EMP Theoretical Notes

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TRANSIENT SPHERICAL WAVES

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Abstract

The expressions in the time domain representing the general solution of the wave equation in spherical polar coordinates are summarized. The boundary-value problem where the tangential component of $\vec{E}$ is given as a function of time on the surface of a sphere is solved for outward-moving waves. The solution of this problem was used to find the radiation field surrounding a conducting sphere irradiated by an intense short pulse of gamma rays. The problem embodies the transient spherical-antenna problem as a special case. Since the solution involves solving ordinary linear differential equations with constant coefficients, it is similar to that of a transient circuit problem and Laplace transforms can be employed in a similar way.

INTRODUCTION

In electromagnetic circuit theory, dynamic signals have been treated traditionally as sinusoidal functions of time or combinations thereof. Many circuits of interest utilized sinusoidal signals or signals of a fairly narrow bandwidth, and the behavior of such circuits was most easily understood in terms of frequency-domain parameters. Furthermore, it was argued that such a treatment was perfectly general since any physically realizeable time function can be Fourier-analyzed. However, the advent of pulsed circuits quickly demonstrated that Fourier analysis was clumsy and hence inappropriate for pulsed-signal problems. The well-known techniques of transient circuit analysis were found useful in treating such problems.

In electromagnetic field theory, dynamic fields are now treated almost entirely under the assumption of sinusoidal time dependence. Most fields of interest are most easily understood in this framework. As in circuit theory, the treatment is quite general. However, there are now pulsed-field problems of interest which are only very clumsily done by Fourier analysis. It is hoped that the transient field theory presented here will be the beginning of the development of mathematical tools appropriate to the treatment of pulsed fields.

In the following section a summary is given of the solution of the wave equation in the time domain in spherical-polar coordinates for electric multipole fields. The derivation is given in another paper [1] which will be referred to in this article as MTTD. The author thought it advisable to summarize the solution here for three reasons: to present simplifications in some of the expressions in MTTD of which he was formerly unaware; to present the solution in terms of real variables (better adapted to problem solving and less suitable for theoretical manipulation); and for the convenience of the reader.

The primary contribution in this paper is the solution of the boundary-value problem where the tangential component of $\vec{E}$ is given on the surface of a sphere as a function of time. Introduction of the Laplace transform casts the problem into a framework familiar to electrical engineers. The special case of the spherical antenna is treated. The more general solution was used by the author to find the radiation field surrounding a conducting sphere irradiated by an intense short pulse of gamma rays.*

SUMMARY OF TRANSIENT WAVE-EQUATION SOLUTION

If ψ satisfies the scalar wave equation, then the spherical wave expansion for ψ for outgoing waves can be written

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* As the backscattered and/or transmitted gamma rays exit from the surface, they cause Compton electrons to leave the surface. If the electron flux is high enough, the resulting electric field (due to space charge) keeps the electrons close to the surface and the electromagnetic effect of the electrons can be viewed as that of a dipole layer. The negative of the gradient of a dipole layer strength where the dipole layer is adjacent to a conducting surface gives the tangential component of $\vec{E}$ just outside the dipole layer. [See J. A. Stratton, Electromagnetic Theory (McGraw-Hill Book Company, Inc., New York, 1941), p. 191].

$$\psi(\vec{x}, t^*) = \sum_{\ell=0}^{\infty} \Xi_\ell(r) \left\{ c_{\ell 0}(t^*) \; \overline{P}_\ell(\cos\theta) \right.$$

$$\left. + \sum_{m=1}^{\ell} [c_{\ell m}(t^*) \cos m\phi + d_{\ell m}(t^*) \sin m\phi] \; \overline{P}_\ell^m(\cos\theta) \right\} \quad . \qquad (1)$$

The general outward-moving, electric-multipole (transverse B field) spherical components of $\vec{E}$ and $\vec{B}$ can be written

$$E_r = \sum_{\ell=1}^{\infty} - \frac{[\ell(\ell+1)]^{\frac{1}{2}}}{r} \Xi_\ell(r) \left\{ a_{\ell 0}(t^*) \; \overline{P}_\ell(\cos\theta) \right.$$

$$\left. + \sum_{m=1}^{\ell} [a_{\ell m}(t^*) \cos m\phi + b_{\ell m}(t^*) \sin m\phi] \; \overline{P}_\ell^m(\cos\theta) \right\} \quad , \qquad (2)$$

$$E_\theta = \sum_{\ell=1}^{\infty} \Lambda_\ell(r) \left\{ a_{\ell 0}(t^*) \; \overline{P}_\ell^1(\cos\theta) \right.$$

$$+ \sum_{m=1}^{\ell} [a_{\ell m}(t^*) \cos m\phi + b_{\ell m}(t^*) \sin m\phi]$$

$$\left. \times \frac{1}{2} \left[\left(\frac{(\ell-m)(\ell+m+1)}{\ell(\ell+1)} \right)^{\frac{1}{2}} \overline{P}_\ell^{m+1}(\cos\theta) - \left(\frac{(\ell+m)(\ell-m+1)}{\ell(\ell+1)} \right)^{\frac{1}{2}} \overline{P}_\ell^{m-1}(\cos\theta) \right] \right\} \quad , \qquad (3)$$

$$E_\phi = \sum_{\ell=1}^{\infty} \frac{1}{2} \left[\frac{2\ell+1}{\ell(\ell+1)(2\ell-1)} \right]^{\frac{1}{2}}$$

$$\times \; \Lambda_\ell(r) \sum_{m=1}^{\ell} \left[a_{\ell m}(t^*) \sin m\phi - b_{\ell m}(t^*) \cos m\phi \right]$$

$$\times \left\{ \left[(\ell - m)(\ell - m - 1) \right]^{\frac{1}{2}} \overline{P}_{\ell-1}^{m+1} (\cos \theta) + \left[(\ell + m)(\ell + m - 1) \right]^{\frac{1}{2}} \overline{P}_{\ell-1}^{m-1} (\cos \theta) \right\} \quad . \quad (4)$$

$$B_r = 0 \quad ,$$

$$B_\theta = \sum_{\ell=1}^{\infty} \frac{1}{2} \left[\frac{2\ell+1}{\ell(\ell+1)(2\ell-1)} \right]^{\frac{1}{2}}$$

