EMP Theoretical Notes
Note XXIII
7 May 1963

Conservations Equations for Electromagnetic Energy Density and Momentum Density

1/Lt Peter M. Livingston
Air Force Weapons Laboratory

Abstract

The equations for conservation of electromagnetic energy density and momentum density are derived in spherical polar coordinates in a Legendre coefficient expansion assuming ϕ symmetry of the field distribution. These represent a possible check on the accuracy of a numerical solution of Maxwell's equations in spherical coordinates, for example, the EMP from a ground burst.

Foreword

The electromagnetic energy density equation derived in this note is used in both the Compute and EMP code at AFWL as an accuracy indicator.

Capt J. H. Darrah
1/Lt C. E. Baum
Editors

Text

Let $W = 1/2\ (\varepsilon E^2 + \mu H^2)$ be the E. M. energy density (Joules/M^3) and consider Maxwell's equations

(1) $\text{curl}\ \underline{H} = \underline{J}_c + \sigma\underline{E} + \varepsilon \frac{\partial}{\partial t} \underline{E}$

(2) $\text{curl}\ \underline{E} = -\mu \frac{\partial}{\partial t} \underline{H}$

If $\underline{E}$ is dotted in eq. 1, and $\underline{H}$ dotted into eq. 2, and the results added -

(3) $\underline{E} \cdot \text{curl}\ \underline{H} - \underline{H} \cdot \text{curl}\ \underline{E} = \underline{E} \cdot (\underline{J}_c + \sigma\underline{E}) + \frac{\partial}{\partial t} \frac{1}{2} (\varepsilon E^2 + \mu H^2)$

This may be simplified by means of the vector identity $\text{div}\ (\underline{E} \times \underline{H}) = -\underline{E} \cdot \text{curl}\ \underline{H} + \underline{H} \cdot \text{curl}\ \underline{E}$. Now let us define the electromagnetic momentum density as

$$\underline{g} = \frac{1}{c^2} (\underline{E} \times \underline{H}) \qquad \left(\frac{Kg}{M^3} \frac{M}{sec}\right)$$

Thus eq. 3 becomes

(4) $\frac{\partial}{\partial t} W + c^2\ \text{div}\ \underline{g} + \underline{E} \cdot (\underline{J}_c + \sigma\underline{E}) = 0$

Thus, the loss of electromagnetic energy density arises from two terms; the first is a global loss from radiation (energy flux through bounding surface at infinity) and the second is Joule heat loss.

The momentum conservation equation can be cast into a form similar to momentum conservation equation of fluid dynamics, with the Maxwell stress tensor playing the role of the sum of the pressure and shear tensors in fluid theory. Of course, in this equation, the transport term is missing, just as it was in 4. We begin with Maxwell's equations;

(5) $\varepsilon\ \text{curl}\ \underline{E} = -\frac{1}{c^2} \frac{\partial}{\partial t} \underline{H}$

(6) $\mu\ \text{curl}\ \underline{H} = \mu\ (\underline{J}_c + \sigma\underline{E}) + \frac{1}{c^2} \frac{\partial}{\partial t} \underline{E}$

Now if we operate the $\underline{E} \times$ on the left in equation (5), and with $\times\ \underline{H}$ on the right of equation (6), then combining results, we obtain -

(7) $-\varepsilon\ (\underline{E} \times \text{curl}\ \underline{E}) + \mu((\text{curl}\ \underline{H}) \times \underline{H}) = \mu\ (\underline{J}_c + \sigma\ \underline{E}) \times \underline{H} + \frac{\partial}{\partial t} \underline{g}$

where $\underline{g} = \frac{\underline{E} \times \underline{H}}{c^2}$ is the electromagnetic momentum density as defined earlier.

The term $\mu(\underline{J}_c + \sigma\ \underline{E}) \times \underline{H}$ is easily identified as a Lorentz force. Now making use of the vector identities -

(8) $(\text{curl}\ \underline{H}) \times \underline{H} = -\underline{H} \times (\text{curl}\ \underline{H}) = -\frac{1}{2}\ \text{grad}\ H^2 + (\underline{H} \cdot \nabla)\ \underline{H}$

(9) $\underline{E} \times (\text{curl}\ \underline{E}) = \frac{1}{2}\ \text{grad}\ E^2 - (\underline{E} \cdot \nabla)\ \underline{E}$

we have -

(10) $$\frac{\partial}{\partial t} \underline{g} + \text{grad} \frac{1}{2} (\varepsilon E^2 + \mu H^2) - \varepsilon (\underline{E} \cdot \nabla) \underline{E} - \mu(\underline{H} \cdot \nabla)\underline{H} + \mu(\underline{J}_c + \sigma\underline{E}) \times \underline{H} = 0$$

