \text{for } n + m \text{ even} \end{cases} \tag{A12}$$

giving

$$\Gamma_1 = \sum_{n=1}^{\infty,2} \frac{2n + 1}{n(n + 1)} \left[P_n^1(0)\right]^2 j_n(ka\nu_-)h_n^{(2)}(ka\nu_+) \tag{A13}$$

6A. [illegible], 3rd ed., McGraw-Hill, [illegible] p. 163.

## Appendix B: Representations for $X(\alpha)$

In equations 131 and 132 we have the function

$$X(\alpha) \equiv \int_0^1 [1 - \xi^2]^{-1/2} e^{\alpha\xi} d\xi = \frac{\pi}{2}[I_0(\alpha) + \mathbf{L}_0(\alpha)] \qquad \text{(B1)}$$

This function has a series expansion which can be found by setting

$$\xi = \sin(\delta) \ , \qquad d\xi = \cos(\delta)d\delta \qquad \text{(B2)}$$

giving[1B,2B]

$$X(\alpha) = \int_0^{\pi/2} e^{\alpha\sin(\delta)} d\delta = \sum_{\ell'=0}^{\infty} \frac{\alpha^{\ell'}}{\ell'!} \int_0^{\pi/2} [\sin(\delta)]^{\ell'} d\delta$$

$$= \frac{\pi}{2} \sum_{\ell'=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)^{\ell'}}{\left[\Gamma\left(\frac{\ell}{2} + 1\right)\right]^2} \qquad \text{(B3)}$$

In another form this series can be written as

$$X(\alpha) = \frac{\pi}{2} \sum_{\ell'=0}^{\infty,2} \frac{\alpha^{\ell'}}{(\ell'!!)^2} + \sum_{\ell'=1}^{\infty,2} \frac{\alpha^{\ell'}}{(\ell'!!)^2} \qquad \text{(B4)}$$

or as

$$X(\alpha) = \frac{\pi}{2} \sum_{\ell'=0}^{\infty} \frac{\alpha^{2\ell'}}{((2\ell')!!)^2} + \sum_{\ell'=0}^{\infty} \frac{\alpha^{2\ell'+1}}{((2\ell' + 1)!!)^2} \qquad \text{(B5)}$$

where the double factorial notation is defined in equations A10. These series expansions are absolutely convergent for all $\alpha$. They can also be found from the series for $I_0$ and $\mathbf{L}_0$ by addition.[3B]

1B. Reference 16, eqn. 858.44.

2B. Reference 8, eqn. 6.1.18.

3B. Reference 8, eqns. 9.6.10 and 12.2.1.

For large $|\alpha|$ we can use asymptotic expansions for $X(\alpha)$ to help characterize the behavior of the function. First we have as $|\alpha| \to \infty$ with $|\arg(\alpha)| < \pi/2$ the results[4B,5B]

$$I_o(\alpha) = \frac{e^{\alpha}}{\sqrt{2\pi\alpha}}\left\{1 + \frac{1}{8\alpha} + O(\alpha^{-2})\right\}$$

$$= \frac{e^{\alpha}}{\sqrt{2\pi\alpha}}\left\{1 + O(\alpha^{-1})\right\}$$

$$\mathbf{L}_o(\alpha) = I_o(\alpha) - \frac{2}{\pi\alpha}\left\{\sum_{\ell'=0}^{\ell''-1} \frac{\left(\frac{1}{2}\right)_{\ell'}(2\ell')!}{\ell'!}\alpha^{-2\ell'} + O(\alpha^{-2\ell''})\right\} \tag{B6}$$

$$= \frac{e^{\alpha}}{\sqrt{2\pi\alpha}}\left\{1 + O(\alpha^{-1})\right\}$$

and thus

$$X(\alpha) = \pi I_o(\alpha) - \frac{1}{\alpha} + O(\alpha^{-3})$$

$$= \sqrt{\frac{\pi}{2\alpha}}\, e^{\alpha}\left\{1 + \frac{1}{8\alpha} + O(\alpha^{-2})\right\}$$

$$= \sqrt{\frac{\pi}{2\alpha}}\, e^{\alpha}\left\{1 + O(\alpha^{-1})\right\} \tag{B7}$$

This can be used for large positive $\alpha$ ($\alpha \to +\infty$) in computation.

Next as $|\alpha| \to \infty$ with $|\arg(-\alpha)| < \pi/2$ we use the relations

$$I_o(\alpha) = I_o(-\alpha) \;, \qquad \mathbf{L}_o(\alpha) = -\mathbf{L}_o(-\alpha) \tag{B8}$$

---

4B. Reference 8, eqns. 9.7.1 and 12.2.6.