$$\times \frac{1}{c} \frac{\partial}{\partial t^*} \Xi_\ell(r) \sum_{m=1}^{\ell} \left[b_{\ell m}(t^*) \cos m\phi - a_{\ell m}(t^*) \sin m\phi \right]$$

$$\times \left\{ \left[(\ell - m)(\ell - m - 1) \right]^{\frac{1}{2}} \overline{P}_{\ell-1}^{m+1} (\cos \theta) + \left[(\ell + m)(\ell + m - 1) \right]^{\frac{1}{2}} \overline{P}_{\ell-1}^{m-1} (\cos \theta) \right\} \quad , \quad (5)$$

$$B_\phi = \sum_{\ell=1}^{\infty} \frac{1}{c} \frac{\partial}{\partial t^*} \Xi_\ell(r) \left\{ a_{\ell 0}(t^*) \; \overline{P}_\ell^1 (\cos \theta) \right.$$

$$+ \sum_{m=1}^{\ell} \left[a_{\ell m}(t^*) \cos m\phi + b_{\ell m}(t^*) \sin m\phi \right]$$

$$\left. \times \frac{1}{2} \left[\left(\frac{(\ell - m)(\ell + m + 1)}{\ell(\ell+1)} \right)^{\frac{1}{2}} \overline{P}_\ell^{m+1} (\cos \theta) - \left(\frac{(\ell + m)(\ell - m + 1)}{\ell(\ell+1)} \right)^{\frac{1}{2}} \overline{P}_\ell^{m-1} (\cos \theta) \right] \right\} \; , (6)$$

where $\Xi_\ell(r)$ is the differential operator (the Hankel operator)

$$\Xi_\ell(r) = \sum_{j=0}^{\ell} \frac{\mu_{\ell j}}{r^{j+1}\, c^{\ell-j}} \frac{\partial^{\ell-j}}{\partial t^{*\ell-j}} ,$$

$$\text{with } \mu_{\ell j} = \frac{\prod_{k=0}^{j} (\ell+k)(\ell-k+1)}{\ell(\ell+1)\, 2^j\, j!} ,$$

and $\Lambda_\ell(r)$ is the linear combination of two Hankel operators

$$\Lambda_\ell(r) = \frac{1}{2\ell+1} \left[\ell\, \Xi_{\ell+1}(r) + \frac{\ell+1}{c^2} \frac{\partial^2}{\partial t^{*2}} \Xi_{\ell-1}(r) \right] ,$$

$$\Lambda_\ell(r) = \sum_{j=0}^{\ell+1} \frac{\nu_{\ell j}}{r^{j+1}\, c^{\ell+1-j}} \frac{\partial^{\ell+1-j}}{\partial t^{*\,\ell+1-j}} ,$$

$$\text{with } \nu_{\ell j} = [\ell \mu_{\ell+1,j} + (\ell+1)\mu_{\ell-1,j}] \Big/ (2\ell+1) ,$$

$$\nu_{\ell j} = \mu_{\ell j} \frac{\ell(\ell+1) + j(j-1)}{\ell(\ell+1) - j(j-1)} , \quad j < \ell+1 ,$$

$$\nu_{\ell,\ell+1} = \nu_{\ell\ell} .$$

The functions $a_{\ell m}$, $b_{\ell m}$, $c_{\ell m}$, and $d_{\ell m}$ are arbitrary real functions of retarded time, $t^* = t - r/c$. The relationship between $a_{\ell m}$ and $b_{\ell m}$ and the functions α_E of MTTD are

$$a_{\ell 0}(t^*) = \alpha_E(\ell, 0, t^*) \Big/ (2\pi)^{\frac{1}{2}} , \tag{7}$$

$$a_{\ell m}(t^*) = [\alpha_E(\ell, m, t^*) + (-1)^m \alpha_E(\ell, -m, t^*)] \Big/ (2\pi)^{\frac{1}{2}} , \quad m > 0 , \tag{8}$$

$$b_{\ell m}(t^*) = i[\alpha_E(\ell, m, t^*) - (-1)^m \alpha_E(\ell, -m, t^*)] \Big/ (2\pi)^{\frac{1}{2}}, \quad (i = \sqrt{-1}). \qquad (9)$$

The function $\overline{P}_\ell^m(\cos\theta)$ is the normalized Legendre function.

The above equations are in Gaussian units. The similar expressions for magnetic multipoles are obtained by replacing $\vec{B}$ with $\vec{E}$ and $\vec{E}$ with $-\vec{B}$. Incoming waves are discussed in Appendix A of MTTD. Note that if the arbitrary functions of t^* are given the form $\exp(-i\omega t^*)$, the classical frequency domain expressions result. If the fields are zero for $t^* < 0$, Eqs. (1) through (6) can easily be Laplace transformed with respect to t^*; the operators $\Xi_\ell(r)$ and $\Lambda_\ell(r)$ become polynomials of order ℓ and $(\ell+1)$ respectively in the transformed variable. Thus many of the transient techniques utilizing the Laplace transforms which are classical to circuit theory can also be applied to transient spherical waves.

SPHERICAL BOUNDARY-VALUE PROBLEM

In MTTD the boundary-value problem was solved where E_r is given on the surface of a sphere that contains electric-multipole sources. The author found this useful for extrapolating pulsed fields that were calculated by a rather tedious numerical solution of Maxwell's equations in the vicinity of a nuclear burst. The other "half" of the spherical boundary-value problem will be solved in this section. That is, given the tangential component of $\vec{E}$ as a function of time on the surface of a sphere that contains the sources of the field, find the implied fields everywhere outside the sphere; or, given

E_θ and E_ϕ as functions of t^*, θ, and ϕ on a sphere of radius R, find the functions $a_{\ell m}(t^*)$, $b_{\ell m}(t^*)$ which appear in Eqs. (2) through (6), and hence $\Xi_\ell(r)\, a_{\ell m}$, $\Xi_\ell(r)\, b_{\ell m}$, $\Lambda_\ell(r)\, a_{\ell m}$, and $\Lambda_\ell(r)\, b_{\ell m}$ where r can take on any value.*