This may be simplified by defining the stress tensor

(11) $$\underline{\underline{T}} = (\varepsilon \underline{E}\,\underline{E} + \mu \underline{H}\,\underline{H}) - \underline{\underline{U}} (\varepsilon E^2 + \mu H^2)/2$$

where $\{ \underline{\underline{U}} \}_{ij} = \delta_{ij}$

If we take the divergence of $\underline{\underline{T}}$, we obtain -

(12) $$\nabla\cdot\underline{\underline{T}} = \varepsilon (\nabla\cdot\underline{E}) \underline{E} + \varepsilon (\underline{E}\cdot\nabla) \underline{E} + \mu (\nabla\cdot\underline{H}) \underline{H} + \mu (\underline{H}\cdot\nabla) \underline{H} - \nabla\cdot\underline{\underline{U}} (\varepsilon E^2 + \mu H^2)/2$$

But $(\nabla\cdot E) = \rho/\varepsilon$, $\nabla\cdot H = 0$,

and therefore eq. 10 becomes

(13) $$\frac{\partial}{\partial t} \underline{g} + \mu (\underline{J}_c + \sigma \underline{E}) \times \underline{H} + \rho\underline{E} = \nabla\cdot\underline{\underline{T}}$$

Thus the time derivative of the momentum density is equal to the divergence of the stress (electromagnetic "pressure") tensor minus the Lorentz body force.

The interpretation of the Poynting vector as an energy flux through an arbitrary surface is open to question if the surface has infinitesimal area. The following paradox arises: It is possible to arrange a static magnetic and electric field such that, on a surface bounding the fields, the Poynting vector may not be zero for all points of the surface. Thus, if its interpretation as an energy flux holds everywhere, this means energy is lost to the static fields. On the other hand, the integral of the Poynting vector for such a case over the bounding surface is zero because the vector is a constant. With this as a physical explanation of what is to follow, we shall then determine the relations on the coefficients of the pertinent Legendre Polynomial explanation by (a) writing out the energy and momentum equations in component form, (b) substituting the expansions for E and H, and (c) then integrating over the products of the polynomials. The equations in question are:

Energy conservation -

(14) $$\frac{\partial}{\partial t} \left\{ \frac{1}{2} (\varepsilon (E_r^2 + E_\theta^2) + \mu H_\phi^2) \right\} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_\theta H_\phi) - \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\, E_r H_\phi) + E_r J_r + E_r^2 \sigma + E_\theta^2 \sigma = 0$$

Momentum conservation; r component

(15) $$\frac{1}{c^2} \frac{\partial}{\partial t} (E_\theta H_\phi) + \mu\sigma E_\theta H_\phi = \varepsilon E_\theta \frac{1}{r} \frac{\partial}{\partial\theta} E_r - \frac{\partial}{\partial r} (\frac{1}{2} \varepsilon E_\theta^2 + \frac{1}{2} \mu H_\phi^2)$$

Momentum conservation; θ component

(16) $$-\frac{1}{c^2}\frac{\partial}{\partial t}(E_r H_\phi) - \mu (J_r + \sigma E_r) H_\phi = \varepsilon E_r \frac{\partial}{\partial r} E_\theta - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{2}\varepsilon E_r^{\,2} + \frac{1}{2}\mu H_\phi^{\,2}\right)$$

The following expansions are introduced:

(17) $$H_\phi = \sum_n H_n P_n^{(1)} (\cos\theta); \quad J_r = \sum_n J_n P_n^{(0)} (\cos\theta)$$

(18) $$E_\theta = \sum_n E_n^{(\theta)} P_n^{(1)} (\cos\theta)$$ μ, ε constant

σ = σ (r, t) only

(19) $$E_r = \sum_n E_n^{(r)} P_n^{(0)} (\cos\theta)$$

If these expressions are introduced into eqs. 14, 15 and 16 and the resulting equations integrated over sin θ d θ, the following averaged equations are obtained:

Energy conservation

(20) $$\frac{\partial}{\partial t}\left\{\frac{1}{2}\left(\varepsilon E_\ell^{(r)^2} + \ell(\ell+1)\left(\varepsilon E_\ell^{(\theta)^2} + \mu H_\ell^{\,2}\right)\right)\right\} + \frac{\ell(\ell+1)}{r^2}\frac{\partial}{\partial r}\left(r^2 E_\ell^{(\theta)} H_\ell\right)$$
$$+ E_\ell^{(r)} J_\ell + \sigma (r, t)\left(E_\ell^{(r)^2} + \ell(\ell+1) E_\ell^{(\theta)^2}\right) = 0$$

