EMP Theoretical Notes

Note XXII

11 April 1963

Spherical Coordinate Expansion of Maxwell's Equations for Computer Solution

1/Lt Peter M. Livingston

Research Directorate

Air Force Special Weapons Center
(presently part of the Air Force Weapons Laboratory)

Abstract

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Capt J. H. Darrah
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Editors

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In MKS units, Maxwell's equations are:

(1) $\text{curl } \underline{B} = \mu \left(\underline{J}_c + \sigma \underline{E} + \varepsilon \frac{\partial}{\partial t} \underline{E}\right)$

(2) $\text{curl } \underline{E} = - \frac{\partial}{\partial t} \underline{B}$

(3) $\text{div } \underline{B} = 0$, and $\text{div } \underline{D} = \rho$; $\underline{D} = \varepsilon \underline{E}$

The EMP problem shall be defined as follows: Consider space divided into two semi-infinite half-spaces, called the "upper" (air) half-space, and the "lower" (ground) half-space. The source of Compton current will be located at the origin at the interface of the two half-spaces. If $\hat{z}$ is taken as the direction perpendicular to the interface, and pointing into the "upper" (air) half-space, then a right-handed polar coordinate system (r, θ, ϕ) may be described.

The EMP problem is further specified by requiring of statement of μ, ε, σ, and $\underline{J}_c$ in all space-time. This will uniquely determine the fields (to first order). Furthermore, if these quantities are independent of the azimuthal angle, ϕ, then the field components will also be independent of ϕ. Thus, for

a. $0 \leq \theta \leq \pi/2$ (air-half-space)

$$\mu = \mu_o;\ \varepsilon^I = \varepsilon_o;\ \sigma^I = \sigma(r, \theta, t);\ \underline{J}_c = \underline{J}_c\ (r, \theta, t)$$

and

b. $\pi/2 \leq \theta \leq \pi$ (ground half-space)

$$\mu = \mu_o;\ \varepsilon^{II} = K\varepsilon_o;\ \sigma^{II} = C (\text{a constant});\ \underline{J}_c = 0$$

where K is the specific inductive capacity of the ground. These specifications imply that

a. $B_r = E_\phi = B_\theta = 0$

b. $\frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\, B_\phi) = \mu \left(J_{c,r} + \sigma E_r + \varepsilon \frac{\partial}{\partial t} E_r\right)$

c. $-\frac{1}{r} \frac{\partial}{\partial r} (r\, B_\phi) = \mu \left(\sigma E_\theta + \varepsilon \frac{\partial}{\partial t} E_\theta\right)$

d. $\frac{\partial}{\partial r} (r\, E_\theta) - \frac{\partial}{\partial\theta} E_r = - \frac{\partial}{\partial t} (r\, B_\phi)$

where, b, c, and d, of course, are equations 1, 2, and 3 written in polar coordinates using condition a.

We shall assume the following about the sources:

a. $\sigma(r, \theta, t) = \sigma(r, t)\, \kappa(\theta, \pi/2) + \sigma^{II}(1 - \kappa(\theta, \pi/2))$

b. $\underline{J}_c\ (r, \theta, t) = J_{c,r}\ (r, t)\, \kappa(\theta, \pi/2)\, \hat{r}$

Here $\kappa(\theta, \pi/2)$ is the Heaviside step function, and σ^{II} is a constant.

Boundary conditions. The initial boundary conditions on the field components is that they are zero on and outside the light cone.

a. $B_\phi\ (r, \theta, t) = E_r\ (r, \theta, t) = E_\theta\ (r, \theta, t) = 0, \ r \geq ct.$

The spatial boundary conditions are taken at the origin, $r = 0$, and on the plane dividing "air" and "ground," $\theta = \pi/2$. In the first case, we shall require that the fields be finite at the origin:

b. $rB_\phi \Big|_{r=0} = r\ E_\theta \Big|_{r=0} = E_r \Big|_{r=0} = 0$

This implies that the driving functions $J_{c,r}\ (r, t)$ and $\sigma\ (r, t)$ cannot diverge faster than $r^{-2+\varepsilon}$, $\varepsilon > 0$ as $r \to 0$. In the second case, we shall assume there are no true sheet currents, nor true surface charge at the $\theta = \pi/2$ interface. This implies the following field connection formulae:

c. $B_\phi^{\ upper} = B_\phi^{\ lower}$; $\left[\sigma\ (r,t) + \varepsilon_o \frac{\partial}{\partial t}\right] E_\theta^{\ upper} = \left[\sigma^{II} + K\varepsilon_o \frac{\partial}{\partial t}\right] E_\theta^{\ lower}$;
$E_r^{\ upper} = E_r^{\ lower}$

all evaluated at $\theta = \pi/2$. This completes the unique and general specification of the EMP problem.