Suppose E_θ and E_ϕ are known on the sphere of radius $r = R$. The ϕ - independent part of the field must be obtained from E_θ since (for electric multipoles) E_ϕ has no ϕ - independent part. Multiplying both sides of Eq. (3) by $\overline{P}^1_{\ell'}(\cos\theta)$ and integrating over a sphere of radius R, one obtains

$$\Lambda_{\ell'}(R)\, a_{\ell' 0}(t^*) = \frac{1}{2\pi} \int_{\substack{\text{sphere of} \\ \text{radius R}}} E_\theta\, \overline{P}^1_{\ell'}(\cos\theta)\, d\Omega \quad . \tag{10}$$

If E_θ is not dependent on ϕ, Eq. (10) becomes

$$\Lambda_{\ell'}(R)\, a_{\ell' 0}(t^*) = \int_0^\pi E_\theta\, \overline{P}^1_{\ell'}(\cos\theta)\, \sin\theta\, d\theta \quad . \tag{11}$$

Equation (10) is an $(\ell'+1)$ - order ordinary linear differential equation with constant coefficients.

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* As the problem is stated and subsequently solved, only electric multipole fields are included in the analysis. Since the spacial dependence of the right-hand side of Eq. (6) is identical to that of the right-hand side of Eq. (3) and the spacial dependence of the right-hand side of Eq. (5) is identical (apart from a change in sign) to that of the right-hand side of Eq. (4), the solution of the same problem for magnetic multipoles is accomplished in a parallel manner with E_ϕ in place of E_θ, $-E_\theta$ in place of E_ϕ, and the operator $(1/c)\cdot \partial/\partial t^* \cdot \Xi_\ell(r)$ in place of $\Lambda_\ell(r)$.

Similar equations for $a_{\ell m}$ and $b_{\ell m}$, $m \neq 0$, will be derived in terms of E_ϕ. From Eq. (4) and the identities given by Condon and Shortley [2], one can write

$$\sin\theta \, E_\phi = \sum_{\ell=1}^{\infty} \frac{1}{[\ell(\ell+1)]^{\frac{1}{2}}}$$

$$\times \Lambda_\ell(r) \sum_{m=1}^{\ell} [b_{\ell m}(t^*)\cos m\phi - a_{\ell m}(t^*)\sin m\phi] \, m \, \overline{P}_\ell^m(\cos\theta) \quad . \qquad (12)$$

Multiplying both sides of Eq. (12) by $\sin m'\phi \, \overline{P}_{\ell'}^{m'}(\cos\theta)$ and integrating over a sphere one obtains

$$\Lambda_{\ell'}(R) \, a_{\ell' m'}(t^*) = -\frac{[\ell'(\ell'+1)]^{\frac{1}{2}}}{\pi m'} \int\limits_{\substack{\text{sphere of}\\ \text{radius R}}} \sin\theta \, E_\phi \sin m'\phi \, \overline{P}_{\ell'}^{m'}(\cos\theta) \, d\Omega \quad . \qquad (13)$$

And similarly, multiplying by $\cos m'\phi \, \overline{P}_{\ell'}^{m'}(\cos\theta)$ and integrating one obtains

$$\Lambda_{\ell'}(R) \, b_{\ell' m'}(t^*) = \frac{[\ell'(\ell'+1)]^{\frac{1}{2}}}{\pi m'} \int\limits_{\substack{\text{sphere of}\\ \text{radius R}}} \sin\theta \, E_\phi \cos m'\phi \, \overline{P}_{\ell'}^{m'}(\cos\theta) \, d\Omega \quad . \qquad (14)$$

To solve the differential equations, Eqs. (10), (13), and (14), and find the resulting fields, Laplace transforms will be employed. The symbol $\mathcal{L}$ will always refer to the Laplace transform with respect to the retarded time, t^*.

Let $\lambda_\ell(z)$ and $\xi_\ell(z)$ be the following polynomials:

$$\lambda_\ell(z) = \sum_{j=0}^{\ell+1} \nu_{\ell j}\, z^{\ell+1-j} \quad , \tag{15}$$

and

$$\xi_\ell(z) = \sum_{j=0}^{\ell} \mu_{\ell j}\, z^{\ell-j} \quad . \tag{16}$$

Then, provided that the operand of Λ_ℓ, its first ℓ derivatives, the operand of Ξ_ℓ, and its first $\ell-1$ derivatives are zero initially,

$$\mathcal{L}\left\{ \Lambda_\ell(r) \right\} = \frac{1}{r^{\ell+2}}\, \lambda_\ell \left(\frac{rs}{c}\right) \quad , \tag{17}$$

and

$$\mathcal{L}\left\{ \Xi_\ell(r) \right\} = \frac{1}{r^{\ell+1}}\, \xi_\ell \left(\frac{rs}{c}\right) \quad . \tag{18}$$

The solution of Eqs. (10) and (13) can be written

$$a_{\ell m}(t^*) = R^{\ell+2}\, \mathcal{L}^{-1} \left\{ F_{\ell m}(s) \Big/ \lambda_\ell \left(\frac{Rs}{c}\right) \right\} \quad , \tag{19}$$

and the solution of Eq. (14) can be written

$$b_{\ell m}(t^*) = R^{\ell+2}\, \mathcal{L}^{-1} \left\{ G_{\ell m}(s) \Big/ \lambda_\ell \left(\frac{Rs}{c}\right) \right\} \quad , \tag{20}$$

where $F_{\ell m}(s)$ is the transform of the right-hand side of Eq. (10) for $m=0$, the transform of the right-hand side of Eq. (13) for $m \neq 0$, and $G_{\ell m}(s)$

is the transform of the right-hand side of Eq. (14). Furthermore, in Eq. (2)

$$\Xi_\ell(r)\, a_{\ell m}(t^*) = \frac{R^{\ell+2}}{r^{\ell+1}} \mathcal{L}^{-1} \left\{ \frac{\xi_\ell\left(\frac{rs}{c}\right)}{\lambda_\ell\left(\frac{Rs}{c}\right)} F_{\ell m}(s) \right\} , \tag{21}$$