Momentum conservation; r component

(21) $$\frac{1}{c^2}\frac{\partial}{\partial t}\left(H_\ell E_\ell^{(\theta)}\right) + \mu\sigma E_\ell^{(\theta)} H_\ell = -\frac{\varepsilon}{r} E_\ell^{(\theta)} E_\ell^{(r)} - \frac{\partial}{\partial r}\left(\frac{1}{2}\varepsilon E_\ell^{(\theta)^2} + \frac{1}{2}\mu H_\ell^{\,2}\right)$$

Momentum conservation; θ component

This equation is a bit more difficult to treat; thus it will be dealt with in detail. Inserting the expansions for E_r and H_ϕ, we obtain -

(22) $$\sum_{k,\ell}\Bigg\{-\frac{1}{c^2}\frac{\partial}{\partial t}\left(E_k^{(r)} H_\ell\right) P_k^{(0)}(\cos\theta) P_\ell^{(1)}(\cos\theta) - \mu\left(J_k + \sigma E_k^{(r)}\right)$$
$$H_\ell P_k^{(0)} P_\ell^{(1)} - \varepsilon E_k^{(r)} \frac{\partial}{\partial r}\left(E_\ell^{(\theta)}\right) P_k^{(0)} P_\ell^{(1)} - \frac{1}{r}\Bigg[\frac{1}{2}\varepsilon E_k^{(r)} E_\ell^{(r)}$$
$$\left(\left(\frac{\partial}{\partial\theta} P_k^{(0)}\right) P_\ell^{(0)} + P_k^{(0)}\left(\frac{\partial}{\partial\theta} P_\ell^{(0)}\right)\right) + \frac{1}{2}\mu H_k H_\ell\Bigg(\left(\frac{\partial}{\partial\theta} P_k^{(1)}\right) P_\ell^{(1)}$$
$$+ P_k^{(1)}\left(\frac{\partial}{\partial\theta} P_\ell^{(1)}\right)\Bigg)\Bigg]\Bigg\} = 0$$

The products of the terms $P_k^{(0)}\ P_\ell^{(1)}$ may be simplified if the equation is multiplied by $\sin\theta$, for

$$(23)\quad \sin\theta\, P_\ell^{(1)}(\cos\theta) = \frac{\ell(\ell+1)}{2\ell+1}\left\{P_{\ell-1}^{(0)}(\cos\theta) - P_{\ell+1}^{(0)}(\cos\theta)\right\}$$

Thus

$$(24)\quad \int_0^\pi \sin\theta\, d\theta\,(\sin\theta\, P_\ell^{(1)}(\cos\theta)\, P_k^{(0)}(\cos\theta))$$

$$= \frac{\ell(\ell+1)}{2\ell+1}\left[\delta_{k,\,\ell-1}\frac{2}{2\ell-1} - \delta_{k,\ell+1}\frac{2}{2\ell+3}\right]$$

Note also that $-\frac{\partial}{\partial\theta} P_k^{(0)} = P_k^{(1)}(\cos\theta)$

Some partial results to be used in reducing the sum from a two-fold to one-fold form:

$$(25)\quad -\frac{1}{r}\left\{\frac{1}{2}\varepsilon\, E_k^{(r)}E_\ell^{(r)}\frac{\partial}{\partial\theta}(P_k^{(0)}\ P_\ell^{(0)})\right\} \times \sin\theta$$

$$= -\frac{1}{r}\left\{(-\tfrac{1}{2}\varepsilon)\, E_k^{(r)}\, E_\ell^{(r)}\,(\sin\theta\, P_k^{(1)}P_\ell^{(0)} + \sin\theta\, P_k^{(0)}P_\ell^{(1)})\right\}$$

$$= -\frac{1}{r}\left\{(-\tfrac{1}{2}\varepsilon)\, E_k^{(r)}E_\ell^{(r)}\left[\frac{k(k+1)}{2k+1}(P_{k-1}^{(0)}P_\ell^{(0)} - P_{k+1}^{(0)}P_\ell^{(0)}) + \frac{\ell(\ell+1)}{2\ell+1}(P_k^{(0)}\, P_{\ell-1}^{(0)} - P_k^{(0)}\, P_{\ell+1}^{(0)})\right]\right\}$$

Here we have used the derivative relation and the $\sin\theta\, P^{(1)}$ operation as given above. Upon integration we obtain the result.

