EMP Theoretical Notes

Note VII

MULTIPOLE THEORY IN THE TIME DOMAIN

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ABSTRACT

Spherical outgoing waves of arbitrary time dependence are first written in the usual way as an integral over ω of a multipole expansion. It is then shown that integrals over ω of $e^{-i\omega t} h_l^{(1)}(kr)$ multiplied by an arbitrary function of ω can be replaced by a differential operator operating on an arbitrary function of retarded time. Thus a form of the multipole expansion is obtained which does not explicitly contain the frequency spectrum of the multipoles. Given the value of E_r (for electric multipoles, or B_r for magnetic multipoles) as a function of time on the surface of a sphere, expressions are given for all the field components at all other points in space as functions of time. The method employs a convolution integral and is useful in problems involving a very wide-band frequency spectrum.

I. INTRODUCTION

The classical treatment of spherical waves in terms of a multipole expansion is usually carried out with an assumed sinusoidal time variation. Since the frequency spectrum and phase are arbitrary, the actual time variation (after mathematically performing a Fourier integral) is also arbitrary. However, in some practical radiation problems it is an advantage to carry out calculations in the time domain. In this paper a multipole expansion will be formulated which does not explicitly contain the frequency spectrum. The Fourier integral of the multipole spectrum multiplied by the spherical Hankel function, which appears in the classical formalism, is replaced in the present treatment by a differential operator and an arbitrary function of retarded time. Using the multipole expansion in this form, the problem of extrapolating to larger radii field values given on the surface of a sphere which contains the source can be solved in the time domain, i.e., without Fourier analysis.

The starting point will be Jackson's form of the multipole expansion.[1] Only the electric multipole field will be considered; the magnetic multipoles can be treated in the identical manner with $\vec{E}$ replacing $\vec{B}$ and $-\vec{B}$ replacing $\vec{E}$. Only outward moving waves will be treated in detail; the corresponding

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[1] Jackson, J. D., Classical Electrodynamics (John Wiley & Sons, Inc., New York, 1962), pp. 545, 546.

expressions for inward moving waves are only slightly different; however, there are theoretical difficulties in applying them. They are discussed in the appendix. Some of the methods presented in this paper have been applied to the dipole by Wicklund[2].

II. MULTIPOLE EXPANSION IN FREQUENCY DOMAIN

The electric multipole field for outgoing waves can be written

$$\vec{B} = \sum_{\ell, m} a_E(\ell, m)\, h_\ell^{(1)}(kr)\, \vec{X}_{\ell, m}(\theta, \phi) \quad , \tag{1}$$

$$\vec{E} = \sum_{\ell, m} \frac{i}{k}\, a_E(\ell, m)\, \nabla \times h_\ell^{(1)}(kr)\, \vec{X}_{\ell m}(\theta, \phi) \quad , \tag{2}$$

where $\vec{X}_{\ell m}(\theta, \phi) = (1/\sqrt{\ell(\ell+1)}\,)\, \vec{L}\, Y_{\ell m}(\theta, \phi)$ (the vector spherical harmonic), the time dependence is $e^{-i\omega t}$, $h_\ell^{(1)}$ is the spherical Hankel function, and $a_E(\ell, m)$ are arbitrary complex functions of ω. Explicit expressions for the spherical components of Eqs. (1) and (2) will be obtained. As an intermediate step, the set of basis vectors $(\vec{-}, \vec{+}, \hat{z})$ as used in angular momentum theory will be employed to simplify the algebra. If $\vec{A}$ and $\vec{B}$ are vectors, the following relations hold:

$$\begin{aligned} A_+ &= A_x + i A_y \\ A_- &= A_x - i A_y \end{aligned} \tag{3}$$

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[2] Wicklund, J. S., Extrapolation of the Electromagnetic Field, Diamond Ordnance Fuze Laboratories, TR-1058, 1962.

$$A_x = \frac{1}{2}(A_- + A_+) \tag{4}$$

$$A_y = \frac{i}{2}(A_- - A_+)$$

$$\vec{A} \times \vec{B} = \frac{i}{2} \begin{vmatrix} \vec{-} & \vec{+} & \hat{z} \\ A_+ & A_- & 2A_z \\ B_+ & B_- & 2B_z \end{vmatrix} \tag{5}$$

$$A_r = \frac{1}{2}\sin\theta\,(e^{i\phi} A_- + e^{-i\phi} A_+) + \cos\theta\, A_z \tag{6}$$

$$A_\theta = \frac{1}{2}\cos\theta\,(e^{i\phi} A_- + e^{-i\phi} A_+) - \sin\theta\, A_z$$

$$A_\phi = \frac{i}{2}(e^{i\phi} A_- - e^{-i\phi} A_+) \quad .$$

The ∇ vector operator can be written:

$$\nabla = \frac{\vec{r}}{r}\frac{\partial}{\partial r} - \frac{i}{r^2}\vec{r} \times \vec{L} \quad , \tag{7}$$

where $\vec{L}$ is defined as $\vec{L} = -i(\vec{r} \times \nabla)$. Thus

$$\nabla_- = \sin\theta\, e^{-i\phi}\frac{\partial}{\partial r} + \frac{1}{r}(\sin\theta\, e^{-i\phi} L_z - \cos\theta\, L_-) \tag{8}$$

$$\nabla_+ = \sin\theta\, e^{i\phi}\frac{\partial}{\partial r} + \frac{1}{r}(\cos\theta\, L_+ - \sin\theta\, e^{i\phi} L_z)$$

$$\nabla_z = \cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{2r}(e^{i\phi} L_- - e^{-i\phi} L_+) \quad .$$

Using Eqs. (6) one can write the spherical components of $\vec{B}$ as

$$B_r = 0$$

$$B_\theta = \sum_{\ell, m} \frac{a_E(\ell, m)\, h_\ell^{(1)}(kr)}{\sqrt{\ell(\ell+1)}} \left[\frac{1}{2} \cos\theta \left(\sqrt{(\ell+m)(\ell-m+1)}\; e^{i\phi}\, Y_{\ell, m-1} + \sqrt{(\ell-m)(\ell+m+1)}\; e^{-i\phi}\, Y_{\ell, m+1} \right) - m \sin\theta\; Y_{\ell m} \right]$$

$$B_\phi = \frac{i}{2} \sum_{\ell, m} \frac{a_E(\ell, m)\, h_\ell^{(1)}(kr)}{\sqrt{\ell(\ell+1)}} \left(\sqrt{(\ell+m)(\ell-m+1)}\; e^{i\phi}\, Y_{\ell, m-1} - \sqrt{(\ell-m)(\ell+m+1)}\; e^{-i\phi}\, Y_{\ell, m+1} \right) . \tag{9}$$

Using Eqs. (5) and (8), the expressions for $\sin\theta$ and $\cos\theta$ times the Legendre function from Condon and Shortley[3], and the identities

$$\frac{h_\ell^{(1)}(kr)}{r} = \frac{k}{2\ell+1} \left[h_{\ell-1}^{(1)}(kr) + h_{\ell+1}^{(1)}(kr) \right] \tag{10}$$

and

$$\frac{\partial\, h_\ell^{(1)}(kr)}{\partial\, r} = \frac{k}{2\ell+1} \left[\ell\, h_{\ell-1}^{(1)}(kr) - (\ell+1)\, h_{\ell+1}^{(1)}(kr) \right] \tag{11}$$

taken from Jackson[4], the $\vec{-}$, $\vec{+}$, and $\hat{z}$ components of $\vec{E}$ can be written as follows:

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[3] Condon, E. U., and Shortley, G. H., Theory of Atomic Spectra (Cambridge University Press, London and New York, 1953), p. 53.