$$\Xi_\ell(r)\, b_{\ell m}(t^*) = \frac{R^{\ell+2}}{r^{\ell+1}} \mathcal{L}^{-1} \left\{ \frac{\xi_\ell\left(\frac{rs}{c}\right)}{\lambda_\ell\left(\frac{Rs}{c}\right)} G_{\ell m}(s) \right\} ; \tag{22}$$

in Eqs. (3) and (4)

$$\Lambda_\ell(r)\, a_{\ell m}(t^*) = \left(\frac{R}{r}\right)^{\ell+2} \mathcal{L}^{-1} \left\{ \frac{\lambda_\ell\left(\frac{rs}{c}\right)}{\lambda_\ell\left(\frac{Rs}{c}\right)} F_{\ell m}(s) \right\} , \tag{23}$$

$$\Lambda_\ell(r)\, b_{\ell m}(t^*) = \left(\frac{R}{r}\right)^{\ell+2} \mathcal{L}^{-1} \left\{ \frac{\lambda_\ell\left(\frac{rs}{c}\right)}{\lambda_\ell\left(\frac{Rs}{c}\right)} G_{\ell m}(s) \right\} ; \tag{24}$$

and in Eqs. (5) and (6)

$$\frac{\partial}{\partial t^*} \Xi_\ell(r)\, a_{\ell m}(t^*) = \frac{R^{\ell+2}}{r^{\ell+1}} \mathcal{L}^{-1} \left\{ \frac{s\xi_\ell\left(\frac{rs}{c}\right)}{\lambda_\ell\left(\frac{Rs}{c}\right)} F_{\ell m}(s) \right\} , \tag{25}$$

$$\frac{\partial}{\partial t^*} \Xi_\ell(r)\, b_{\ell m}(t^*) = \frac{R^{\ell+2}}{r^{\ell+1}} \mathcal{L}^{-1} \left\{ \frac{s\xi_\ell\left(\frac{rs}{c}\right)}{\lambda_\ell\left(\frac{Rs}{c}\right)} G_{\ell m}(s) \right\} . \tag{26}$$

Thus transfer functions are defined that relate all the field components at an arbitrary radius r to the tangential component of $\vec{E}$ at radius R. They can be used to find the delta-function response and the delta-function response used to write Eqs. (21) through (26) in their convolution-integral form if that is desired (see MTTD for this kind of treatment of a slightly different problem). To find the inverse transforms indicated above and similar ones for related problems, one must know the roots of the polynomials $\lambda_\ell(z)$ and $\xi_\ell(z)$; a short table of these roots is given as Table I. They were computed on the CDC 6600 computer at the Weapons Laboratory, Kirtland Air Force Base, Albuquerque, New Mexico.

SPHERICAL ANTENNA PROBLEM

A very simple application of the above theory is the transient analysis of the spherical antenna. An antenna consisting of two perfectly-conducting hemispherical shells whose equators are infinitesimally separated is considered. The antenna is driven by a voltage v(t) applied between the equators which are taken to be equipotential circles. The fields and the driving function v(t) are taken to be zero initially (that is, for the retarded time $t^* = t - r/c < 0$).

In the proposed problem the fields are ϕ independent; therefore, their general time-domain form is given by the m=0 terms of Eqs. (2), (3), and (6). Since the tangential component of $\vec{E}$ must vanish on the surface of the sphere except across the infinitesimal separation of the equators of the hemispheres, E_θ, on the surface of the sphere, must have the form

$$E_\theta(t^*) = \frac{v(t^*+R/c)\ \delta(\theta-\pi/2)}{R} \quad , \tag{27}$$

where R is the radius of the spherical antenna and δ is the Dirac delta function. The function $\delta(\theta-\pi/2)$ can be expanded in terms of the normalized Legendre function $\overline{P}^1_\ell$ as follows:

$$\delta(\theta-\pi/2) = \sum_{\ell=1,3,5,\cdots}^{\infty} \overline{P}^1_\ell(0)\ \overline{P}^1_\ell(\cos\theta) \quad . \tag{28}$$

(In the following, all indicated summations over ℓ are to include only odd values of ℓ.)

Thus

$$E_\theta(t^*)\Big|_{r=R} = \frac{v(t^*+R/c)}{R} \sum_{\ell=1}^{\infty} \overline{P}^1_\ell(0)\ \overline{P}^1_\ell(\cos\theta) \quad . \tag{29}$$

Equating coefficients of $\overline{P}^1_\ell(\cos\theta)$ in Eq. (29) and Eq. (3) for each value of ℓ (with r=R), one obtains

$$\Lambda_\ell(R)\ a_\ell(t^*) = \frac{v(t^*+R/c)}{R}\ \overline{P}^1_\ell(0) \quad . \tag{30}$$

Solution of the differential equation (30) for $a_\ell(t^*)$ and substitution of this solution into Eqs. (2), (3), and (6) yields the multipole expansion of the field surrounding the antenna as a function of time. Hence,

$$a_\ell(t^*) = \overline{P}_\ell^1(0)\ R^{\ell+1}\quad \mathcal{L}^{-1}\left\{ V(s) \Big/ \lambda_\ell\left(\frac{Rs}{c}\right)\right\} \quad , \tag{31}$$

where $$V(s) = \mathcal{L}\left\{ v(t^* + R/c)\right\} .$$

The transform input admittance of the antenna will be obtained omitting the admittance due to field lines interior to the sphere. The charge on the upper half sphere is given by

$$q = \frac{1}{4\pi} \int_{\substack{\text{upper half}\\ \text{sphere}}} E_r\, da$$

$$q = \frac{R}{2} \sum_\ell - [\ell(\ell+1)]^{\frac{1}{2}}\ \Xi_\ell(R)\ a_\ell(t^*) \int_0^{\pi/2} \overline{P}_\ell(\cos\theta)\sin\theta\, d\theta\ . \tag{32}$$