$$(26)\quad \text{term} = -\frac{1}{r}\left\{(-\varepsilon)\left[E_{\ell+1}^{(r)}\, E_\ell^{(r)}\frac{2(\ell+1)}{(2\ell+1)(2\ell+3)} + \frac{2\ell\, E_{\ell-1}^{(r)}\, E_\ell^{(r)}}{(2\ell+1)(2\ell-1)}\right] + \right\}$$

A somewhat more complicated procedure may be applied to the product $\frac{\partial}{\partial\theta} H_\phi^2$ as follows -

$$(27)\quad \sum_{k,\ell}\frac{1}{2}\ \mu H_k H_\ell \sin\theta\frac{\partial}{\partial\theta}(P_k^{(1)}P_\ell^{(1)}) \qquad \left\{\text{sin } \theta \text{ comes from multiplying whole eq. by sin } \theta\right\}$$

Now

$$(28)\quad \sin\theta\frac{\partial}{\partial\theta}(P_k^{(1)}\ P_\ell^{(1)}) = \frac{\partial}{\partial\theta}(\sin\theta\, P_k^{(1)})\, P_\ell^{(1)} + \cos\theta\, P_k^{(1)}\, P_\ell^{(1)}$$

where

$$(29)\quad \cos\theta\, P_k^{(1)} = \frac{k}{2k+1}P_{k+1}^{(1)} + \frac{k+1}{2k+1}P_{k-1}^{(1)}$$

and

$$(30)\quad \frac{\partial}{\partial\theta}(\sin\theta\, P_\ell^{(1)}) = \frac{\partial}{\partial\theta}\left\{\frac{\ell(\ell+1)}{2\ell+1}(P_{\ell-1}^{(0)} - P_{\ell+1}^{(0)})\right\}$$
$$= \frac{\ell(\ell+1)}{2\ell+1}(P_{\ell+1}^{(1)} - P_{\ell-1}^{(1)})$$

Thus

$$(31)\quad \sin\theta\frac{\partial}{\partial\theta}(P_\ell^{(1)} P_k^{(1)}) = \frac{k(k+1)}{2k+1}P_{k+1}^{(1)}P_\ell^{(1)} - \frac{(k-1)(k+1)}{2k+1}P_{k-1}^{(1)}P_\ell^{(1)}$$
$$+ \frac{\ell(\ell+1)}{2\ell+1}P_k^{(1)}P_{\ell+1}^{(1)} - \frac{(\ell-1)(\ell+1)}{2\ell+1}P_{\ell-1}^{(1)}P_k^{(1)}$$

Upon integration, we have -

$$(32)\quad \left\langle\sum_{k,\ell}\frac{1}{2}\mu H_k H_\ell \sin\theta\frac{\partial}{\partial\theta}(P_k^{(1)} P_\ell^{(1)})\right\rangle$$

$$= \mu\left[H_\ell H_{\ell-1}\frac{\ell(\ell-1)(\ell+1)}{(2\ell-1)(2\ell+1)} + H_\ell H_{\ell+1}\frac{\ell(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)}\right]$$

Upon inserting these results in the equation and setting each term to zero (condition on the coefficients) -

$$(33)\quad -\frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\ell(\ell+1)}{2\ell+1}\frac{2}{2\ell-1}E_{\ell-1}^{(r)}H_\ell - \frac{\ell(\ell+1)}{2\ell+1}\frac{2}{2\ell+3}E_{\ell+1}^{(r)}H_\ell\right]$$

$$-\mu\frac{\ell(\ell+1)}{2\ell+1}\left[\frac{2}{2\ell-1}(J_{\ell-1} + \sigma E_{\ell-1}^{(r)})H_\ell - \frac{2}{2\ell+3}(J_{\ell+1} + \sigma E_{\ell+1}^{(r)})H_\ell\right]$$

$$-\varepsilon\frac{\ell(\ell+1)}{2\ell+1}\left[\frac{2}{2\ell-1}E_{\ell-1}^{(r)}\frac{\partial}{\partial r}E_\ell^{(\theta)} - \frac{2}{2\ell+3}E_{\ell+1}^{(r)}\frac{\partial}{\partial r}E_\ell^{(\theta)}\right]$$

$$-\frac{1}{r}\left\{-\frac{2\varepsilon}{2\ell+1}\left(E_{\ell+1}^{(r)}E_\ell^{(r)}\frac{(\ell+1)}{2\ell+3} + \ell\frac{E_{\ell-1}^{(r)}E_\ell^{(r)}}{2\ell-1}\right) + \mu\frac{\ell(\ell+1)}{(2\ell+1)}\left(H_\ell H_{\ell-1}\frac{(\ell-1)}{2\ell-1}\right.\right.$$
$$\left.\left. + H_\ell H_{\ell+1}\frac{(\ell+2)}{2\ell+3}\right)\right\} = 0$$

EMP Theoretical Notes
Note XXIII
7 May 1963

Conservations Equations for Electromagnetic Energy Density and Momentum Density

1/Lt Peter M. Livingston
Air Force Weapons Laboratory

Abstract

The equations for conservation of electromagnetic energy density and momentum density are derived in spherical polar coordinates in a Legendre coefficient expansion assuming ϕ symmetry of the field distribution. These represent a possible check on the accuracy of a numerical solution of Maxwell's equations in spherical coordinates, for example, the EMP from a ground burst.