[4] Jackson, p. 540.

$$E_- = \sum_{\ell, m} \frac{a_E(\ell, m)}{\sqrt{\ell(\ell+1)}} \left[\sqrt{\frac{(\ell+m)(\ell+m-1)}{(2\ell-1)(2\ell+1)}} \quad (\ell+1)\, h^{(1)}_{\ell-1}(kr)\, Y_{\ell-1, m-1} \right.$$

$$\left. - \sqrt{\frac{(\ell-m+1)(\ell-m+2)}{(2\ell+1)(2\ell+3)}} \quad \ell\, h^{(1)}_{\ell+1}(kr)\, Y_{\ell+1, m-1} \right] ,$$

$$E_+ = \sum_{\ell, m} \frac{a_E}{\sqrt{\ell(\ell+1)}} \left[\sqrt{\frac{(\ell+m+1)(\ell+m+2)}{(2\ell+1)(2\ell+3)}} \quad \ell\, h^{(1)}_{\ell+1}(kr)\, Y_{\ell+1, m+1} \right.$$

$$\left. - \sqrt{\frac{(\ell-m)(\ell-m-1)}{(2\ell-1)(2\ell+1)}} \quad (\ell+1)\, h^{(1)}_{\ell-1}(kr)\, Y_{\ell-1, m+1} \right] ,$$

$$E_z = \sum_{\ell, m} \frac{-a_E(\ell, m)}{\sqrt{\ell(\ell+1)}} \left[\sqrt{\frac{(\ell-m+1)(\ell+m+1)}{(2\ell+1)(2\ell+3)}} \quad \ell\, h^{(1)}_{\ell+1}(kr)\, Y_{\ell+1, m} \right.$$

$$\left. + \sqrt{\frac{(\ell-m)(\ell+m)}{(2\ell-1)(2\ell+1)}} \quad (\ell+1)\, h^{(1)}_{\ell-1}(kr)\, Y_{\ell-1, m} \right] . \qquad (12)$$

The expressions for E_-, E_+, and E_z given in Eqs. (12) can be substituted in Eqs. (6) to yield the spherical components. The identities from Condon and Shortley[5] and those given by Eqs. (10) and (11) can again be applied to yield the following expressions for E_r, E_θ, and E_ϕ.

$$E_r = \sum_{\ell, m} -a_E(\ell, m) \sqrt{\ell(\ell+1)} \frac{h^{(1)}_\ell(kr)}{kr} Y_{\ell m}$$

$$E_\theta = \sum_{\ell, m} \frac{a_E(\ell, m)}{2\sqrt{\ell(\ell+1)}} \frac{1}{(2\ell+1)} \left[\ell\, h^{(1)}_{\ell+1}(kr) - (\ell+1)\, h^{(1)}_{\ell-1}(kr) \right]$$

$$\times \left(\sqrt{(\ell-m)(\ell+m+1)}\; e^{-i\phi}\, Y_{\ell, m+1} - \sqrt{(\ell+m)(\ell-m+1)}\; e^{+i\phi}\, Y_{\ell, m-1} \right)$$

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[5] Condon and Shortley, p. 53.

$$E_\phi = \frac{i}{2} \sum_{\ell, m} \frac{-a_E(\ell, m)}{\sqrt{\ell(\ell+1)}} \left\{ \frac{\ell h^{(1)}_{\ell+1}(kr)}{\sqrt{(2\ell+1)(2\ell+3)}} \right.$$

$$\times \left[\sqrt{(\ell+m+1)(\ell+m+2)} \; e^{-i\phi} \, Y_{\ell+1, m+1} \right.$$

$$\left. + \sqrt{(\ell-m+1)(\ell-m+2)} \; e^{+i\phi} \, Y_{\ell+1, m-1} \right]$$

$$- \frac{(\ell+1) h^{(1)}_{\ell-1}(kr)}{\sqrt{(2\ell-1)(2\ell+1)}} \left[\sqrt{(\ell-m)(\ell-m-1)} \; e^{-i\phi} \, Y_{\ell-1, m+1} \right.$$

$$\left. \left. + \sqrt{(\ell+m)(\ell+m-1)} \; e^{+i\phi} \, Y_{\ell-1, m-1} \right] \right\} . \qquad (13)$$

III. MULTIPOLE EXPANSION IN THE TIME DOMAIN

In general, the multipole amplitudes $a_E(\ell, m)$ can be taken to be arbitrary complex functions of ω. A Fourier transform to the time domain of Eqs. (9) and Eqs. (13) can then be performed to yield multipole expansions of the field components in the time domain. Thus, in Eqs. (9), if $a_E(\ell, m) h^{(1)}_\ell(kr)$ is replaced by

$$\alpha_{EB}(\ell, m, r, t) = \int_{-\infty}^{+\infty} e^{-i\omega t} \, a_E(\ell, m) h^{(1)}_\ell(kr) \, d\omega \qquad (14)$$

and, in Eqs. (13), if $a_E(\ell, m) h^{(1)}_\ell(kr) / kr$, $a_E(\ell, m) h^{(1)}_{\ell+1}(kr)$, and $a_E(\ell, m) h^{(1)}_{\ell-1}(kr)$ are replaced by

$$\alpha_{Er}(\ell, m, r, t) = \int_{-\infty}^{+\infty} e^{-i\omega t} \; a_E(\ell, m) \, \frac{h_\ell^{(1)}(kr)}{kr} \, d\omega \quad , \tag{15}$$

$$\alpha_{E+}(\ell, m, r, t) = \int_{-\infty}^{+\infty} e^{-i\omega t} \; a_E(\ell, m) \, h_{\ell+1}^{(1)}(kr) \; d\omega \quad , \tag{16}$$

$$\alpha_{E-}(\ell, m, r, t) = \int_{-\infty}^{+\infty} e^{-i\omega t} \; a_E(\ell, m) \, h_{\ell-1}^{(1)}(kr) \; d\omega \quad , \tag{17}$$

respectively, the resultant multipole expansions are in the time domain. In the following, expressions will be found for α_{EB}, α_{Er}, α_{E+}, and α_{E-} which do not contain explicitly the frequency spectrum $a_E(\ell, m)$ of the multipoles, but instead contain arbitrary functions of retarded time $\alpha_E(\ell, m, t^*)$.