5B. Reference 9, p. 115.

so that we can use equations B6 with $\alpha$ replaced by $-\alpha$ and appropriate changes in sign. Thus as $|\alpha| \to \infty$ with $|\arg(-\alpha)| < \pi/2$ we have

$$I_o(\alpha) = \frac{e^{-\alpha}}{\sqrt{-2\pi\alpha}}\left\{1 - \frac{1}{8\alpha} + O(\alpha^{-2})\right\}$$

(B9)

$$\mathbf{L}_o(\alpha) = -I_o(\alpha) - \frac{2}{\pi\alpha}\left\{\sum_{\ell'=o}^{\ell''-1} \frac{\left(\frac{1}{2}\right)_{\ell'}(2\ell')!}{\ell'!}\alpha^{-2\ell'} + O(\alpha^{-2\ell''})\right\}$$

and

$$X(\alpha) = \frac{\pi}{2}[I_o(\alpha) + \mathbf{L}_o(\alpha)]$$

$$= -\frac{1}{\alpha}\left\{\sum_{\ell'=o}^{\ell''-1} \frac{\left(\frac{1}{2}\right)_{\ell'}(2\ell')!}{\ell'!}\alpha^{-2\ell'} + O(\alpha^{-2\ell''})\right\}$$

$$= -\frac{1}{\alpha}\left\{\sum_{\ell'=o}^{\ell''-1} [2\ell' - 1)!!]^2\alpha^{-2\ell'} + O(\alpha^{-2\ell''})\right\}$$

$$= -\frac{1}{\alpha} - \frac{1}{\alpha^3} + O(\alpha^{-5})$$

$$= -\frac{1}{\alpha} + O(\alpha^{-3}) \qquad \text{(B10)}$$

This can be used for large negative $\alpha$ ($\alpha \to -\infty$) in computation. This result can also be obtained by a direct asymptotic expansion of the integral in equation B1.

## Appendix C: Behavior of $\Omega_n$ Series for Large $\ell$

Consider some of the features of $\Omega_n$. From equation 135

$$\Omega_n = \begin{cases} \displaystyle\sum_{\ell=0}^{\infty} B_{n,\ell}\left\{X\left(-\frac{b}{a}(2\ell+2)\right) + X\left(-\frac{b}{a}(2\ell+1-n)\right)\right\} & \text{for } n \neq 1 \\ 0 & \text{for } n = 1 \end{cases} \tag{C1}$$

$$B_{n,\ell} = 2\,\frac{\left(\frac{1-n}{2}\right)_{\ell+1}\left(\frac{1-n}{2}\right)_{\ell}}{(\ell+1)!\,\ell!} = 2\,\frac{(\eta)_{\ell+1}(\eta)_{\ell}}{(\ell+1)!\,\ell!}$$

where we have, as in equation 113,

$$\eta \equiv \frac{1-n}{2} \tag{C2}$$

In calculating $\Omega_n$ there are two cases to consider.

Case 1: n odd

In this case $\eta$ is a negative integer or zero. In calculating $B_{n,\ell}$ we have the Pochhammer symbols

$$\begin{aligned}(\eta)_\ell &= (\eta)(\eta+1)\cdots(\eta+\ell-1) \\ &= (-1)^\ell(-\eta)(-\eta-1)\cdots(-\eta-\ell+1) \\ &= (-1)^\ell\frac{(-\eta)!}{(-\eta-\ell)!} \\ &= (-1)^\ell\frac{\left(\frac{n-1}{2}\right)!}{\left(\frac{n-1}{2}-\ell\right)!}\end{aligned} \tag{C3}$$

$$(\eta)_{\ell+1} = (\eta)(\eta+1) \cdots (\eta+\ell)$$

$$= (-1)^{\ell+1} \frac{\left(\frac{n-1}{2}\right)!}{\left(\frac{n-1}{2} - \ell - 1\right)!}$$

Note for $\ell \geq 1 - \eta$ that $(\eta)_\ell = 0$ and for $\ell \geq -\eta$ that $(\eta)_{\ell+1} = 0$. Thus the sum over $\ell$ truncates giving

$$\Omega_n = \sum_{\ell=0}^{\frac{n-3}{2}} B_{n,\ell}\left[X\left(-\frac{b}{a}(2\ell+2)\right) + X\left(-\frac{b}{a}(2\ell+1-n)\right)\right] \tag{C4}$$

Thus for any given odd n there are only a finite number of terms needed to calculate $\Omega_n$. Note that if we use the convention that if the upper limit on the summation is less than the lower limit the sum shall be taken equal to zero, then this sum gives $\Omega_1 = 0$ as required.