From Abromowitz and Stegun [3] the integral in Eq. (32) can be evaluated as follows:

$$\int_0^{\pi/2} \overline{P}_\ell(\cos\theta)\sin\theta\, d\theta = \left[\frac{2\ell+1}{2}\right]^{\frac{1}{2}} \int_0^1 P_\ell(x)\, dx$$

$$= \frac{[(2\ell+1)\ \pi]^{\frac{1}{2}}}{2^{3/2}\ \Gamma(1-\ell/2)\ \Gamma(\ell/2+3/2)}$$

$$= \frac{[2(2\ell+1)]^{\frac{1}{2}}}{3\, B(1-\ell/2,\ 3/2+\ell/2)} \quad , \tag{33}$$

where $B(n,m) = \Gamma(m)\,\Gamma(n) \Big/ \Gamma(m+n)$ = the Beta function. The current entering the upper half sphere is the time derivative of q, hence

$$I(s) = \frac{-1}{6} \sum_{\ell} \frac{[2\ell(\ell+1)(2\ell+1)]^{\frac{1}{2}}\, s}{B(1-\ell/2,\ 3/2+\ell/2)\, R^{\ell}}\ \xi_{\ell}\left(\frac{Rs}{c}\right) A_{\ell}(s) \quad , \tag{34}$$

$$\text{where } A_{\ell}(s) = \mathcal{L}\left\{a_{\ell}(t^{*})\right\} = \overline{P}_{\ell}^{1}(0)\ R^{\ell+1}\ V(s) \Big/ \lambda_{\ell}\left(\frac{Rs}{c}\right) .$$

The input admittance is given by

$$Y(s) = I(s) \Big/ V(s) \quad ,$$

$$Y(s) = \frac{-R}{6} \sum_{\ell} \frac{\overline{P}_{\ell}^{1}(0)\ [2\ell(\ell+1)(2\ell+1)]^{\frac{1}{2}}\ s\xi_{\ell}\left(\frac{Rs}{c}\right)}{B(1-\ell/2,\ 3/2+\ell/2)\ \lambda_{\ell}\left(\frac{Rs}{c}\right)} \quad . \tag{35}$$

To illustrate application of the theory, the dipole component of the field surrounding the spherical antenna will be calculated when a unit step-function voltage is applied. Let

$$v(t) = v(t^{*}+R/c) = u(t^{*}-t_{o}^{*}) \ , \quad t_{o}^{*} \geq 0 \quad ,$$

where u is the unit step function. Then

$$V(s) = \mathcal{L}\left\{v(t^{*}+R/c)\right\} = e^{-t_{o}^{*}s} \Big/ s \quad .$$

The dipole terms of Eqs. (2), (3), and (6) can be written (with $\overline{P}_{1}(\cos\theta) = (3/2)^{\frac{1}{2}}\cos\theta$, $\overline{P}_{1}^{1}(\cos\theta) = -(\sqrt{3}/2)\sin\theta$, $\overline{P}_{1}^{1}(0) = -\sqrt{3}/2)$,

$$E_r = -(\sqrt{3}/r)\ \Xi_1(r)\, a_1(t^*) \cos\theta \quad , \tag{36}$$

$$E_\theta = -(\sqrt{3}/2)\ \Lambda_1(r)\, a_1(t^*) \sin\theta \quad , \tag{37}$$

$$B_\phi = -\frac{\sqrt{3}}{2c}\ \frac{\partial}{\partial t^*}\ \Xi_1(r)\, a_1(t^*) \sin\theta \quad , \tag{38}$$

where

$$\Xi_1(r) = \frac{1}{rc}\ \frac{\partial}{\partial t^*} + \frac{1}{r^2} \quad , \tag{39}$$

$$\xi_1(z) = z + 1 \quad , \tag{40}$$

$$\Lambda_1(r) = \frac{1}{rc^2}\ \frac{\partial^2}{\partial t^{*2}} + \frac{1}{r^2 c}\ \frac{\partial}{\partial t^*} + \frac{1}{r^3} \quad , \tag{41}$$

$$\lambda_1(z) = z^2 + z + 1 = (z + \tfrac{1}{2})^2 + 3/4 \quad . \tag{42}$$

Equation (31) becomes (for $\ell = 1$)

$$a_1(t^*) = -\frac{\sqrt{3}}{2}\ R^2\ \mathcal{L}^{-1} \left\{ \frac{e^{-t_o^* s}}{s} - \frac{e^{-t_o^* s}\,(s + c/R)}{(s + c/2R)^2 + 3c^2/4R^2} \right\}$$

$$a_1(t^*) = -\frac{\sqrt{3}}{2}\ R^2\ u(t^* - t_o^*) \left\{ 1 - e^{-\frac{c}{2R}(t^* - t_o^*)} \times \left[\cos \frac{\sqrt{3}\,c}{2R}\,(t^* - t_o^*) + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}\,c}{2R}\,(t^* - t_o^*) \right] \right\} \quad . \tag{43}$$

Differentiating Eq. (43) with respect to t^*, one obtains

$$a_1'(t^*) = -Rcu(t^* - t_o^*)\ e^{-\frac{c}{2R}(t^* - t_o^*)} \sin \frac{\sqrt{3}\,c}{2R}(t^* - t_o^*) \quad , \tag{44}$$

and

$$a_1''(t^*) = -\frac{\sqrt{3}}{2}\, c^2 u(t^* - t_o^*)\ e^{-\frac{c}{2R}(t^* - t_o^*)}$$

$$\times \left[\cos \frac{\sqrt{3}\,c}{2R}(t^* - t_o^*) - \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}\,c}{2R}(t^* - t_o^*)\right] \quad . \tag{45}$$

The dipole field is then given by

$$E_r = -\sqrt{3}\left(\frac{a_1'}{r^2 c} + \frac{a_1}{r^3}\right) \cos\theta \quad , \tag{46}$$

$$E_\theta = -\frac{\sqrt{3}}{2}\left(\frac{a_1''}{rc^2} + \frac{a_1'}{r^2 c} + \frac{a_1}{r^3}\right) \sin\theta \quad , \tag{47}$$

$$B_\phi = -\frac{\sqrt{3}}{2}\left(\frac{a_1''}{rc^2} + \frac{a_1'}{r^2 c}\right) \sin\theta \quad ; \tag{48}$$

the radiating part of the dipole field is

$$E_r = 0$$

$$E_\theta = B_\phi = \frac{3}{4r}\, u(t^* - t_o^*)\ e^{-\frac{c}{2R}(t^* - t_o^*)}$$

$$\times \left[\cos \frac{\sqrt{3}\,c}{2R}(t^* - t_o^*) - \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}\,c}{2R}(t^* - t_o^*)\right] \sin\theta \quad . \tag{49}$$