The spherical Hankel function can be written

$$h_n^{(1)}(kr) = e^{i\omega t} \, \overleftrightarrow{\Xi}_n(r) \left[\frac{e^{-i(\omega t^* + \pi/2)}}{k^{n+1}} \right] \quad , \tag{18}$$

where $t^* = t - r/c$ and $\overleftrightarrow{\Xi}_n(r)$ is the differential operator

$$\Xi_n(r) = \Bigg[\frac{1}{r c^n} \frac{d^n}{dt^{*n}} + \frac{n(n+1)}{2 \cdot 1!\, r^2 c^{n-1}} \frac{d^{n-1}}{dt^{*n-1}} + \frac{n(n^2-1)(n+2)}{2^2 \cdot 2!\, r^3 c^{n-2}} \frac{d^{n-2}}{dt^{*n-2}} + \cdots$$

$$+ \frac{n(n^2-1)(n^2-4)\cdots[n^2-(j-1)^2]\,(n+j)}{2^j \cdot j!\, r^{j+1} c^{n-j}} \frac{d^{n-j}}{dt^{*n-j}} + \cdots$$

$$+ \frac{n(n^2-1)(n^2-4)\cdots[n^2-(n-1)^2]\,2n}{2^n n!\, r^{n+1}} \Bigg] \quad . \tag{19}$$

The variable t is a dummy in Eq. (18). If, at a given constant radius r, a dimensionless retarded time τ is defined by the equation $\tau = ct^*/r$, the operator $\Xi_n(r)$ can be written in the simpler form

$$\Xi_n(r) = \frac{1}{r^{n+1}} \Bigg[\frac{d^n}{d\tau^n} + \frac{n(n+1)}{2 \cdot 1!} \frac{d^{n-1}}{d\tau^{n-1}} + \frac{n(n^2-1)(n+2)}{2^2 \cdot 2!} \frac{d^{n-2}}{d\tau^{n-2}} + \cdots$$

$$+ \frac{n(n^2-1)(n^2-4)\cdots[n^2-(j-1)^2]\,(n+j)}{2^j j!} \frac{d^{n-j}}{d\tau^{n-j}} + \cdots$$

$$+ \frac{n(n^2-1)(n^2-4)\cdots[n^2-(n-1)^2]\,2n}{2^n n!} \Bigg] \quad . \tag{20}$$

For simplicity of notation, let μ_{nj} be the coefficients of Eq. (20) so that that equation may be written

$$\Xi_n(r) = \frac{1}{r^{n+1}} \sum_{j=0}^{n} \mu_{nj} \frac{d^{n-j}}{d\tau^{n-j}} \quad . \tag{21}$$

Substitution of the expression for $h_n^{(1)}$ in Eq. (18) for $h_n^{(1)}$ in Eq. (15) and associating the dummy t with time, one obtains

$$\alpha_{Er}(\ell, m, r, t) = \int_{-\infty}^{+\infty} a_E(\ell, m) \frac{1}{r} \overleftrightarrow{\Xi}_\ell(r) \frac{e^{-i(\omega t^* + \pi/2)}}{k^{\ell+2}} d\omega \quad . \tag{22}$$

The operator, $\frac{1}{r} \overleftrightarrow{\Xi}_\ell(r)$, may be taken out from under the integral sign since it is not a function of ω (assuming that $a_E(\ell, m)$ are well enough behaved functions of ω to allow the change in order of integration and differentiation). The functions $\alpha_E(\ell, m, t^*)$ of retarded time are now defined by

$$\alpha_E(\ell, m, t^*) = \int_{-\infty}^{+\infty} \left[a_E(\ell, m) \; e^{-i(\omega t^* + \pi/2)} / k^{\ell+2} \right] d\omega \quad . \tag{23}$$

Equation (22) can then be written

$$\alpha_{Er}(\ell, m, r, t) = \frac{1}{r} \overleftrightarrow{\Xi}_\ell(r) \; \alpha_E(\ell, m, t^*) \quad . \tag{24}$$

Since the functions $\alpha_E(\ell, m, t^*)$ are Fourier transforms of arbitrary functions of ω, they are arbitrary functions of retarded time. Similarly, one can write

$$\alpha_{EB}(\ell, m, r, t) = \frac{i}{c} \frac{\partial}{\partial t^*} \overleftrightarrow{\Xi}_\ell(r) \; \alpha_E(\ell, m, t^*) \quad , \tag{25}$$

$$\alpha_{E+}(\ell, m, r, t) = \overleftrightarrow{\Xi}_{\ell+1}(r) \; \alpha_E(\ell, m, t^*) \quad , \tag{26}$$

$$\alpha_{E-}(\ell, m, r, t) = \frac{-1}{c^2} \frac{\partial^2}{\partial t^{*2}} \overleftrightarrow{\Xi}_{\ell-1}(r)\ \alpha_E(\ell, m, t^*) \quad . \tag{27}$$

The multipole expansion for the $\vec{B}$ field (Eqs. (9)) can now be written in the time domain as follows:

$$B_r = 0$$

$$B_\theta = \sum_{\ell, m} \frac{1}{\sqrt{\ell(\ell+1)}} \frac{i}{c} \frac{\partial}{\partial t^*} \overleftrightarrow{\Xi}_\ell(r)\ \alpha_E(\ell, m, t^*)$$

$$\times \left[\frac{1}{2} \cos\theta \left(\sqrt{(\ell+m)(\ell-m+1)}\ e^{i\phi}\ Y_{\ell, m-1} + \sqrt{(\ell-m)(\ell+m+1)}\ e^{-i\phi}\ Y_{\ell, m+1} \right) - m \sin\theta\, Y_{\ell m} \right]$$

$$B_\theta = \sum_{\ell, m} \frac{1}{2\sqrt{\ell(\ell+1)}} \frac{i}{c} \frac{\partial}{\partial t^*} \overleftrightarrow{\Xi}_\ell(r)\ \alpha_E(\ell, m, t^*)$$

$$\times \left[\frac{\ell}{\sqrt{(2\ell+1)(2\ell+3)}} \left(\sqrt{(\ell+m+1)(\ell+m+2)}\ e^{-i\phi}\ Y_{\ell+1, m+1} + \sqrt{(\ell-m+1)(\ell-m+2)}\ e^{i\phi} Y_{\ell+1, m-1} \right) \right.$$

$$\left. + \frac{\ell+1}{\sqrt{(2\ell-1)(2\ell+1)}} \left(\sqrt{(\ell-m)(\ell-m-1)}\ e^{-i\phi}\ Y_{\ell-1, m+1} + \sqrt{(\ell+m)(\ell+m-1)}\ e^{i\phi}\ Y_{\ell-1, m-1} \right) \right]$$

$$B_\phi = \sum_{\ell, m} \frac{-1}{2\sqrt{\ell(\ell+1)}} \frac{1}{c} \frac{\partial}{\partial t^*} \overleftrightarrow{\Xi}_\ell(r)\ \alpha_E(\ell, m, t^*)$$

$$\times \left(\sqrt{(\ell+m)(\ell-m+1)}\ e^{i\phi}\ Y_{\ell, m-1} - \sqrt{(\ell-m)(\ell+m+1)}\ e^{-i\phi}\ Y_{\ell, m+1} \right) \tag{28}$$