Case 2: n even

In this case the sum over $\ell$ does not terminate. The Pochhammer symbols have the form

$$(\eta)_\ell = (\eta)(\eta+1) \cdots (\eta+\ell-1)$$

$$= (-1)^\ell(-\eta)(-\eta-1) \cdots (-\eta-\ell+1)$$

$$= (-1)^\ell \frac{\Gamma(-\eta+1)}{\Gamma(-\eta-\ell+1)}$$

$$= (-1)^\ell \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2} - \ell\right)}$$

(C5)

$$(\eta)_{\ell+1} = (\eta)(\eta+1) \cdots (\eta+\ell)$$

$$= (-1)^{\ell+1} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2} - \ell - 1\right)}$$

and also

$$(\eta)_\ell = \frac{\Gamma(\eta+\ell)}{\Gamma(\eta)} = \frac{\Gamma\left(\frac{1-n}{2} + \ell\right)}{\Gamma\left(\frac{1-n}{2}\right)}$$

(C6)

$$(\eta)_{\ell+1} = \frac{\Gamma(\eta+\ell+1)}{\Gamma(\eta)} = \frac{\Gamma\left(\frac{1-n}{2} + \ell + 1\right)}{\Gamma\left(\frac{1-n}{2}\right)}$$

Since n is even the Γ functions here have half integer arguments and are then finite and non zero even for negative arguments. For these Γ functions of negative argument we can also use[1C]

$$\frac{1}{\Gamma(w)} = \frac{1}{\pi} \sin(\pi w)\ \Gamma(1 - w) \tag{C7}$$

For large arguments we have the asymptotic form of the Γ functions known as the Stirling approximation[2C] which has as $w \to \infty$ with $|\arg(w)| < \pi$

$$\Gamma(w) = e^{-w}\ w^{w-\frac{1}{2}}\ (2\pi)^{1/2}\ [1 + O(w^{-1})] \tag{C8}$$

Another asymptotic form for $w \to \infty$ with $|\arg(w)| < \pi$ is given for fixed $w_1$, $w_2$ by[3C]

$$\frac{\Gamma(w + w_1)}{\Gamma(w + w_2)} = w^{w_1 - w_2}\ [1 + O(w^{-1})] \tag{C9}$$

1C. Reference 8, eqn. 6.1.17.

2C. Reference 8, eqn. 6.1.37.

3C. Reference 8, eqn. 6.1.47.

Next consider $B_{n,\ell}$ for large $\ell$. As $\ell \to \infty$ we have

$$B_{n,\ell} = 2\,\frac{(\eta)_{\ell+1}(\eta)_\ell}{(\ell+1)!\,\ell!} = \frac{2}{[\Gamma(\eta)]^2}\,\frac{\Gamma(\eta+\ell+1)}{\Gamma(\ell+2)}\,\frac{\Gamma(\eta+\ell)}{\Gamma(\ell+1)}$$

$$= \frac{2}{[\Gamma(\eta)]^2}\,\ell^{\eta-1}\,\ell^{\eta-1}\,[1 + O(\ell^{-1})]$$

$$= \frac{2}{\pi^2}[\sin(\pi\eta)]^2\,[\Gamma(1-\eta)]^2\,\ell^{2\eta-2}\,[1+O(\ell^{-1})]$$

$$= \frac{2}{\pi^2}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2\,\ell^{-n-1}\,[1+O(\ell^{-1})] \qquad \text{(C10)}$$

Now write $\Omega_n$ as

$$\Omega_n = \sum_{\ell=0}^{\ell_1} B_{n,\ell}\left\{X\left(-\frac{b}{a}(2\ell+2)\right) + X\left(-\frac{b}{a}(2\ell+1-n)\right)\right\} + \Delta_1 \qquad \text{(C11)}$$

where the remainder after considering only the first $\ell_1$ terms is

$$\Delta_1 \equiv \sum_{\ell=\ell_1+1}^{\infty} B_{n,\ell}\left\{X\left(-\frac{b}{a}(2\ell+2)\right) + X\left(-\frac{b}{a}(2\ell+1-n)\right)\right\} \qquad \text{(C12)}$$

Let $\ell_1$ be some large positive integer and substitute for $B_{n,\ell}$ from equation C10 and for X from equation B10 to give as $\ell_1 \to \infty$

$$\Delta_1 = \frac{2}{\pi^2}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2 \sum_{\ell=\ell_1+1}^{\infty} \ell^{-n-1}[1+O(\ell^{-1})]\left\{\frac{a}{b}\,\frac{1}{2\ell}+\frac{a}{b}\,\frac{1}{2\ell}+O(\ell^{-2})\right\}$$

$$= \frac{2}{\pi^2}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2 \frac{a}{b} \sum_{\ell=\ell_1+1}^{\infty} [\ell^{-n-2}+O(\ell^{-n-3})] \qquad \text{(C13)}$$