Foreword

The electromagnetic energy density equation derived in this note is used in both the Compute and EMP code at AFWL as an accuracy indicator.

Capt J. H. Darrah
1/Lt C. E. Baum
Editors

Text

Let $W = 1/2\ (\varepsilon E^2 + \mu H^2)$ be the E. M. energy density (Joules/M^3) and consider Maxwell's equations

(1) $$\text{curl } \underline{H} = \underline{J}_c + \sigma\underline{E} + \varepsilon \frac{\partial}{\partial t} \underline{E}$$

(2) $$\text{curl } \underline{E} = -\mu \frac{\partial}{\partial t} \underline{H}$$

If $\underline{E}$ is dotted in eq. 1, and $\underline{H}$ dotted into eq. 2, and the results added -

(3) $$\underline{E} \cdot \text{curl } \underline{H} - \underline{H} \cdot \text{curl } \underline{E} = \underline{E} \cdot (\underline{J}_c + \sigma\underline{E}) + \frac{\partial}{\partial t} \frac{1}{2} (\varepsilon E^2 + \mu H^2)$$

This may be simplified by means of the vector identity $\text{div } (\underline{E} \times \underline{H}) = -\underline{E} \cdot \text{curl } \underline{H} + \underline{H} \cdot \text{curl } \underline{E}$. Now let us define the electromagnetic momentum density as

$$\underline{g} = \frac{1}{c^2} (\underline{E} \times \underline{H}) \qquad \left(\frac{Kg}{M^3} \frac{M}{sec}\right)$$

Thus eq. 3 becomes

(4) $$\frac{\partial}{\partial t} W + c^2 \text{ div } \underline{g} + \underline{E} \cdot (\underline{J}_c + \sigma\underline{E}) = 0$$

Thus, the loss of electromagnetic energy density arises from two terms; the first is a global loss from radiation (energy flux through bounding surface at infinity) and the second is Joule heat loss.

The momentum conservation equation can be cast into a form similar to momentum conservation equation of fluid dynamics, with the Maxwell stress tensor playing the role of the sum of the pressure and shear tensors in fluid theory. Of course, in this equation, the transport term is missing, just as it was in 4. We begin with Maxwell's equations;

(5) $$\varepsilon \text{ curl } \underline{E} = -\frac{1}{c^2} \frac{\partial}{\partial t} \underline{H}$$

(6) $$\mu \text{ curl } \underline{H} = \mu (\underline{J}_c + \sigma\underline{E}) + \frac{1}{c^2} \frac{\partial}{\partial t} \underline{E}$$

Now if we operate the $\underline{E} \times$ on the left in equation (5), and with $\times \underline{H}$ on the right of equation (6), then combining results, we obtain -

(7) $$-\varepsilon (\underline{E} \times \text{curl } \underline{E}) + \mu((\text{curl } \underline{H}) \times \underline{H}) = \mu (\underline{J}_c + \sigma \underline{E}) \times \underline{H} + \frac{\partial}{\partial t} \underline{g}$$

where $\underline{g} = \dfrac{\underline{E} \times \underline{H}}{c^2}$ is the electromagnetic momentum density as defined earlier.

The term $\mu(\underline{J}_c + \sigma \underline{E}) \times \underline{H}$ is easily identified as a Lorentz force. Now making use of the vector identities -

(8) $$(\text{curl } \underline{H}) \times \underline{H} = - \underline{H} \times (\text{curl } \underline{H}) = -\frac{1}{2} \text{ grad } H^2 + (\underline{H} \cdot \nabla) \underline{H}$$

(9) $$\underline{E} \times (\text{curl } \underline{E}) = \frac{1}{2} \text{ grad } E^2 - (\underline{E} \cdot \nabla) \underline{E}$$

we have -

(10) $\frac{\partial}{\partial t} \underline{g} + \text{grad} \frac{1}{2} (\varepsilon E^2 + \mu H^2) - \varepsilon (\underline{E} \cdot \nabla) \underline{E} - \mu(\underline{H} \cdot \nabla)\underline{H} + \mu(\underline{J}_c + \sigma\underline{E}) \times \underline{H} = 0$