Likewise, the multipole expansion for the $\vec{E}$ field (Eqs. (13)) can be written

$$E_r = \sum_{\ell, m} - \sqrt{\ell(\ell+1)} \; \frac{1}{r} \overleftrightarrow{\Xi}_{\ell}(r) \; \alpha_E(\ell, m, t^*) \; Y_{\ell m}$$

$$E_\theta = \sum_{\ell, m} \frac{1}{2\sqrt{\ell(\ell+1)}} \frac{1}{(2\ell+1)} \left[\ell \overleftrightarrow{\Xi}_{\ell+1}(r) + \frac{(\ell+1)}{c^2} \frac{\partial^2}{\partial t^{*2}} \overleftrightarrow{\Xi}_{\ell-1}(r) \right] \alpha_E(\ell, m, t^*)$$

$$\times \left(\sqrt{(\ell-m)(\ell+m+1)} \; e^{-i\phi} \; Y_{\ell, m+1} - \sqrt{(\ell+m)(\ell-m+1)} \; e^{+i\phi} \; Y_{\ell, m-1} \right)$$

$$E_\phi = \frac{i}{2} \sum_{\ell, m} \frac{-1}{\sqrt{\ell(\ell+1)}} \left\{ \frac{\ell}{\sqrt{(2\ell+1)(2\ell+3)}} \overleftrightarrow{\Xi}_{\ell+1}(r) \; \alpha_E(\ell, m, t^*) \right.$$

$$\times \left[\sqrt{(\ell+m+1)(\ell+m+2)} \; e^{-i\phi} \; Y_{\ell+1, m+1} + \sqrt{(\ell-m+1)(\ell-m+2)} \; e^{+i\phi} \; Y_{\ell+1, m-1} \right]$$

$$+ \frac{\ell+1}{\sqrt{(2\ell-1)(2\ell+1)}} \frac{1}{c^2} \frac{\partial^2}{\partial t^{*2}} \overleftrightarrow{\Xi}_{\ell-1}(r) \; \alpha_E(\ell, m, t^*)$$

$$\left. \times \left[\sqrt{(\ell-m)(\ell-m-1)} \; e^{-i\phi} \; Y_{\ell-1, m+1} + \sqrt{(\ell+m)(\ell+m-1)} \; e^{+i\phi} \; Y_{\ell-1, m-1} \right] \right\} \quad . \quad (29)$$

If the field to be described is ϕ independent, then m=0 and the multipole expansion reduces to

$$B_r = 0$$

$$B_\theta = 0$$

$$B_\phi = \frac{1}{\sqrt{2\pi}} \sum_\ell \frac{1}{c} \frac{\partial}{\partial t^*} \overleftrightarrow{\Xi}_\ell(r)\ \alpha_E(\ell, 0, t^*)\ \overline{P_\ell^1}$$

$$E_r = \frac{1}{\sqrt{2\pi}} \sum_\ell - \sqrt{\ell(\ell+1)}\ \frac{1}{r} \overleftrightarrow{\Xi}_\ell(r)\ \alpha_E(\ell, 0, t^*)\ \overline{P_\ell^0}$$

$$E_\theta = \frac{1}{\sqrt{2\pi}} \sum_\ell \frac{1}{(2\ell+1)} \left[\ell\ \overleftrightarrow{\Xi}_{\ell+1}(r) + \frac{(\ell+1)}{c^2} \frac{\partial^2}{\partial t^{*2}} \overleftrightarrow{\Xi}_{\ell-1}(r) \right] \alpha_E(\ell, 0, t^*)\ \overline{P_\ell^1} \qquad (30)$$

$$E_\phi = 0 \quad ,$$

where $\overline{P_\ell^m}$ is the normalized Legendre function. Only in the special case of $m=0$, the field values are real if $\alpha_E(\ell, 0, t^*)$ is real. For general values of m the functions $\alpha_E(\ell, m, t^*)$ can be arbitrary complex functions of t^*. The real or imaginary parts of the expressions given in Eqs. (28) and (29) then represent the actual values of the field components. Since the operator $\overleftrightarrow{\Xi}_\ell(r)$ plays a role similar to that of the Hankel function in the frequency domain expressions, it will be referred to as the Hankel operator.

IV. SPHERICAL BOUNDARY-VALUE PROBLEM

Suppose E-M field components are known as functions of time on the surface of a sphere which contains all the sources of the field of interest. Then (it will be shown that) the functions $\alpha_E(\ell, m, t^*)$ and $O_\ell(r)\ \alpha_E(\ell, m, t^*)$, where $O_\ell(r)$ is any of the operators in Eqs. (28) or (29), can be expressed as integrals of the given field. Since integration is a standard operation for electronic computers, the technique presented here lends itself well to numerical techniques.

Only the components E_r will be used to obtain the functions $\alpha_E(\ell, m, t^*)$ and $O_\ell\ \alpha_E(\ell, m, t^*)$. The reason for this is two-fold. First, the dependence of the terms of the expansion of E_r on the angle coordinates θ, ϕ is given simply by the spherical harmonic. Thus, on a sphere of radius r_o, E_r can be expressed in the form

$$E_r(t^*) = \sum_{\ell, m} - \sqrt{\ell(\ell+1)}\ \frac{1}{r_o}\ \beta_E(\ell, m, t^*)\ Y_{\ell, m} \quad , \tag{31}$$

where $\beta_E(\ell, m, t^*)$ is given by

$$\beta_E(\ell, m, t^*) = - \frac{r_o}{\sqrt{\ell(\ell+1)}} \int_{\substack{\text{sphere of} \\ \text{radius } r_o}} E_r(t^*)\ Y^*_{\ell, m}\ d\Omega \quad . \tag{32}$$

Second, the component E_r is due only to the electric multipole; even if a magnetic multipole is present, it does not contribute to E_r. Thus, if both types of multipole sources are assumed present in the same problem, the electric part will be selected from the total field if E_r is used to analyze the fields. The magnetic multipole part can be analyzed in the identical manner by replacing $\vec{E}$ by $-\vec{B}$ and $\vec{B}$ by $\vec{E}$. Thus B_r only would be used to analyze the magnetic multipoles. All fields (and their derivatives) are assumed to be zero initially, i.e., at $t^* = 0$.

Equating the coefficients of $Y_{\ell, m}$ in Eqs. (31) and the first of Eqs. (29) (with $r = r_o$) one obtains

$$\Xi_{\ell}(r_o)\ \alpha_E(\ell, m, t^*) = \beta_E(\ell, m, t^*) \quad . \tag{33}$$

Defining the dimensionless retarded time $\tau_o = t^* c/r_o$, one can write Eq. (33) as

$$\Xi_{\ell}(r_o)\ \alpha'_E(\ell, m, \tau_o) = \beta'_E(\ell, m, \tau_o) \quad , \tag{34}$$

where $\alpha'_E(\ell, m, \tau_o) = \alpha_E(\ell, m, \tau_o r_o/c)$, and $\beta'_E(\ell, m, \tau_o) = \beta_E(\ell, m, \tau_o r_o/c)$.

To solve Eqs. (34) Green's functions will be found that satisfy the equations

$$\Xi_{\ell}(r_o) G_{\ell}(\tau_o, \tau_o') = \delta(\tau_o - \tau_o'), \quad \ell = 0, \cdots \infty \tag{35}$$

with the initial conditions, $G_{\ell}(0, \tau_o') = G_{\ell}^{(1)}(0, \tau_o') = \cdots = G_{\ell}^{(\ell-1)}(0, \tau_o') = 0$,

where $G_{\ell}^{(j)}(0, \tau_o') = d^j G_{\ell}(\tau_o, \tau_o') / d\tau_o^j \Big|_{\tau_o = 0}$. The functions $\alpha'_E(\ell, m, \tau_o)$ will then be given by

$$\alpha'_E(\ell, m, \tau_o) = \int_0^{\tau_o} G_{\ell}(\tau_o, \tau_o')\ \beta'_E(\ell, m, \tau_o')\, d\tau_o' \quad . \tag{36}$$