The dipole contribution to the admittance is

$$Y(s) = \frac{3c}{8} \frac{s(s + c/R)}{s^2 + sc/R + c^2/R^2} \quad . \tag{50}$$

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TABLE I

ROOTS OF $\xi_\ell(z) = 0$ AND $\lambda_\ell(z) = 0$

	$\xi_\ell(z) = 0$		$\lambda_\ell(z) = 0$	
Order (ℓ)	Real part of z	Imaginary part of z	Real part of z	Imaginary part of z
1	-1	0	$-\frac{1}{2}$	$\sqrt{3}/2$
2	- 3/2	$\sqrt{3}/2$	-1.596071638	0
			-0.7019641810	1.807339494
3	-2.322185355	0	-2.157137812	0.8705692254
	-1.838907323	1.754380960	-0.8428621876	2.757855949
4	-2.896210603	0.8672341289	-2.948741846	0
	-2.103789397	2.657418042	-2.571399191	1.752302755
			-0.9542298856	3.714784350
5	-3.646738595	0	-3.544264727	0.8689259641
	-3.351956399	1.742661416	-2.908061830	2.644316257
	-2.324674303	3.571022920	-1.047673443	4.676410473
6	-4.248359396	0.8675096732	-4.284595129	0
	-3.735708356	2.626272311	-4.033560546	1.743039771
	-2.515932248	4.492672954	-3.195236371	3.544886180
			-1.128905519	5.641635033
7	-4.971786859	0	-4.897196456	0.8683913178
	-4.758290528	1.739286061	-4.454008033	2.623305066
	-4.070139164	3.517174048	-3.447592085	4.452564752
	-2.685676879	5.420694131	-1.201203426	6.609715253
8	-5.587886043	0.8676144454	-5.615557382	0
	-5.204840791	2.616175153	-5.426268729	1.739810862
	-4.368289217	4.414442500	-4.825424448	3.509503129
	-2.838983949	6.353911299	-3.673889391	5.366221890
			-1.266638741	7.580125416
9	-6.297019182	0	-6.238321191	0.8681549411
	-6.129367904	1.737848383	-5.895441151	2.615285471
	-5.604421820	3.498156918	-5.159793630	4.401147200
	-4.638439887	5.317271675	-3.879831702	6.284977175
	-2.979260798	7.291463688	-1.326612326	8.552478962
10	-6.922044905	0.8676651955	-6.944430489	0
	-6.615290965	2.611567921	-6.792207399	1.738299850

Order (ℓ)	$\xi_\ell(z) = 0$ Real part of z	Imaginary part of z	$\lambda_\ell(z) = 0$ Real part of z	Imaginary part of z
	-5.967528329	4.384947189	-6.319093633	3.495019225
	-4.886219567	6.224985482	-5.464999504	5.297720483
	-3.108916234	8.232699459	-4.069364260	7.208135960
			-1.382119961	9.526482648
11	-7.622339846	0	-7.573931598	0.8680302983
	-7.484229861	1.737102821	-7.293547002	2.611314297
	-7.057892388	3.489014504	-6.706737525	4.378914907
	-6.301337455	5.276191744	-5.746544433	6.198743398
	-5.115648284	7.137020759	-4.245342190	8.135141649
	-3.229722090	9.177111569	-1.433897251	10.50190823
12	-8.253422011	0.8676935720	-8.272218847	0
	-7.997270600	2.609066537	-8.144817101	1.737469494
	-7.465571240	4.370169593	-7.753179545	3.487518640
	-6.611004250	6.171534993	-7.065042865	5.266763671
	-5.329708591	8.052906864	-6.008441122	7.103789672
	-3.343023308	10.12429681	-4.409905440	9.065541502
			-1.482504503	11.47857419
13	-8.947709674	0	-8.906513703	0.8679566810
	-8.830252084	1.736666400	-8.669061587	2.609045336
	-8.470591771	3.483868451	-8.178729525	4.366979417
	-7.844380277	5.254903407	-7.398895045	6.158321510
	-6.900372826	7.070644312	-6.253717282	8.012486055
	-5.350683983	8.972247775	-4.564703571	9.998962132
	-3.449867221	11.07392855	-1.528379288	12.45633355
14	-9.583171394	0.8677110289	-9.599371821	0
	-9.363145852	2.607553324	-9.489786996	1.736963741
	-8.911000555	4.361604178	-9.155106803	3.483095215
	-8.198846970	6.143041071	-8.575796737	5.249642938
	-7.172395962	7.973217354	-7.711994862	7.053341694
	-5.720352384	9.894707597	-6.484720470	8.924507011
	-3.551086883	12.02573803	-4.711037728	10.93509174
			-1.571870493	13.43506541
15	-10.27310967	0	-10.23725187	0.8679096214
	-10.17091400	1.736388919	-10.03119976	2.607623288
	-9.859567228	3.480671211	-9.609164972	4.359754421
	-9.323599321	5.242258895	-8.948633922	6.135398676
	-8.532459052	7.034393626	-8.007222183	7.951588510
	-7.429396993	8.878982621	-6.703311930	9.839568384
	-5.900151714	10.81999914	-4.849954149	11.87366708
	-3.647356862	12.97950107	-1.613261215	14.41466900