This may be simplified by defining the stress tensor

(11) $\underline{\underline{T}} = (\varepsilon \underline{E}\,\underline{E} + \mu \underline{H}\,\underline{H}) - \underline{\underline{U}} (\varepsilon E^2 + \mu H^2)/2$

where $\{ \underline{\underline{U}} \}_{ij} = \delta_{ij}$

If we take the divergence of $\underline{\underline{T}}$, we obtain -

(12) $\nabla \cdot \underline{\underline{T}} = \varepsilon (\nabla \cdot \underline{E}) \underline{E} + \varepsilon (\underline{E} \cdot \nabla) \underline{E} + \mu (\nabla \cdot \underline{H}) \underline{H} + \mu (\underline{H} \cdot \nabla) \underline{H}$

$- \nabla \cdot \underline{\underline{U}} (\varepsilon E^2 + \mu H^2)/2$

But $(\nabla \cdot E) = \rho/\varepsilon$, $\nabla \cdot H = 0$,

and therefore eq. 10 becomes

(13) $\frac{\partial}{\partial t} \underline{g} + \mu (\underline{J}_c + \sigma \underline{E}) \times \underline{H} + \rho\underline{E} = \nabla \cdot \underline{\underline{T}}$

Thus the time derivative of the momentum density is equal to the divergence of the stress (electromagnetic "pressure") tensor minus the Lorentz body force.

The interpretation of the Poynting vector as an energy flux through an arbitrary surface is open to question if the surface has infinitesimal area. The following paradox arises: It is possible to arrange a static magnetic and electric field such that, on a surface bounding the fields, the Poynting vector may not be zero for all points of the surface. Thus, if its interpretation as an energy flux holds everywhere, this means energy is lost to the static fields. On the other hand, the integral of the Poynting vector for such a case over the bounding surface is zero because the vector is a constant. With this as a physical explanation of what is to follow, we shall then determine the relations on the coefficients of the pertinent Legendre Polynomial explanation by (a) writing out the energy and momentum equations in component form, (b) substituting the expansions for E and H, and (c) then integrating over the products of the polynomials. The equations in question are:

Energy conservation -

(14) $\frac{\partial}{\partial t} \left\{ \frac{1}{2} (\varepsilon (E_r^2 + E_\theta^2) + \mu H_\phi^2) \right\} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_\theta H_\phi)$

$- \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta\, E_r H_\phi) + E_r J_r + E_r^2 \sigma + E_\theta^2 \sigma = 0$

Momentum conservation; r component

(15) $\frac{1}{c^2} \frac{\partial}{\partial t} (E_\theta H_\phi) + \mu\sigma E_\theta H_\phi = \varepsilon E_\theta \frac{1}{r} \frac{\partial}{\partial \theta} E_r - \frac{\partial}{\partial r} (\frac{1}{2} \varepsilon E_\theta^2 + \frac{1}{2} \mu H_\phi^2)$

Momentum conservation; θ component

$$(16)\quad -\frac{1}{c^2}\frac{\partial}{\partial t}(E_r H_\phi) - \mu (J_r + \sigma E_r) H_\phi = \varepsilon E_r \frac{\partial}{\partial r} E_\theta - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{2}\varepsilon E_r^2 + \frac{1}{2}\mu H_\phi^2\right)$$

The following expansions are introduced:

$$(17)\quad H_\phi = \sum_n H_n P_n^{(1)} (\cos\theta);\; J_r = \sum_n J_n P_n^{(0)} (\cos\theta)$$

$$(18)\quad E_\theta = \sum_n E_n^{(\theta)} P_n^{(1)} (\cos\theta)$$

μ, ε constant

σ = σ (r, t) only

$$(19)\quad E_r = \sum_n E_n^{(r)} P_n^{(0)} (\cos\theta)$$

If these expressions are introduced into eqs. 14, 15 and 16 and the resulting equations integrated over sin θ d θ, the following averaged equations are obtained:

Energy conservation

$$(20)\quad \frac{\partial}{\partial t}\left\{\frac{1}{2}\left(\varepsilon E_\ell^{(r)2} + \ell(\ell+1)\left(\varepsilon E_\ell^{(\theta)2} + \mu H_\ell^2\right)\right)\right\} + \frac{\ell(\ell+1)}{r^2}\frac{\partial}{\partial r}\left(r^2 E_\ell^{(\theta)} H_\ell\right) + E_\ell^{(r)} J_\ell + \sigma (r,t)\left(E_\ell^{(r)2} + \ell(\ell+1) E_\ell^{(\theta)2}\right) = 0$$