To solve Eqs. (35) the homogeneous equations must first be solved. Since the equations are linear with constant coefficients, this is merely a matter of finding the roots of the auxiliary equation which, for Eq. (35), is

$$F_\ell(z) = 0 \quad , \tag{37}$$

where
$$F_\ell(z) = \sum_{j=0}^{\ell} \mu_{\ell j} \, z^{\ell - j} \quad .$$

The roots of $F_\ell(z)$ are the roots of $H^{(1)}_{\ell+\frac{1}{2}}(iz)$ where $H^{(1)}_{\ell+\frac{1}{2}}(iz)$ is the half-odd-integer order Hankel function of the first kind. In Jahnke and Emde's notation[6], $F_\ell(z) = z^\ell S_{\ell+\frac{1}{2}}(2z) = z^\ell \sqrt{\frac{1}{2}\pi z}\, \exp(z)\,(i)^{\ell+3/2} H^{(1)}_{\ell+\frac{1}{2}}(iz)$. Note that $H^{(1)}_{\ell+\frac{1}{2}}(iz)$ has a singular point at $z=0$ which annihilates the zero and branch point in its coefficient; thus $F_\ell(z)$ is analytic and nonzero at $z=0$. The general behavior of the roots of $H^{(1)}_{\ell+\frac{1}{2}}(iz)$ can be deduced from the graph on p. 243 of Jahnke and Emde (Ref. 6). It is found that for ℓ odd, $F_\ell(z)$ has one real negative root and $(\ell-1)$ complex roots which appear in complex conjugate pairs and have negative real parts. For ℓ even, all of the ℓ roots of $F_\ell(z)$ are complex (appearing, of course, in complex conjugate pairs) and have negative real parts. Numerical values of the roots for $\ell \leq 16$ are given in the appendix. It is significant to note that all the roots are distinct. Thus, the solution of the homogeneous equation can now be written explicitly. Let $D_\ell(\tau_o)$ satisfy the equation $\overleftrightarrow{\Xi}_\ell(r_o)\, D_\ell(\tau_o) = 0$, then

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[6] Jahnke, E., and Emde, F., Tables of Functions (Dover Publications, 1945), pp. 136-137.

$$D_\ell(\tau_o) = \sum_{j=1}^{\substack{\ell/2,\ \ell\ \text{even} \\ (\ell-1)/2,\ \ell\ \text{odd}}} \exp(p_{\ell j}\tau_o)\,(c_j' \sin q_{\ell j}\tau_o + d_j' \cos q_{\ell j}\tau_o)$$

$$+ f' \exp(p_{\ell,\frac{1}{2}(\ell+1)}\tau_o) \quad ; \tag{38}$$

where $f' = 0$ if ℓ is even; c_j', d_j', and f' are arbitrary constants; the complex roots of $F_\ell(z)$ are given by $p_{\ell j} \pm iq_{\ell j}$; the real root of $F_\ell(z)$ (if ℓ is odd) is $P_{\ell,\frac{1}{2}(\ell+1)}$. To find the Green's functions, the arbitrary constants in Eq. (38) must be evaluated such that the resulting expression satisfies Eqs. (35) with their initial conditions. To facilitate this operation it is convenient to define the constants slightly differently. Let the Green's function be given by

$$G_\ell(\tau_o, \tau_o') = r_o^{\ell+1} \left\{ \sum_{j=1}^{\substack{\ell/2,\ \ell\ \text{even} \\ (\ell-1)/2,\ \ell\ \text{odd}}} \exp\left[p_{\ell j}(\tau_o - \tau_o')\right] \left[c_{\ell j} \sin q_{\ell j}(\tau_o - \tau_o') \right.\right.$$

$$\left.\left. + d_{\ell j} \cos q_{\ell j}(\tau_o - \tau_o')\right] + f_\ell \exp\left[p_{\ell,\frac{1}{2}(\ell+1)}(\tau_o - \tau_o')\right] \right\} , \tag{39}$$

where, again, $f_\ell = 0$ if ℓ is even. The derivatives of this expression for $G_\ell(\tau_o, \tau_o')$ can be written

$$\frac{d^k G_\ell(\tau_o, \tau_o')}{d\tau_o^k} = r_o^{\ell+1} \left\{ \sum_{j=1}^{\substack{\ell/2,\ \ell\ \text{even} \\ (\ell-1)/2,\ \ell\ \text{odd}}} r_{\ell j}^k \exp\left[p_{\ell j}(\tau_o - \tau_o')\right] \right.$$

$$\times \left[(d_{\ell j} \cos k\theta_{\ell j} + c_{\ell j} \sin k\theta_{\ell j}) \quad \cos q_{\ell j} (\tau_o - \tau_o') \right. \tag{40}$$

$$\left. + \quad (c_{\ell j} \cos k\, \theta_{\ell j} - d_{\ell j} \sin k\theta_{\ell j}) \quad \sin q_{\ell j} (\tau_o - \tau_o') \right]$$

$$\left. + \quad f_\ell \, p^k_{\ell, \frac{1}{2}(\ell+1)} \; \exp \left[p_{\ell, \frac{1}{2}(\ell+1)} (\tau_o - \tau_o') \right] \right\} \; ,$$

where $r_{\ell j} \exp (i\theta_{\ell j}) = p_{\ell j} + iq_{\ell j}$. The initial conditions below Eq. (35) state that, if two sets of constants $c_{\ell j}$, $d_{\ell j}$, and f_ℓ are defined, the first applying for $0 \leq \tau_o < \tau_o'$ and another for $0 < \tau_o' < \tau_o$, then the first set is given by

$$\sum_{j=1}^{\substack{\ell/2,\ \ell \text{ even} \\ (\ell-1)/2,\ \ell \text{ odd}}} r^k_{\ell j} \exp (-p_{\ell j}\, \tau_o') \left[(d_{\ell j} \cos k\theta_{\ell j} + c_{\ell j} \sin k\theta_{\ell j}) \cos (-q_{\ell j}\, \tau_o') \right.$$

$$\left. + \quad (c_{\ell j} \cos k\theta_{\ell j} - d_{\ell j} \sin k\theta_{\ell j}) \sin (-q_{\ell j}\, \tau_o') \right] \tag{41}$$

$$+ \quad f_\ell \, p^k_{\ell, \frac{1}{2}(\ell+1)} \exp (-p_{\ell, \frac{1}{2}(\ell+1)}\, \tau_o') = 0, \; k = 0, 1, \cdots, \ell - 1$$

for all τ_o' (>0). Thus, for $\tau_o < \tau_o'$, all the constants are zero and the Green's function is identically zero. Integration of both sides of Eq. (35) from $\tau_o' - \epsilon$ to $\tau_o' + \epsilon$ and taking the limit as $\epsilon \longrightarrow 0$ reveals that $d^{\ell-1} G_\ell (\tau_o, \tau_o') / d\tau_o^{\ell-1}$ must have a positive discontinuous jump of magnitude $r_o^{\ell+1}$ at the point $\tau_o = \tau_o'$. Thus

$$\lim_{\epsilon \longrightarrow 0} \; d^{\ell-1} G_\ell (\tau_o, \tau_o') / d\,\theta_o^{\ell-1} \Bigg|_{\tau_o = \tau_o' + \epsilon} = r_o^{\ell+1} \quad .$$

Since the lower order derivatives must be continuous at $\tau_o = \tau_o'$ to satisfy Eq. (35), the equations determining the constants in the Green's function for $\tau_o > \tau_o'$ are

$$\sum_{j=1}^{\substack{\ell/2,\ \ell \text{ even} \\ (\ell-1)/2,\ \ell \text{ odd}}} r_{\ell j}^{k} \left(d_{\ell j} \cos k\theta_{\ell j} + c_{\ell j} \sin k\theta_{\ell j}\right) + f_{\ell}\, p_{\ell, \frac{1}{2}(\ell+1)}^{k} = \delta_{k, \ell-1} \quad , \tag{42}$$

$$k = 0,\ 1,\ \cdots,\ \ell - 1 \quad ,$$

where δ_{ij} is the Kronecker delta.