Order (ℓ)	$\xi_\ell(z) = 0$ Real part of z	Imaginary part of z	$\lambda_\ell(z) = 0$ Real part of z	Imaginary part of z
16	-10.91188608	0.8677225274	-10.92612065	0
	-10.71898582	2.606567007	-10.82995926	1.736632693
	-10.32511960	4.356163381	-10.53750416	3.480256180
	-9.712326333	6.125760891	-10.03596240	5.239049239
	-8.847968197	7.928772856	-9.300558297	7.024111048
	-7.673240791	9.787697438	-8.286867899	8.852842627
	-6.071241383	11.74787494	-6.910995074	10.75742137
	-3.739231797	13.93502848	-4.982307935	12.81446362
			-1.652784641	15.39505933
17	-11.59852949	0	-11.56678323	0.8678777307
	-11.50807678	1.736201538	-11.38478098	2.606671836
	-11.23343682	3.478543891	-11.01384447	4.355023741
	-10.76413418	5.234074902	-10.43919184	6.120946018
	-10.08029444	7.012009983	-9.634214533	7.915635886
	-9.147588678	8.825998301	-8.552788135	9.756902574
	-7.905449596	10.69914508	-7.109003617	11.67784725
	-6.234580978	12.67812023	-5.108807833	13.75728820
	-3.827173785	14.89215892	-1.690635357	16.37616397
18	-12.23990214	0.8677305005	-12.25259650	0
	-12.06813584	2.605887882	-12.16691738	1.736404110
	-11.71894880	4.352479754	-11.90708711	3.478321054
	-11.18003902	6.114394093	-11.46424513	5.231993548
	-10.43001297	7.900893103	-10.82180201	7.005378298
	-9.433132221	9.725900314	-9.951749883	8.809829094
	-8.127283945	11.61313175	-8.806510338	10.66358444
	-6.390972784	13.61054735	-7.298363798	12.60065288
	-3.911572291	15.85075360	-5.230048494	14.70197329
			-1.726977595	17.35792065
19	-12.92396306	0	-12.89548140	0.8678551283
	-12.84282780	1.736069051	-12.73238558	2.606003361
	-12.59706281	3.477054900	-12.40135067	4.351747709
	-12.17923126	5.228450548	-11.89192355	6.111176852
	-11.57560107	6.997076375	-11.18619059	7.892262322
	-10.76353844	8.792293022	-10.25493485	9.706551135
	-9.706102401	10.62832110	-9.049308382	11.57272077
	-8.339800719	12.52948382	-7.479939417	13.52566693
	-6.541095062	14.54499130	-5.346534194	15.64837254
	-3.992758918	16.81069206	-1.761951366	18.34027538
20	-13.56742428	0.8677362550	-13.57887930	0
	-13.41259714	2.605400147	-13.50161525	1.736239623

Order (ℓ)	$\xi_\ell(z) = 0$ Real part of z	Imaginary part of z	$\lambda_\ell(z) = 0$ Real part of z	Imaginary part of z
	-13.09882247	4.349864912	-13.26778316	3.476940997
	-12.61728132	6.106479870	-12.87098107	5.227039945
	-11.95309080	7.882058434	-12.29949881	6.992550739
	-11.08258033	9.686093242	-11.53433738	8.781506126
	-9.967762479	11.53311473	-10.54524877	10.60566926
	-8.543895727	13.44804527	-9.282257065	12.48415921
	-6.685526878	15.48130619	-7.654465117	14.45273675
	-4.071018562	17.77186907	-5.458696640	16.59635726
			-1.795677094	19.32318100
21	-14.24940652	0	-14.22358003	0.8678385292
	-14.17584550	1.735971921	-14.07584419	2.605515492
	-13.95340920	3.475971113	-13.77679047	4.349380969
	-13.57662086	5.224408900	-13.31879695	6.104235294
	-13.03556064	6.986558406	-12.68913589	7.876071653
	-12.31439774	8.769266832	-11.86789883	9.673014935
	-11.38857706	10.58218072	-10.82394195	11.50705844
	-10.21918526	12.44014662	-9.506272503	13.39776101
	-8.740335564	14.36867549	-7.822571456	15.38172575
	-6.824766934	16.41936230	-5.566908539	17.54581363
	-4.146597975	18.73419204	-1.828259189	20.30659606
22	-14.89458435	0.8677405436	-14.90502055	0
	-14.75364244	2.605037951	-14.83466244	1.736117299
	-14.46866184	4.347939382	-14.62205230	3.475921338
	-14.03309328	6.100731104	-14.26240844	5.223420553
	-13.43611972	7.868656137	-13.74713019	6.983338528
	-12.66111359	9.658623316	-13.06264801	8.761683857
	-11.68275194	11.48044728	-12.18827681	10.56669437
	-10.46129048	13.34929282	-11.09208189	12.41060167
	-8.929781865	15.29124738	-9.722142623	14.31339969
	-6.959248085	17.35904377	-7.984804097	16.31251122
	-4.219712426	19.69757906	-5.671494101	18.49664040
			-1.859788826	21.29048391
23	-15.57485737	0	-15.55123282	0.8678259817
	-15.50757534	1.735898592	-15.41619986	2.605148406
	-15.30439066	3.475157256	-15.14340874	4.347613375
	-14.96114989	5.221402104	-14.72712151	6.099112417
	-14.47043646	6.978844723	-14.15793539	7.864333101
	-13.82068737	8.752730166	-13.42157121	9.649324823
	-12.99459350	10.55004813	-12.49666964	11.46245235
	-11.96615514	12.38079035	-11.35058814	13.31618977