Momentum conservation; r component

$$(21)\quad \frac{1}{c^2}\frac{\partial}{\partial t}\left(H_\ell E_\ell^{(\theta)}\right) + \mu\sigma E_\ell^{(\theta)} H_\ell = -\frac{\varepsilon}{r} E_\ell^{(\theta)} E_\ell^{(r)} - \frac{\partial}{\partial r}\left(\frac{1}{2}\varepsilon E_\ell^{(\theta)2} + \frac{1}{2}\mu H_\ell^2\right)$$

Momentum conservation; θ component

This equation is a bit more difficult to treat; thus it will be dealt with in detail. Inserting the expansions for E_r and H_ϕ, we obtain -

$$(22)\quad \sum_{k,\ell}\Bigg\{-\frac{1}{c^2}\frac{\partial}{\partial t}\left(E_k^{(r)} H_\ell\right) P_k^{(0)}(\cos\theta)\, P_\ell^{(1)}(\cos\theta) - \mu\left(J_k + \sigma E_k^{(r)}\right) H_\ell P_k^{(0)} P_\ell^{(1)} - \varepsilon E_k^{(r)}\frac{\partial}{\partial r}\left(E_\ell^{(\theta)}\right) P_k^{(0)} P_\ell^{(1)} - \frac{1}{r}\Bigg[\frac{1}{2}\varepsilon E_k^{(r)} E_\ell^{(r)} \left(\left(\frac{\partial}{\partial \theta} P_k^{(0)}\right) P_\ell^{(0)} + P_k^{(0)}\left(\frac{\partial}{\partial \theta} P_\ell^{(0)}\right)\right) + \frac{1}{2}\mu H_k H_\ell \left(\left(\frac{\partial}{\partial \theta} P_k^{(1)}\right) P_\ell^{(1)} + P_k^{(1)}\left(\frac{\partial}{\partial \theta} P_\ell^{(1)}\right)\right)\Bigg]\Bigg\} = 0$$

The products of the terms $P_k^{(0)}\ P_\ell^{(1)}$ may be simplified if the equation is multiplied by $\sin\theta$, for

$$(23)\quad \sin\theta\, P_\ell^{(1)}(\cos\theta) = \frac{\ell(\ell+1)}{2\ell+1}\left\{P_{\ell-1}^{(0)}(\cos\theta) - P_{\ell+1}^{(0)}(\cos\theta)\right\}$$

Thus

$$(24)\quad \int_0^\pi \sin\theta\, d\theta\, (\sin\theta\, P_\ell^{(1)}(\cos\theta)\, P_k^{(0)}(\cos\theta))$$

$$= \frac{\ell(\ell+1)}{2\ell+1}\left[\delta_{k,\ell-1}\frac{2}{2\ell-1} - \delta_{k,\ell+1}\frac{2}{2\ell+3}\right]$$

Note also that $-\frac{\partial}{\partial\theta} P_k^{(0)} = P_k^{(1)}(\cos\theta)$

Some partial results to be used in reducing the sum from a two-fold to one-fold form:

$$(25)\quad -\frac{1}{r}\left\{\frac{1}{2}\varepsilon E_k^{(r)}E_\ell^{(r)}\frac{\partial}{\partial\theta}(P_k^{(0)}P_\ell^{(0)})\right\} \times \sin\theta$$

$$= -\frac{1}{r}\left\{(-\tfrac{1}{2}\varepsilon)\, E_k^{(r)}\, E_\ell^{(r)}\, (\sin\theta\, P_k^{(1)}P_\ell^{(0)} + \sin\theta\, P_k^{(0)}P_\ell^{(1)})\right\}$$

$$= -\frac{1}{r}\left\{(-\tfrac{1}{2}\varepsilon)\, E_k^{(r)}E_\ell^{(r)}\left[\frac{k(k+1)}{2k+1}(P_{k-1}^{(0)}P_\ell^{(0)} - P_{k+1}^{(0)}P_\ell^{(0)}) + \frac{\ell(\ell+1)}{2\ell+1}(P_k^{(0)}P_{\ell-1}^{(0)} - P_k^{(0)}P_{\ell+1}^{(0)})\right]\right\}$$

Here we have used the derivative relation and the $\sin\theta\, P^{(1)}$ operation as given above. Upon integration we obtain the result.

$$(26)\quad \text{term} = -\frac{1}{r}\left\{(-\varepsilon)\left[E_{\ell+1}^{(r)}\, E_\ell^{(r)}\frac{2(\ell+1)}{(2\ell+1)(2\ell+3)} + \frac{2\ell\, E_{\ell-1}^{(r)}\, E_\ell^{(r)}}{(2\ell+1)(2\ell-1)}\right] + \right\}$$