Now that the Green's functions are determined, the next step will be to find explicit expressions for $O_{\ell}(r)\ \alpha_E'(\ell, m, \tau_o)$, where again, $O_{\ell}(r)$ is any of the operators appearing in Eqs. (28) and (29). Terms of the form

$$\frac{1}{r^{\ell+1}} \frac{d^k \alpha_E'(\ell, m, \tau_o)}{d\tau^k} \quad , \quad k = 1,\ 2,\ \cdots,\ \ell+1$$

must be evaluated. By succeeding differentiations of Eq. (36) one obtains

$$\frac{d^k \alpha_E'(\ell, m, \tau_o)}{d\tau_o^k} = \int_0^{\tau_o} G_{\ell}^{(k)}(\tau_o, \tau_o')\, \beta_E'(\ell, m, \tau_o')\, d\tau_o', \quad k = 1, \cdots, \ell-1 \tag{43}$$

$$\frac{d^{\ell} \alpha_E'(\ell, m, \tau_o)}{d\tau_o^{\ell}} = \int_0^{\tau_o} G_{\ell}^{(\ell)}(\tau_o, \tau_o')\, \beta_E'(\ell, m, \tau_o')\, d\tau_o')\, d\tau_o' + r_o^{\ell+1}\, \beta_E'(\ell, m, \tau_o)$$

$$\frac{d^{\ell+1} \alpha_E'(\ell, m, \tau_o)}{d\tau_o^{\ell+1}} = \int_0^{\tau_o} G_{\ell}^{(\ell+1)}(\tau_o, \tau_o')\, \beta_E'(\ell, m, \tau_o')\, d\tau_o'$$

$$+ \quad r_o^{\ell+1} \left[\frac{d\beta_E'(\ell, m, \tau_o)}{d\tau_o} - \frac{1}{2}\ell(\ell+1)\beta_E'(\ell, m, \tau_o) \right] ,$$

where $G_\ell^{(j)}(\tau_o, \tau_o') = d^j G_\ell(\tau_o, \tau_o') / d\tau_o^j$. Noting that $\tau_o = \tau r/r_o$ and hence

that $\frac{d^k}{d\tau^k} = \left(\frac{r}{r_o}\right)^k \frac{d^k}{d\tau_o^k}$, one can write

$$\frac{1}{r^{\ell+1}} \frac{d^k \alpha_E'}{d\tau^k} = \frac{1}{r^{\ell-k+1} r_o^k} \left\{ \int_0^{\tau r/r_o} G_\ell^{(k)}(\tau r/r_o, \tau_o')\beta_E'(\ell, m, \tau_o')\, d\tau_o' \right. \tag{44}$$

$$\left. + \delta_{k\ell}\, r_o^{\ell+1} \beta_E'(\ell, m, \tau_o) + \delta_{k,\ell+1}\, r_o^{\ell+1} \left[\frac{d\beta_E'(\ell, m, \tau_o)}{d\tau_o} - \frac{1}{2}\ell(\ell+1)\beta_E'(\ell, m, \tau_o) \right] \right\} ,$$

$$k = 1, 2, \cdots, \ell+1 .$$

To obtain the final expression for the terms of $O_\ell(r)\, \alpha_E'$, the expression for $G_\ell^{(k)}$ given by Eq. (40) is substituted in Eq. (44) which yields

$$\frac{1}{r^{\ell+1}} \frac{d^k \alpha_E'}{d\tau^k} = \left(\frac{r_o}{r}\right)^{\ell-k+1} \left\{ \int_0^{t^* c/r_o} \left[\sum_{j=1}^{\substack{\ell/2,\ \ell \text{ even} \\ (\ell-1)/2,\ \ell \text{ odd}}} r_{\ell j}^k \right. \right.$$

$$\times \exp\left[p_{\ell j} \left(\frac{t^* c}{r_o} - \tau_o' \right) \right] \left[(d_{\ell j} \cos k\theta_{\ell j} + c_{\ell j} \sin k\theta_{\ell j}) \cos q_{\ell j} \right.$$

$$\times \left(\frac{t^* c}{r_o} - \tau_o' \right) + (c_{\ell j} \cos k\theta_{\ell j} - d_{\ell j} \sin k\theta_{\ell j}) \sin q_{\ell j} \left. \left(\frac{t^* c}{r_o} - \tau_o' \right) \right] \tag{45}$$

$$\left. + f_\ell\, p_{\ell, \frac{1}{2}(\ell+1)}^k \exp\left[p_{\ell, \frac{1}{2}(\ell+1)} \left(\frac{t^* c}{r_o} - \tau_o' \right) \right] \right] \beta_E'(\ell, m, \tau_o')\, d\tau_o'$$

$$+ \quad \delta_{k\ell}\, \beta_E'(\ell, m, t^* c/r_o) + \delta_{k,\ell+1} \left[\frac{d\beta_E'(\ell, m, \tau_o')}{d\tau_o} - \frac{1}{2}\ell(\ell+1)\beta_E'(\ell, m, \tau_o)\right]\Bigg\},$$

$$k = 1, 2, \cdots, \ell+1 \quad .$$

Note that the integral in Eq. (45) is independent of r, that is, independent of the radius of observation of the field. Hence, for a given source, the integration need only be performed once to give field values everywhere outside the sphere.

The integrals required in Eq. (45) are

$$I_s(\ell, m, j, t^*) = \int_0^{\frac{t^* c}{r_o}} \sin q_{\ell j}\left(\frac{t^* c}{r_o} - \tau_o'\right) \exp\left[p_{\ell j}\left(\frac{t^* c}{r_o} - \tau_o'\right)\right] \beta_E'(\ell, m, \tau_o')\, d\tau_o' \tag{46}$$

$$I_c(\ell, m, j, t^*) = \int_0^{\frac{t^* c}{r_o}} \cos q_{\ell j}\left(\frac{t^* c}{r_o} - \tau_o'\right) \exp\left[p_{\ell j}\left(\frac{t^* c}{r_o} - \tau_o'\right)\right] \beta_E'(\ell, m, \tau_o')\, d\tau_o', \tag{47}$$

where $j = 1, 2, \cdots, \ell/2$ if ℓ is even; if ℓ is odd $j = 1, 2, \cdots, (\ell+1)/2$ and $q_{\ell,\frac{1}{2}(\ell+1)}$ is taken to be zero ($p_{\ell,\frac{1}{2}(\ell+1)}$ is, of course, the real root of $F_\ell(y) = 0$). The variable of integration in Eqs. (46) and (47) can be changed back to t^* if that is more convenient in a practical problem.