Replacing the sums by integrals and treating $\ell$ as a continuous variable for this purpose gives as $\ell_1 \to \infty$

$$\int_{\ell_1}^{\infty} \ell^{-n-2}\, d\ell > \sum_{\ell=\ell_1+1}^{\infty} \ell^{-n-2} > \int_{\ell_1+1}^{\infty} \ell^{-n-2}\, d\ell$$

$$\frac{\ell_1^{-n-1}}{n+1} > \sum_{\ell=\ell_1+1}^{\infty} \ell^{-n-2} > \frac{(\ell_1+1)^{-n-1}}{n+1} = \frac{\ell_1^{-n-1}}{n+1}\left[1+O\left(\ell_1^{-1}\right)\right]$$

$$\sum_{\ell=\ell_1+1}^{\infty} \ell^{-n-2} = \frac{\ell_1^{-n-1}}{n+1}\left[1+O\left(\ell_1^{-1}\right)\right] \tag{C14}$$

and

$$\sum_{\ell=\ell_1+1}^{\infty} O(\ell^{-n-3}) = \int_{\ell_1+1}^{\infty} O(\ell^{-n-3})\, d\ell = O\left(\ell_1^{-n-2}\right) \tag{C15}$$

Thus we have as $\ell_1 \to \infty$

$$\Delta_1 = \frac{2}{\pi^2}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2 \frac{a}{b} \frac{\ell_1^{-n-1}}{n+1}\left[1+O\left(\ell_1^{-1}\right)\right] \tag{C16}$$

Then for any given $n \geq 0$ this can be used as an error estimate for sufficiently large $\ell_1$ so as to calculate $\Omega_n$ to some desired accuracy.

## Appendix D: Behavior of $y_a$ Series for Large n

Now we wish to consider the error involved in truncating the sum over n in calculating the admittance. From equation 135 we write

$$y_a = ika \sum_{n=o}^{n_2} \frac{(-ika)^n}{n!} \Omega_n + \Delta_2 \tag{D1}$$

where the remainder after including only the first $n_2$ terms is

$$\Delta_2 = ika \sum_{n=n_2+1}^{\infty} \frac{(-ika)^n}{n!} \Omega_n \tag{D2}$$

From equations 127, 112, 113, 73, and 76 we have

$$\Omega_n \equiv \frac{a}{b} \int_o^{\infty} f_E \nu \Lambda_n d\nu = \frac{1}{\pi} \int_{-1}^{1} [1 - \xi^2]^{-1/2} e^{\frac{b}{a}\xi} \Lambda_n d\nu$$

$$\Lambda_n = \int_o^{2\pi} q^{n-1} \cos(\beta) d\beta = 2\int_o^{\pi} q^{n-1} \cos(\beta) d\beta \tag{D3}$$

$$q \equiv [1 + \nu^2 - 2\nu \cos(\beta)]^{1/2}$$

with

$$\eta = \frac{1 - n}{2}, \qquad \nu = e^{\frac{b}{a}\xi}$$

$$f_E = \begin{cases} \frac{1}{\pi}[1 - \xi^2]^{-1/2} e^{-\frac{b}{a}\xi} & \text{for } |\xi| < 1 \\ 0 & \text{for } |\xi| > 1 \end{cases} \tag{D4}$$

To look at the behavior of $\Delta_2$ for large $n_2$ we first consider $\Lambda_n$ for large n. For convenience define a change of variables from $\beta$ to $\alpha$ as

$$q^2 = 1 + \nu^2 - 2\nu \cos(\beta) = 2\nu[\psi - \cos(\beta)]$$

$$= 2\nu(\psi + 1)e^{-\alpha}$$

$$e^{-\alpha} = \frac{\psi - \cos(\beta)}{\psi + 1} \tag{D5}$$

$$\sin(\beta)d\beta = -(\psi + 1)e^{-\alpha}\, d\alpha$$

where we have defined

$$\psi \equiv \frac{1}{2}[\nu + \nu^{-1}] = \frac{1}{2}\left[e^{\frac{b}{a}\xi} + e^{-\frac{b}{a}\xi}\right] = \cosh\left(\frac{b}{a}\,\xi\right) \tag{D6}$$

Then from equations D3 for $\Lambda_n$ we have

$$\Lambda_n = 2(2\nu)^{-\eta}\int_0^{\pi} [\psi - \cos(\beta)]^{-\eta} \cos(\beta)d\beta$$

$$= 2(2\nu)^{-\eta}(\psi + 1)^{-\eta+1}\int_0^{\alpha_o} e^{(\eta-1)\alpha}\, \Phi(\alpha)d\alpha \tag{D7}$$

with

$$\Phi(\alpha) \equiv [\psi - (\psi + 1)e^{-\alpha}]\left\{1 - [\psi - (\psi + 1)e^{-\alpha}]^2\right\}^{-1/2}$$

$$\alpha_o \equiv \ln\left(\frac{\psi + 1}{\psi - 1}\right) = 2\,\ln\left(\left|\frac{\nu + 1}{\nu - 1}\right|\right) \tag{D8}$$