	$\xi_\ell(z) = 0$		$\lambda_\ell(z) = 0$	
Order (ℓ)	Real part of z	Imaginary part of z	Real part of z	Imaginary part of z
	-10.69487327	14.26043912	-9.930550559	15.23095973
	-9.112810041	16.21564568	-8.141638718	17.24498247
	-7.089348684	18.30024661	-5.772737281	19.44874709
	-4.290550955	20.66195721	-1.890346140	22.27481193
24	-16.22147109	0.8677438252	-16.23105494	0
	-16.09212072	2.604761558	-16.16646701	1.736023852
	-15.83103289	4.346479202	-15.97151786	3.475146171
	-15.43320455	6.096415282	-15.64253880	5.220691602
	-14.89050383	7.858742209	-15.17301358	6.976478969
	-14.19073675	9.638729842	-14.55286968	8.747188643
	-13.31600205	11.44346205	-13.76721980	10.53892869
	-12.23969579	13.28311312	-12.79411080	12.36020014
	-10.92062625	15.17348031	-11.60025909	14.22372108
	-9.289923966	17.14176557	-10.13209287	16.15033540
	-7.215401550	19.24287714	-8.293492744	18.17903929
	-4.359280561	21.62726133	-5.870888319	20.40205247
			-1.920001978	23.25955100
25	-16.90031386	0	-16.87854503	0.8678162671
	-16.83832203	1.735841876	-16.75419504	2.604865201
	-16.65130125	3.474530310	-16.50337285	4.346257320
	-16.33602500	5.219102091	-16.12164579	6.095217004
	-15.88679381	6.973006459	-15.60185246	7.855521926
	-15.29487578	8.740402143	-14.93335200	9.631865966
	-14.54753484	10.52660014	-14.10072829	11.43042853
	-13.62634842	12.33878770	-13.08149866	13.25985294
	-12.50416676	14.18732458	-11.84179296	15.13310098
	-11.13915683	16.08831926	-10.32729399	17.07142975
	-9.461567618	18.06951145	-8.440734688	19.11459068
	-7.337701148	20.18685057	-5.966168988	21.35648327
	-4.426049574	22.59343287	-1.948819316	24.24467496
26	-17.54814642	0.8677463921	-17.55700666	0
	-17.42862048	2.604545817	-17.49731255	1.735950851
	-17.18768926	4.345344928	-17.31729606	3.474542867
	-16.82144133	6.093088204	-17.01407933	5.218580968
	-16.32359672	7.851183322	-16.58259988	6.971223131
	-15.68491245	9.623793917	-16.01515286	8.736228808
	-14.89218044	11.41628390	-15.30060811	10.51832044
	-13.92651355	13.23595022	-14.42308377	12.32375776
	-12.76026418	15.09333929	-13.35961964	14.16133013
	-11.35100096	17.00486627	-12.07580422	16.04424145

Order (ℓ)	$\xi_\ell(z) = 0$ Real part of z	$\xi_\ell(z) = 0$ Imaginary part of z	$\lambda_\ell(z) = 0$ Real part of z	$\lambda_\ell(z) = 0$ Imaginary part of z
	-9.628134422	18.99879586	-10.51661734	17.99415368
	-7.456509390	21.13208994	-8.583691671	20.05155370
	-4.490990401	23.56041900	-6.058776846	22.31197320
			-1.976854405	25.23016024
27	-18.22577477	0	-18.20559118	0.8678085929
	-18.16830108	1.735797105	-18.09035215	2.604642087
	-17.99504811	3.474036970	-17.85818874	4.345193638
	-17.70345844	5.217302036	-17.50560276	6.092182182
	-17.28905250	6.968474094	-17.02700423	7.848724068
	-16.74505909	8.730939652	-16.41422400	9.618577430
	-16.06179412	10.50888095	-15.65570518	11.40650432
	-15.22563346	12.30772339	-14.73515106	13.21885111
	-14.21727168	14.13487778	-13.62916663	15.06455532
	-13.00860300	16.00107715	-12.30283686	16.95706054
	-11.55663418	17.92303835	-10.70047523	18.91842512
	-9.789974806	19.92953860	-8.722655531	20.98985255
	-7.572060361	22.07852515	-6.148888716	23.26846200
	-4.554221779	24.52817191	-2.004157724	26.21598548
28	-18.87465487	0.8677484376	-18.88289312	0
	-18.76356269	2.604374174	-18.82740233	1.735892734
	-18.53987200	4.344445905	-18.66018061	3.474063973
	-18.20046501	6.090466764	-18.37892791	5.216913752
	-17.74038387	7.845277773	-17.97962880	6.967101388
	-17.15243625	9.612259487	-17.45624527	8.727720011
	-16.42655259	11.39562296	-16.80020640	10.50253905
	-15.54873816	13.20086124	-15.99957891	12.29636829
	-14.49930730	15.03550169	-15.03769272	14.11564518
	-13.24972941	16.91046307	-13.89075359	15.96945629
	-11.75648060	18.84275858	-12.52337501	17.87148189
	-9.947402385	20.86166593	-10.87923576	19.84416834
	-7.684564217	23.02609210	-8.857887827	21.92941772
	-4.615850634	25.49664813	-6.236663553	24.22589477
			-2.030774757	27.20213130
29	-19.55123919	0	-19.53242552	0.86780243
	-19.49766924	1.735761145	-19.42504964	2.60446316
	-19.33628646	3.473641709	-19.20893692	4.34434356
	-19.06502886	5.215866017	-18.88127655	6.08976971
	-18.68030877	6.964881144	-18.43761580	7.84336101
	-18.17676044	8.723503809	-17.87153133	9.60820123
	-17.54682610	10.49512009	-17.17410059	11.38808057

Order (ℓ)	$\xi_\ell(z) = 0$ Real part of z	Imaginary part of z	$\lambda_\ell(z) = 0$ Real part of z	Imaginary part of z
	-16.78009582	12.28397748	-16.33305484	13.18786260
	-15.86224149	14.09564116	-15.33138501	15.01407880
	-14.77322894	15.93775658	-14.14492745	16.87596486
	-13.48413102	17.82142666	-12.73785159	18.78743432
	-11.95092018	19.76395557	-11.05322930	20.77131330
	-10.10069906	21.79510986	-8.98962399	22.87018535
	-7.794210402	23.97473208	-6.32224483	25.18422128
	-4.675973637	26.46580802	-2.05674665	28.18858003
30	-20.20102930	0.8677500940	-20.20872716	0
	-20.09725659	2.604235367	-20.15688589	1.73584571
	-19.88848512	4.343721058	-20.00075176	3.47367740
	-19.57218844	6.088363125	-19.73845697	5.21557316
	-19.14437983	7.840570124	-19.36676051	6.96380598
	-18.59934154	9.603146996	-18.88083532	8.72097023
	-17.92919531	11.37949408	-18.27392292	10.49015091
	-17.12322763	13.17390105	-17.53678983	12.27516618
	-16.16680792	14.99200835	-16.65686559	14.08093778
	-15.03958026	16.84158025	-15.61683080	15.91409322
	-13.71224513	18.73390191	-14.39217775	17.78401667
	-12.14029484	20.68656294	-12.94665553	19.70485137
	-10.25011923	22.72980761	-11.22275354	21.69979509
	-7.901170350	24.92439109	-9.11807674	23.81209657
	-4.734678501	27.43561533	-6.40576251	26.14339550
			-2.08211075	29.17531550

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