One can now write the expansions for B_θ, B_ϕ, E_r, E_θ, and E_ϕ at arbitrary radius r in terms of the integrals (46) and (47). Let

$$B_{\ell m}(t^*) = \frac{1}{c}\, \frac{\partial}{\partial t^*}\, \Xi_\ell(r)\, \alpha_E(\ell, m, t^*)$$

$$= \frac{1}{r} \frac{\partial}{\partial \tau} \overleftrightarrow{\Xi}_{\ell}(r) \; \alpha_E(\ell, m, t^*)$$

$$E_{r\ell m} = \frac{1}{r} \overleftrightarrow{\Xi}_{\ell}(r) \; \alpha_E(\ell, m, t^*)$$

$$E_{\ell m+} = \overleftrightarrow{\Xi}_{\ell+1}(r) \; \alpha_E(\ell, m, t^*)$$

$$E_{\ell m-} = \frac{1}{c^2} \frac{\partial^2}{\partial t^{*2}} \overleftrightarrow{\Xi}_{\ell-1}(r) \; \alpha_E(\ell, m, t^*)$$

$$= \frac{1}{r^2} \frac{\partial^2}{\partial \tau^2} \overleftrightarrow{\Xi}_{\ell-1}(r) \; \alpha_E(\ell, m, t^*) \qquad . \tag{48}$$

Substituting the expressions for the derivatives of α_E from Eqs. (45) into Eq. (21) and using I_s and I_c to represent the integrals of Eqs. (46) and (47), one can write $B_{\ell m}$, $E_{r\ell m}$, $E_{\ell m+}$ and $E_{\ell m-}$ as follows:

$$B_{\ell m}(t^*) = \frac{1}{r} \left[\frac{1}{2} \ell(\ell+1) \left(\frac{r_o}{r} - 1 \right) \beta_E(\ell, m, t^*) + \frac{r_o}{c} \frac{d\beta_E(\ell, m, t^*)}{d t^*} \right]$$

$$+ \frac{1}{r} \sum_{i=0}^{\ell} \mu_{\ell i} \, F_{\ell i m}(r, t^*) \tag{49}$$

$$E_{r\ell m}(t^*) = \frac{r_o}{r^2}\,\beta_E(\ell, m, t^*)$$

$$+ \quad \frac{1}{r}\sum_{i=0}^{\ell} \mu_{\ell i}\, F_{\ell, i+1, m}(r, t^*) \tag{50}$$

$$E_{\ell m+} = \frac{(\ell+1)}{2r}\left[(\ell+2)\,\frac{r_o}{r} - \ell\right]\beta_E(\ell, m, t^*) + \frac{r_o}{rc}\,\frac{d\beta_E(\ell, m, t^*)}{dt^*}$$

$$+ \quad \frac{1}{r}\sum_{i=0}^{\ell+1} \mu_{\ell+1, i}\, F_{\ell i m}(r, t^*) \quad , \tag{51}$$

$$E_{\ell m-} = \frac{\ell}{2r}\left[(\ell-1)\,\frac{r_o}{r} - \ell - 1\right]\beta_E(\ell, m, t^*) + \frac{r_o}{rc}\,\frac{d\beta_E(\ell, m, t^*)}{dt^*}$$

$$+ \quad \frac{1}{r}\sum_{i=0}^{\ell-1} \mu_{\ell-1, i}\, F_{\ell i m}(r, t^*) \quad , \tag{52}$$

where

$$F_{\ell i m}(r, t^*) = \left(\frac{r_o}{r}\right)^i \left\{ \sum_{j=1}^{\substack{\ell/2,\ \ell \text{ even} \\ (\ell-1)/2,\ \ell \text{ odd}}} r_{\ell j}^{(\ell-i+1)} \right.$$

$$\times \left[\left(d_{\ell j}\cos(\ell-i+1)\ \theta_{\ell j} + c_{\ell j}\sin(\ell-i+1)\ \theta_{\ell j}\right) I_c(\ell, m, j, t^*) \right.$$

$$\left. + \left(c_{\ell j}\cos(\ell-i+1)\ \theta_{\ell j} - d_{\ell j}\sin(\ell-i+1)\ \theta_{\ell j}\right) I_s(\ell, m, j, t^*) \right]$$

$$\left. + \quad f_\ell\, p_{\ell, \frac{1}{2}(\ell+1)}^{(\ell-i+1)} \quad I_c(\ell, m, \tfrac{1}{2}(\ell+1), t^*) \right\} \quad . \tag{53}$$

The field components are then given by Eqs. (28) and (29) with $B_{\ell m}$, $E_{r\ell m}$, $E_{\ell m+}$ and $E_{\ell m-}$ substituted for the quantities they represent.

V. SUMMARY OF STEPS TO SOLVE A FIELD EXTRAPOLATION PROBLEM

This section is a summary of steps that must be taken to solve a field extrapolation problem using the methods given.

1. The roots of the Hankel function (eq. (37)) must first be determined for the orders of interest. These are given in the appendix for $\ell = 1, 2, \cdots, 16$.

2. The Green's function constants must be found. That is, for each order (ℓ), the system of ℓ simultaneous linear equations given by Eqs. (42) must be solved for the ℓ unknowns denoted by $d_{\ell j}$, $c_{\ell j}$ and f_ℓ.

3. The spherical harmonic expansion of E_r on the sphere on which it is known must be obtained; that is, the coefficients $\beta_E(\ell, m, t^*)$ of Eq. (31) must be obtained by evaluating the integral of Eq. (32) or by some other means.

4. The convolution integrals given by Eqs. (46) and (47) must then be evaluated.

5. The quantity in the braces of Eq. (53) can then be evaluated as a function of the indices ℓ, i and m and the independent variable t^*.

These five steps can be carried out before choosing a point to which the field is to be extrapolated. After choosing such a point (r, θ, ϕ), one can proceed to step six.

6. The quantities $F_{\ell i m}(r, t^*)$ can be computed by multiplying the results of step five by $(r_o/r)^i$. The functions $F_{\ell i m}(r, t^*)$ can then be substituted into Eqs. (49) through (52).

7. The extrapolated field values are then given by (from Eqs. (28) and (29))

$$B_\theta = \sum_{\ell, m} \frac{i B_{\ell m}}{2\sqrt{\ell(\ell+1)}} \left[\frac{\ell}{\sqrt{(2\ell+1)(2\ell+3)}} \left(\sqrt{(\ell+m+1)(\ell+m+2)}\; e^{-i\phi}\; Y_{\ell+1, m+1} \right.\right.$$

$$\left. + \sqrt{(\ell-m+1)(\ell-m+2)}\; e^{i\phi}\; Y_{\ell+1, m-1} \right)$$

$$+ \frac{\ell+1}{\sqrt{(2\ell-1)(2\ell+1)}} \left(\sqrt{(\ell-m)(\ell-m-1)}\; e^{-i\phi}\; Y_{\ell-1, m+1} \right.$$

$$\left.\left. + \sqrt{(\ell+m)(\ell+m-1)}\; e^{i\phi}\; Y_{\ell-1, m-1} \right) \right]$$

$$B_\phi = \sum_{\ell, m} \frac{-B_{\ell m}}{2\sqrt{\ell(\ell+1)}} \left[\sqrt{(\ell+m)(\ell-m+1)}\; e^{i\phi}\; Y_{\ell, m-1} \right.$$

$$\left. - \sqrt{(\ell-m)(\ell+m+1)}\; e^{-i\phi}\; Y_{\ell, m+1} \right]$$

$$E_r = \sum_{\ell, m} - \sqrt{\ell(\ell+1)}\; E_{r\ell m}\; Y_{\ell m}$$

$$E_\theta = \sum_{\ell, m} \frac{1}{2\sqrt{\ell(\ell+1)}} \frac{1}{(2\ell+1)} \left[\ell\, E_{\ell m+} + (\ell+1)\, E_{\ell m-} \right]$$