Now we find the asymptotic form of the integral in equation D7 as $\eta \to -\infty$ following a procedure similar to Watson's lemma.[1D] Since there is a limit on the range of $\nu$ of interest we can restrict

$$e^{-\frac{b}{a}} \leq \nu \leq e^{\frac{b}{a}}$$

$$1 \leq \psi \leq \cosh\left(\frac{b}{a}\right) \qquad \text{(D9)}$$

$$\alpha_o = 2\ \ln\left(\left|\coth\left(\frac{b}{2a}\ \xi\right)\right|\right) \geq 2\ \ln\left(\coth\left(\frac{b}{2a}\right)\right)$$

so that $\alpha_o$ has a lower bound, independent of $\eta$ (or n). Since the integral of $\Phi(\alpha)$ over $0 \leq \alpha \leq \alpha_o$ exists and is independent of n then we can write as $\eta \to -\infty$

$$\int_o^{\alpha_o} e^{(\eta-1)\alpha}\ \Phi(\alpha)d\alpha = \int_o^{\alpha_1} e^{(\eta-1)\alpha}\ \Phi(\alpha)d\alpha + O\left(e^{(\eta-1)\alpha_1}\right) \qquad \text{(D10)}$$

where $\alpha_1$ is some number independent of n with $0 < \alpha_1 < \alpha_o$. Then we expand $\Phi$ for $0 < \alpha \leq \alpha_1$ (since $\Phi$ is finite in this range) as the asymptotic form for $\alpha \to 0$ given by

$$\Phi(\alpha) = [-1 + O(\alpha)]\left\{1 - [-1 + (\psi + 1)\alpha + O(\alpha^2)]^2\right\}^{-1/2}$$

$$= -[1 + O(\alpha)]\left\{2(\psi + 1)\alpha + O(\alpha^2)\right\}^{-1/2}$$

$$= -[2(\psi + 1)\alpha]^{-1/2}\ [1 + O(\alpha)] \qquad \text{(D11)}$$

Using this result in equation D10 we obtain as $\eta \to -\infty$ or $n \to +\infty$

1D. E. T. Copson, Asymptotic Expansions, Cambridge, 1965, pp. 48-50.

$$\int_0^{\alpha_0} e^{(\eta-1)\alpha}\,\Phi(\alpha)d\alpha = -[2(\psi+1)]^{-1/2}\,\Gamma\left(\tfrac{1}{2}\right)(1-\eta)^{-1/2}\,[1+O(\eta^{-1})]$$

$$= -\left[\frac{\pi}{\psi+1}\right]^{1/2} (n+1)^{-1/2}\,[1+O(n^{-1})] \tag{D12}$$

Thus from equation D7 we have as $n \to +\infty$

$$\Lambda_n = -2\pi^{1/2}\,(2\nu)^{\frac{n-1}{2}}\,(\psi+1)^{\frac{n}{2}}\,(n+1)^{-\frac{1}{2}}\,[1+O(n^{-1})]$$

$$= -\left(\frac{2\pi}{\nu}\right)^{1/2} (\nu+1)^n\, n^{-\frac{1}{2}}\,[1+O(n^{-1})] \tag{D13}$$

This result applies for all $\nu$ in our range of interest.

For $\Omega_n$ as $n \to +\infty$ we then have

$$\Omega_n = -\left(\frac{2}{\pi n}\right)^{1/2} [1+O(n^{-1})]\int_{-1}^{1} e^{\frac{b}{2a}\xi}\left[e^{\frac{b}{a}\xi}+1\right]^n [1-\xi^2]^{-1/2}\,d\xi \tag{D14}$$

Make a change of variable from $\xi$ to $\alpha'$ defined by

$$e^{-\alpha'} \equiv \frac{e^{\frac{b}{a}\xi} + 1}{e^{\frac{b}{a}} + 1}$$

(D15)

$$-e^{-\alpha'}\,d\alpha' = \frac{b}{a}\,\frac{e^{\frac{b}{a}\xi}\,d\xi}{e^{\frac{b}{a}} + 1}$$

so that we can write $\Omega_n$ as

$$\Omega_n = -\left(\frac{2}{\pi n}\right)^{1/2}[1+O(n^{-1})]\frac{a}{b}\left[e^{\frac{b}{a}}+1\right]^{n+1}\int_0^{\alpha_o'} e^{-(n+1)\alpha'}\Phi'(\alpha')d\alpha' \qquad \text{(D16)}$$

with

$$\Phi'(\alpha') = e^{-\frac{b}{2a}\xi}[1-\xi^2]^{-1/2}$$

$$= \left[\left(e^{\frac{b}{a}}+1\right)e^{-\alpha'}-1\right]^{-1/2}\left\{1-\left[\frac{a}{b}\ln\left[\left(e^{\frac{b}{a}}+1\right)e^{-\alpha'}-1\right]\right]^2\right\}^{-1/2} \qquad \text{(D17)}$$

$$\alpha_o' = \ln\left[\frac{e^{\frac{b}{a}}+1}{e^{-\frac{b}{a}}+1}\right] = \frac{b}{a}$$