Angle Dependence. In preparing these equations along with the boundary conditions for machine solution, it is convenient to eliminate the θ dependence. This is accomplished by expanding the field quantities in Legendre polynomials while noting the following raising and lowering operations:

$$(4) \quad P_\ell^{(1)}\ (\cos\ \theta) = -\frac{d}{d\theta} P_\ell^{(0)}\ (\cos\ \theta)$$

$$(5) \quad \ell(\ell + 1)\ P_\ell^{(0)}\ (\cos\ \theta) = \frac{1}{\sin\ \theta} \frac{d}{d\theta} (\sin\ \theta\ P_\ell^{(1)}\ (\cos\ \theta))$$

The field quantities, and the step function are expanded:

a. $$B_\phi\ (r, \theta, t) = \sum_{n=1} B_n(r, t)\ P_n^{(1)}\ (\cos\ \theta)$$

b. $$E_\theta\ (r, \theta, t) = \sum_{n=1} E_n^{(\theta)}\ (r, t)\ P_n^{(1)}\ (\cos\ \theta)$$

c. $$E_r\ (r, \theta, t) = \sum_{n=0} E_n^{(r)}\ (r, t)\ P_n^{(0)}\ (\cos\ \theta)$$

d. $$\kappa\ (\theta, \pi/2) = \frac{1}{2} - \sum_{\substack{n=1 \\ (odd)}} \frac{2n + 1}{2n(n+1)}\ P_n^{(1)}(0)\ \ P_n^{(0)}\ (\cos\ \theta)$$

In order to represent products like $\sigma \underline{E} (r, \theta, t)$, it is necessary to have the following decomposition rule:

$$(6) \quad P_{\ell}^{(1)} (\cos \theta) P_{\ell'}^{(0)} (\cos \theta) = \sum_{L} S_{L,0,0}^{\ell,\ell'} S_{L,1,0}^{\ell,\ell'} P_{L}^{(1)} (\cos \theta) \left[\frac{\ell(\ell+1)}{L(L+1)}\right]^{1/2}$$

$$(7) \quad P_{\ell}^{(0)} (\cos \theta) P_{\ell'}^{(0)} (\cos \theta) = \sum_{L} (S_{L,0,0}^{\ell,\ell'})^2 P_{L}^{(0)} (\cos \theta)$$

where S_{cde}^{ab} are the five-index Wigner coefficients. This rule follows from the fact that the product of irreducible representations is reducible and is a linear combination of them. Upon introduction of these series into Maxwell's equations, we obtain

$$(8) \quad \frac{n(n+1)}{r} B_n (r,t) = \mu \left[J_n (r,t) + \varepsilon \frac{\partial}{\partial t} E_n^{(r)} (r,t) + \sum_{P,\ell} \sigma_P E_\ell^{(r)} (r,t) (S_{n,0,0}^{P,\ell})^2\right]$$

$$(9) \quad -\frac{1}{r}\frac{\partial}{\partial r} (r B_n (r,t)) = \mu\left[\sum_{P,\ell} \sigma_P E_\ell^{(\theta)} (r,t) S_{n,0,0}^{\ell,P} S_{n,1,0}^{\ell,P} \left[\frac{\ell(\ell+1)}{n(n+1)}\right]^{1/2} + \varepsilon \frac{\partial}{\partial t} E_n^{(\theta)} (r,t)\right]$$

$$(10) \quad \frac{\partial}{\partial r} (r E_n^{(\theta)} (r,t)) + E_n^{(r)} (r,t) = -\frac{\partial}{\partial t} (r B_n (r,t))$$

In the above equations,

$$(11) \quad J_n = J_{c,r}(r,t) \frac{1}{2}; \quad n = 0$$

$$J_n = J_{c,r}(r,t) (-1) \frac{2n+1}{2n(n+1)} P_n^{(1)}(0) P_n^{(0)} (\cos \theta); \quad n = \text{odd}$$

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This concludes the discussion of the general case.

The special case of method of images on an infinitely conducting ground plane. This assumption greatly simplifies the above equations, because now $\sigma(r,\theta,t)$ is taken to be <u>independent</u> of θ; thus eliminating the product of Legendre functions that occurs in $\sigma\underline{E}$. In the method of images, the current sources above the plane ($\theta = \pi/2$) are imaged below the plane with signs reversed. This implies that the new boundary conditions are: (at $\theta = \pi/2$)

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where for this image case -

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