$$\times \left[\sqrt{(\ell-m)(\ell+m+1)}\; e^{-i\phi}\, Y_{\ell,\, m+1} - \sqrt{(\ell+m)(\ell-m+1)}\; e^{i\phi}\, Y_{\ell,\, m-1} \right]$$

$$E_\phi = \frac{i}{2} \sum_{\ell,\, m} \frac{-1}{\sqrt{\ell(\ell+1)}} \left\{ \frac{\ell\, E_{\ell m+}}{\sqrt{(2\ell+1)(2\ell+3)}} \left[\sqrt{(\ell+m+1)(\ell+m+2)}\; e^{-i\phi}\, Y_{\ell+1,\, m+1} \right. \right.$$

$$\left. + \sqrt{(\ell-m+1)(\ell-m+2)}\; e^{+i\phi}\, Y_{\ell+1,\, m-1} \right]$$

$$+ \frac{(\ell+1)\, E_{\ell m-}}{\sqrt{(2\ell-1)(2\ell+1)}} \left[\sqrt{(\ell-m)(\ell-m-1)}\; e^{-i\phi}\, Y_{\ell-1,\, m+1} \right.$$

$$\left. \left. + \sqrt{(\ell+m)(\ell+m-1)}\; e^{i\phi}\, Y_{\ell-1,\, m-1} \right] \right\} .$$

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APPENDIX

A. <u>Incoming Waves</u>. If Eqs. (1) and (2) are rewritten with $h_\ell^{(2)}$ substituted for $h_\ell^{(1)}$, they then represent incoming waves. All the equations in Section II can be rewritten with $h_\ell^{(2)}$ instead of $h_\ell^{(1)}$; they then apply to incoming waves. Since $h_\ell^{(2)}$ (kr) can be written

$$h_\ell^{(2)}(kr) = e^{i\omega t}\,\overleftrightarrow{\Xi}_\ell^{(2)}(r)\left[\frac{e^{-i(\omega t_a^* - \pi/2)}}{k^{\ell+1}}\right] \quad , \qquad (54)$$

where $\overleftrightarrow{\Xi}_\ell^{(2)}(r) = \overleftrightarrow{\Xi}_\ell(r)\Big|_{t^* = -t_a^*}$

$$= \sum_{j=0}^{\ell} \mu_{\ell j}(-1)^{\ell-j}\,\frac{d^{\ell-j}}{dt_a^{*\,\ell-j}} \quad ,$$

$$t_a^* = t + r/c \quad ,$$

the equations of Section III can be written for incoming waves by substituting t_a^* for t^*, $\overleftrightarrow{\Xi}_\ell^{(2)}(r)$ for $\overleftrightarrow{\Xi}_\ell(r)$ (or equivalently $\mu_{\ell j}(-1)^{\ell-j}$ for $\mu_{\ell j}$),

and $\left[\dfrac{e^{-i(\omega t_a^* - \pi/2)}}{k^{\ell+1}}\right]$ for $\left[\dfrac{e^{-i(\omega t^* + \pi/2)}}{k^{\ell+1}}\right]$.

In solving the boundary-value problem of Section IV, the auxiliary equation (37) is the same except for a change in sign of the coefficients of the odd powers of z. Thus the roots are the same as for outgoing waves except for a change of sign of the real parts. Thus the Green's function contains exponentials increasing in time instead of decreasing as in the outgoing wave treatment. This comes about because incoming waves are related to outgoing waves basically by a time reversal. The Green's functions are not time reversals to each other because the boundary conditions forced on them are not related by a time reversal. For outgoing waves $G_\ell(\tau_o, \tau_o')$ was found to be zero for $\tau_o \overline{<} \tau_o'$; for $\tau_o > \tau_o'$ it is nonzero, but exponentially decaying. For incoming waves the Green's functions $G_\ell^{(2)}(\tau_o, \tau_o')$ are again zero for $\tau_o \overline{<} \tau_o'$; for $\tau_o > \tau_o'$, however, the functions $G_\ell^{(2)}$ exponentially rise.

B. Solutions of Scalar Wave Equation. Note that $(r E_r)$ satisfies the scalar wave equation. It follows that the formalism and solution of the spherical boundary-value problem for $(r E_r)$ can be applied to any quantity satisfying the scalar wave equation, i.e., the rectangular field or vector potential components.

C. Roots of the Hankel Function. Using double precision on an IBM 7044 computer, roots of Eq. (37) were obtained through the sixteenth order. Greater computer precision would be needed to obtain them beyond the sixteenth order. For ℓ=1, the root is -1, for ℓ=2, the roots are $-3/2 \pm i\sqrt{3}$. Table I contains the roots that were found numerically.

TABLE I

Roots of $H^{(1)}_{\ell+\frac{1}{2}}(iz) = 0$

Order (ℓ)	Real part of z	Imaginary part of z
3	-2. 322185	0.
3	-1. 838907	1. 754381
4	-2. 896211	0. 867234
4	-2. 103789	2. 657418
5	-3. 646739	0.
5	-3. 351956	1. 742661
5	-2. 324674	3. 571023
6	-4. 248359	0. 867510
6	-3. 735708	2. 626272
6	-2. 515932	4. 492673
7	-4. 971787	0.
7	-4. 758290	1. 739286
7	-4. 070139	3. 517174
7	-2. 685677	5. 420694
8	-5. 587886	0. 867614
8	-5. 204841	2. 616175
8	-4. 368289	4. 414442
8	-2. 838984	6. 353911
9	-6. 297019	0.
9	-6. 129368	1. 737848
9	-5. 604422	3. 498157
9	-4. 638440	5. 317272
9	-2. 979261	7. 291464
10	-6. 922050	0. 867690
10	-6. 615282	2. 611555
10	-5. 967534	4. 384950
10	-4. 886218	6. 224985
10	-3. 108916	8. 232699
11	-7. 622450	0.
11	-7. 484148	1. 737140
11	-7. 057923	3. 488977
11	-6. 301334	5. 276207
11	-5. 115647	7. 137018
11	-3. 229722	9. 177112

Order (ℓ)	Real part of z	Imaginary part of z
12	-8. 253457	0. 867839
12	-7. 997204	2. 608989
12	-7. 465614	4. 370186
12	-6. 610991	6. 171537
12	-5. 329710	8. 052905
12	-3. 343023	10. 124297
13	-8. 947802	0.
13	-8. 830184	1. 736704
13	-8. 470615	3. 483830
13	-7. 844380	5. 254921
13	-6. 900370	7. 070641
13	-5. 530681	8. 972248
13	-3. 449867	11. 073928
14	-9. 583335	0. 868314
14	-9. 362826	2. 607241
14	-8. 911220	4. 361654
14	-8. 198775	6. 143068
14	-7. 172405	7. 973204
14	-5. 720353	9. 894709
14	-3. 551087	12. 025738
15	-10. 273503	0.
15	-10. 170628	1. 736566
15	-9. 859659	3. 480484
15	-9. 323611	5. 242350
15	-8. 532440	7. 034373
15	-7. 429402	8. 878983
15	-5. 900151	10. 819999
15	-3. 647357	12. 979501
16	-10. 914145	0. 875305
16	-10. 714492	2. 602741
16	-10. 328305	4. 356535
16	-9. 711228	6. 126361
16	-8. 848105	7. 928469
16	-7. 673256	9. 787751
16	-6. 071237	11. 747872
16	-3. 739232	13. 935028