We can now expand this integral for large n in the same manner as equation D7. Then write as $n \to +\infty$

$$\int_0^{\alpha_o'} e^{-(n+1)\alpha'}\Phi'(\alpha')d\alpha' = \int_0^{\alpha_1'} e^{-(n+1)\alpha'}\Phi'(\alpha')d\alpha' + O\left(e^{-(n+1)\alpha_1'}\right) \qquad \text{(D18)}$$

where $0 < \alpha_1' < \alpha_o'$ with $\alpha_1'$ independent of n. Expand $\Phi'$ for $0 < \alpha' \leq \alpha_1'$. Since $\Phi'$ is finite for all $\alpha'$ with $0 < \alpha' \leq \alpha_1'$ then expand $\Phi'$ as $\alpha' \to 0$ giving

$$\Phi'(\alpha') = \left[e^{\frac{b}{a}}+O(\alpha')\right]^{-1/2}\left\{1-\left[\frac{a}{b}\ln\left[e^{\frac{b}{a}}-\left(e^{\frac{b}{a}}+1\right)\alpha'+O(\alpha'^2)\right]\right]^2\right\}^{-1/2}$$

$$= \left[e^{\frac{b}{a}}+O(\alpha')\right]^{-1/2}\left\{1-\left[1+\frac{a}{b}\ln\left[1-\left(1+e^{-\frac{b}{a}}\right)\alpha'+O(\alpha'^2)\right]\right]^2\right\}^{-1/2}$$

$$= \left[e^{\frac{b}{a}}+O(\alpha')\right]^{-1/2}\left\{1-\left[1-\frac{a}{b}\left(1+e^{-\frac{b}{a}}\right)\alpha'+O(\alpha'^2)\right]^2\right\}^{-1/2}$$

$$= \left[e^{\frac{b}{a}}+O(\alpha')\right]^{-1/2}\left[2\frac{a}{b}\left(1+e^{-\frac{b}{a}}\right)\alpha'+O(\alpha'^2)\right]^{-1/2}$$

$$= \left[2\frac{a}{b}\left(e^{\frac{b}{a}}+1\right)\alpha'\right]^{-1/2}[1+O(\alpha')] \qquad \text{(D19)}$$

Using this in equation D18 we have as $n \to \infty$

$$\int_0^{\alpha'_0} e^{-(n+1)\alpha_\phi'(\alpha')}d\alpha' = \left[2\frac{a}{b}\left(e^{\frac{b}{a}}+1\right)\right]^{-1/2}\Gamma\left(\frac{1}{2}\right)(n+1)^{-1/2}[1+O(n^{-1})]$$

$$= \left[\frac{\pi}{2\frac{a}{b}\left(e^{\frac{b}{a}}+1\right)}\right]^{1/2} n^{-1/2}[1+O(n^{-1})] \qquad \text{(D20)}$$

Then from equation D16 we have as $n \to \infty$

$$\Omega_n = -\left(\frac{a}{b}\right)^{1/2}\left[e^{\frac{b}{a}}+1\right]^{n+\frac{1}{2}} n^{-1}[1+O(n^{-1})]$$

$$= -\left[\frac{b}{a}\left(e^{\frac{b}{a}}+1\right)\right]^{-1/2}\left[e^{\frac{b}{a}}+1\right]^{n+1}(n+1)^{-1}[1+O(n^{-1})] \qquad \text{(D21)}$$

Now substitute this result into equation D2 to give as $n_2 \to +\infty$

$$\Delta_2 = \left[\frac{b}{a}\left(e^{\frac{b}{a}}+1\right)\right]^{-1/2} \sum_{n=n_2+1}^{\infty} \frac{(-ika)^{n+1}\left[e^{\frac{b}{a}}+1\right]^{n+1}}{(n+1)!}[1+O(n^{-1})]$$

$$= \left[\frac{b}{a}\left(e^{\frac{b}{a}}+1\right)\right]^{-1/2} \sum_{n=n_2+2}^{\infty} \frac{\left[-ika\left(e^{\frac{b}{a}}+1\right)\right]^{n}}{n!}[1+O(n^{-1})] \qquad \text{(D22)}$$