A somewhat more complicated procedure may be applied to the product $\frac{\partial}{\partial\theta} H_\phi^2$ as follows -

$$(27)\quad \sum_{k,\ell}\frac{1}{2}\mu H_k H_\ell \sin\theta\frac{\partial}{\partial\theta}(P_k^{(1)}P_\ell^{(1)}) \qquad \left\{\begin{array}{l}\sin\theta \text{ comes from multiplying}\\ \text{whole eq. by } \sin\theta\end{array}\right\}$$

Now

$$(28)\quad \sin\theta\frac{\partial}{\partial\theta}(P_k^{(1)}P_\ell^{(1)}) = \frac{\partial}{\partial\theta}(\sin\theta\, P_k^{(1)})\, P_\ell^{(1)} + \cos\theta\, P_k^{(1)}\, P_\ell^{(1)}$$

where

$$(29)\quad \cos\theta\, P_k^{(1)} = \frac{k}{2k+1}P_{k+1}^{(1)} + \frac{k+1}{2k+1}P_{k-1}^{(1)}$$

and

$$(30)\quad \frac{\partial}{\partial\theta}(\sin\theta\, P_\ell^{(1)}) = \frac{\partial}{\partial\theta}\left\{ \frac{\ell(\ell+1)}{2\ell+1}(P_{\ell-1}^{(0)} - P_{\ell+1}^{(0)}) \right\}$$

$$= \frac{\ell(\ell+1)}{2\ell+1}(P_{\ell+1}^{(1)} - P_{\ell-1}^{(1)})$$

Thus

$$(31)\quad \sin\theta \frac{\partial}{\partial\theta}(P_\ell^{(1)} P_k^{(1)}) = \frac{k(k+1)}{2k+1} P_{k+1}^{(1)} P_\ell^{(1)} - \frac{(k-1)(k+1)}{2k+1} P_{k-1}^{(1)} P_\ell^{(1)}$$

$$+ \frac{\ell(\ell+1)}{2\ell+1} P_k^{(1)} P_{\ell+1}^{(1)} - \frac{(\ell-1)(\ell+1)}{2\ell+1} P_{\ell-1}^{(1)} P_k^{(1)}$$

Upon integration, we have -

$$(32)\quad \left\langle \sum_{k,\ell} \frac{1}{2}\mu H_k H_\ell \sin\theta \frac{\partial}{\partial\theta}(P_k^{(1)} P_\ell^{(1)}) \right\rangle$$

$$= \mu\left[H_\ell H_{\ell-1} \frac{\ell(\ell-1)(\ell+1)}{(2\ell-1)(2\ell+1)} + H_\ell H_{\ell+1} \frac{\ell(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)} \right]$$

Upon inserting these results in the equation and setting each term to zero (condition on the coefficients) -

$$(33)\quad -\frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\ell(\ell+1)}{2\ell+1}\frac{2}{2\ell-1} E_{\ell-1}^{(r)} H_\ell - \frac{\ell(\ell+1)}{2\ell+1}\frac{2}{2\ell+3} E_{\ell+1}^{(r)} H_\ell \right]$$

$$- \mu \frac{\ell(\ell+1)}{2\ell+1}\left[\frac{2}{2\ell-1}(J_{\ell-1} + \sigma E_{\ell-1}^{(r)}) H_\ell - \frac{2}{2\ell+3}(J_{\ell+1} + \sigma E_{\ell+1}^{(r)}) H_\ell \right]$$

$$- \varepsilon \frac{\ell(\ell+1)}{2\ell+1}\left[\frac{2}{2\ell-1} E_{\ell-1}^{(r)} \frac{\partial}{\partial r} E_\ell^{(\theta)} - \frac{2}{2\ell+3} E_{\ell+1}^{(r)} \frac{\partial}{\partial r} E_\ell^{(\theta)} \right]$$

$$- \frac{1}{r}\left\{ - \frac{2\varepsilon}{2\ell+1}\left(E_{\ell+1}^{(r)} E_\ell^{(r)} \frac{(\ell+1)}{2\ell+3} + \ell \frac{E_{\ell-1}^{(r)} E_\ell^{(r)}}{2\ell-1}\right) + \mu \frac{\ell(\ell+1)}{(2\ell+1)}\left(H_\ell H_{\ell-1} \frac{(\ell-1)}{2\ell-1}\right.\right.$$

$$\left.\left. + H_\ell H_{\ell+1} \frac{(\ell+2)}{2\ell+3}\right)\right\} = 0$$