This is basically the remainder in truncating an exponential series. Rewrite this equation as

$$\Delta_2 = \left[\frac{b}{a}\left(e^{\frac{b}{a}}+1\right)\right]^{-1/2} \frac{\left[-ika\left(e^{\frac{b}{a}}+1\right)\right]^{n_2+2}}{(n_2+2)!} \sum_{n=n_2+2}^{\infty} \phi_n''$$

$$\phi_n'' = \frac{(n_2+2)!}{n!}\left[-ika\left(e^{\frac{b}{a}}+1\right)\right]^{n-n_2-2} [1+O(n^{-1})] \qquad \text{(D23)}$$

Define

$$n_3 \equiv n - n_2 - 2 \qquad \text{(D24)}$$

and note that as $n_2 \to \infty$ with $n \geq n_2 + 2$

$$\phi_{n_2+2} = 1 + O\left(n_2^{-1}\right)$$

$$\frac{\phi_{n+1}}{\phi_n} = \frac{-ika\left(e^{\frac{b}{a}}+1\right)}{n+1}\left[1 + O\left(n_2^{-1}\right)\right] \qquad \text{(D25)}$$

Then for $n + 1 > |ka|(e^{b/a} + 1)$ we can bound the series for $n \geq n_2 + 3$ in equation D23 by a geometric series so that as $n_2 \to \infty$

$$\left| \sum_{n=n_2+3}^{\infty} \Phi_n'' \right| < \sum_{n=n_2+3}^{\infty} |\Phi_n''|$$

$$< \sum_{n_3=1}^{\infty} \left[ |ka| \left( e^{\frac{b}{a}}+1 \right) \right]^{n_3} (n_2+3)^{-n_3} [1+O(n_2^{-1})]$$

$$= \frac{|ka|\left(e^{\frac{b}{a}}+1\right)}{n_2+3} \left\{ 1 - \frac{|ka|\left(e^{\frac{b}{a}}+1\right)}{n_2+3} \right\}^{-1} [1+O(n_2^{-1})]$$

$$= O(n_2^{-1}) \tag{D26}$$

as $n_2 \to \infty$. Thus we have for $\Delta_2$ as $n_2 \to \infty$

$$\Delta_2 = \left[ \frac{b}{a}\left(e^{\frac{b}{a}}+1\right) \right]^{-1/2} \frac{\left[ -ika\left(e^{\frac{b}{a}}+1\right) \right]^{n_2+2}}{(n_2+2)!} [1+O(n_2^{-1})] \tag{D27}$$

For fixed ka and b/a then for sufficiently large $n_2$ this result can be used to estimate the error in calculating $y_a$.

## Appendix E: Numerical Techniques for Computer Calculation

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The numerical calculations for the graphs in this note were done on the CDC 6600 computer at AFWL. Most of the calculations are straightforward, but the admittance computations involved some special techniques, and the methods used are described below.

The first step in the calculation was to write a computer function which would calculate the $X(\alpha)$ of equation 132 for any value of $\alpha$. A series expansion for this function was developed in appendix B and is given by equation B5. The two summations in equation B5 are treated separately. Let $X_1(\alpha)$ be the first part of the function, so that

$$X_1(\alpha) = \frac{\pi}{2} \sum_{\ell'=0}^{\infty} \frac{\alpha^{2\ell'}}{((2\ell')!!)^2} \qquad \text{(E1)}$$

This may be expressed for computational purposes as the summation of a pi product

$$X_1(\alpha) = \frac{\pi}{2}\left[1 + \left(\sum_{\ell'=1}^{m} \prod_{k=1}^{\ell'} \frac{\alpha^2}{4k^2}\right) + R_m\right] \qquad \text{(E2)}$$

Since this series expansion is absolutely convergent by the ratio test an upper bound on the remainder may be expressed as[1E]

$$|R_m| \leq \frac{|a_{m-1}|^2}{|a_{m-1}| - |a_m|} = \varepsilon \qquad \text{(E3)}$$

where $a_m$ is the last term in the summation. The relative error due to truncation may be expressed as

$$\varepsilon_r = \left|\frac{\varepsilon}{S_m}\right| \leq E \qquad \text{(E4)}$$

where $S_m$ is the summation carried out to $\ell' = m$. For the calculations in this note E was set at $10^{-13}$ to insure accuracy.

1E. W. Kaplan, Advanced Calculus, Addison-Wesley, 1952, pp. 328, 329.

Similarly, let the second part of equation B5 be $X_2(\alpha)$, so that

$$X_2(\alpha) = \sum_{\ell'=0}^{\infty} \frac{\alpha^{2\ell'+1}}{((2\ell'+1)!!)^2} \tag{E5}$$

Expressed as a pi product for the computer, this is

$$X_2(\alpha) = \alpha + \sum_{\ell'=1}^{m} \alpha \left( \prod_{k=1}^{\ell'} \frac{\alpha^2}{4k^2 + 4k + 1} \right) + R_m \tag{E6}$$

This series is similar to the one in equation E2 and the error criterion is treated in the same manner.

Note that $X_1(\alpha)$ is an even function and $X_2(\alpha)$ is odd, so that with negative $\alpha$ one actually has a subtraction when algebraically adding $X_1(\alpha)$ and $X_2(\alpha)$. This causes round-off error in the computer when the values of $X_1(\alpha)$ and $X_2(\alpha)$ are in the order of $10^9$. So at the point where round-off error starts being significant one can switch over to the asymptotic form of $X(\alpha)$ for negative $\alpha$ as derived in appendix B and given by equation B10. It was determined numerically that the round-off error is significant at $\alpha < -25$. The relative difference at $\alpha = -22$ between the asymptotic form and the straight calculation is in the order of .001. The actual switchover point is at $\alpha = -22.7$, with a relative difference of .0011.

With positive $\alpha$ the two parts of $X(\alpha)$ are positive and no round-off error occurs in the addition, however with large positive $\alpha$ the calculations require more computer time and it is desirable to switch to the asymptotic form for positive numbers at some point. The asymptotic expression is given in equation B7. The switchover occurs at $\alpha = 200$ with a relative difference between the straight calculation and the asymptotic form of .0006.

The next step in the admittance calculation involves the $\Omega_n$ function. From equation 135 we have

$$\Omega_n = \sum_{\ell=0}^{\infty} B_{n,\ell} \left\{ X\left(-\frac{b}{a}(2\ell + 2)\right) + X\left(-\frac{b}{a}(2\ell + 1 - n)\right) \right\} \tag{E7}$$

The $B_{n,\ell}$ term is given in equation 135 and for computational purposes in the computer may be represented by a pi product as

$$B_{n,\ell} = \frac{(1 - n + 2\ell)}{\ell + 1} \prod_{k=1}^{\ell} \left(1 - \frac{n + 1}{2k}\right)^2 \quad \text{(E8)}$$

if $\ell \neq 0$.

If $\ell = 0$

$$B_{n,\ell} = 1 - n \quad \text{(E9)}$$

In appendix C it was seen that if n is odd the series for $\Omega_n$ is finite and truncates at $\ell = (n - 3)/2$. This allows the computation to proceed to this point without the necessity of checking for truncation error. If n is even and the series is truncated at some $\ell_1$ there is a remainder $\Delta_1$ to consider as shown in equation C11. This remainder is derived in appendix C and given in equation C16. In this note the summation for $\Omega_n$ is carried to the point where the relative error is

$$\varepsilon = \left|\frac{\Delta_1}{\Omega_n}\right| \leq .001 \quad \text{(E10)}$$

The number of terms required for convergence for the n = 0 term was large for small b/a. Table E1 summarizes the calculations performed for $\Omega_o$. The asymptotic form listed is that of equation 145 for $\Omega_o$ at small b/a.

| b/a | No. of terms | $\Omega_o$ by series | Asymptotic form | Relative difference |
|---|---|---|---|---|
| .001 | 41443 | 15.3613 | 15.3607 | .00004 |
| .01 | 5919 | 10.7564 | 10.7555 | .0001 |
| .1 | 1034 | 6.1621 | 6.1503 | .0016 |

Table E1. Calculations for $\Omega_o$

For n > 1 the number of terms needed for convergence varied, but the summation was usually completed by the time n/2 terms were reached, except for the first few values of n, which required more terms.

The asymptotic form for $\Omega_n$ for large n is given in equation D21. In the calculations for n = 200 and b/a = .1 the asymptotic form differed from the calculations with a relative difference of .003. For smaller b/a this relative difference is larger for the same n.

The final step in the admittance calculation is the solution of the admittance itself. This is given in equation D1 by

$$y_a = ika \sum_{n=o}^{n_2} \frac{(-ika)^n}{n!} \Omega_n + \Delta_2 \tag{E11}$$

The remainder $\Delta_2$ is given in equation D25. Again the error term may be represented as

$$\varepsilon = \left|\frac{\Delta_2}{y_a}\right| \leq E \tag{E12}$$

It was found that this series converges rapidly and greater accuracy could be obtained by carrying the sum out farther. So, for this series E was set equal to $10^{-8}$. The number of terms required for convergence depends on n, but the largest number used was 67 at b/a = .1 and ka = 10. The accuracy obtained by using this criterion for convergence is in the order of .0001.

Since $\Omega_n$ is dependent only on n and not on ka it was found convenient to store $\Omega_n$ for $0 \leq n \leq 70$ in an array and use the values stored for all ka. This decreased the computer running time significantly as compared to calculating the $\Omega_n$ every time, for every ka.