EMP Theoretical Notes

Note L

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Diffusion Approximation Validity Analysis

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Abstract

The following is a derivation of a criterion for evaluating the surface burst nuclear electromagnetic pulse (EMP) fields using the formalism of the one-dimensional, inhomogeneous diffusion equation. The demonstration consists of finding the conditions on air conductivity required to insure that the solution to the diffusion equation satisfies, approximately, Maxwell's equations (and hence the inhomogeneous wave equation). This solution to the inhomogeneous diffusion equation is the general solution because the homogeneous solution is zero (we expect no fields without the driving current).

The following is a demonstration of the validity of evaluating the surface burst nuclear electromagnetic pulse (EMP) fields using the formalism of the one-dimensional, inhomogeneous diffusion equation. The demonstration consists of finding the conditions on air conductivity required to insure that the solution to the diffusion equation satisfies, approximately, Maxwell's equations (and hence the inhomog. wave equation). This solution to the inhomogeneous diffusion equation is the general solution because the homogeneous solution is zero (we expect no fields without the driving current).

First let's establish the criteria which insure that the EMP fields are described approximately by the diffusion equation. In regions much nearer the ground than the nuclear burst point, the field components and spatial dependences are conveniently described in a cylindrical coordinate system.* We will assume azimuthal (φ) symmetry of the radiation source and the environment. In this case, Maxwell's equations reduce to (if only radial driving currents are considered)

$$\frac{-\partial B\varphi}{\partial z} = \mu(J_r + \sigma E_r + \epsilon \dot{E}_r) \tag{1}$$

$$\frac{1}{r}\frac{\partial(rB\varphi)}{\partial r} = \mu(\sigma E_z + \epsilon \dot{E}_z) \tag{2}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial B\varphi}{\partial t} \tag{3}$$

If we can be assured that

$$|\epsilon \dot{E}_r| \ll |\sigma E_r| \tag{4a}$$

$$|\epsilon \dot{E}_z| \ll |\sigma E_z| \tag{4b}$$

and

$$\left|\frac{\partial E_r}{\partial z}\right| \gg \left|\frac{\partial E_z}{\partial r}\right| \tag{5}$$

then, our set of equations becomes

$$-\frac{\partial B\varphi}{\partial z} = \mu(J_r + \sigma E_r) \tag{6}$$

*With the z axis an outward normal to the earth and the origin at the projection of the detonation point on the earth's surface.

$$\frac{1}{r}\frac{\partial(rB\phi)}{\partial r} = \mu\sigma E_z \tag{7}$$

and

$$\frac{\partial E_r}{\partial z} = -\frac{\partial B\phi}{\partial t} \tag{8}$$

If we take the derivative of equation (6) with respect to z and use equation (8) to eliminate $(\partial E_r/\partial z)$, we obtain

$$-\frac{\partial^2 B\phi}{\partial z^2} = \mu\left[\left(\frac{\partial J_r}{\partial z} + \sigma\left(-\frac{\partial B\phi}{\partial t}\right)\right)\right] \tag{9}$$

in regions where the electrical properties are not functions of z. This equation can be rearranged to the following form:

$$\frac{\partial^2 B\phi}{\partial z^2} - \mu\sigma\frac{\partial B\phi}{\partial t} = -\mu\frac{\partial J_r}{\partial z}\ . \tag{10}$$

This is the one-dimensional inhomogeneous diffusion equation. I intend to develop a criterion for how late in time diffusion theory is applicable. This demonstration will consist of showing that the field solution to the diffusion equation (equation 10) for a step-function driver insures that the criteria (4a), (4b), and (5) are met for certain levels of conductivity. And, therefore, the solution to the diffusion equation is also an approximate solution to Maxwell's equations.

The solution of equation (10) with a step function current and conductivity driver has been obtained by Babb and Graham using the Green's function method, and by Baum using Laplace transforms (for an infinitely conducting ground). This solution is

$$B\phi = 2\mu J\ c\ \frac{\varepsilon_o}{\sqrt{\sigma}}\left[\sqrt{\frac{t-\frac{r}{c}}{\pi\varepsilon_o}}\ e^{-\frac{\mu_o\sigma z^2}{4(t-\frac{r}{c})}} - 2\sqrt{\frac{\mu_o\sigma z^2}{4\varepsilon_o}}\ \mathrm{erfc}\left(\sqrt{\frac{\mu\sigma z^2}{4(t-\frac{r}{c})}}\right)\right] \tag{11}$$

In particular when z = 0

$$B\phi(z=0) = 2\mu J\ c\ \frac{\varepsilon_o}{\sqrt{\sigma}}\sqrt{\frac{t-\frac{r}{c}}{\pi\varepsilon_o}} = \frac{2\sqrt{\mu}\ J}{\sqrt{\sigma\pi}}\sqrt{t-\frac{r}{c}} \tag{12}$$

for $t - \frac{r}{c} > 0$.

I will investigate the conditions under which the fields given by equations (12), (6), and (7) satisfy the criteria (4a), (4b), and (5). These conditions concern times of applicability and conductivity levels. Presumably, these same conditions are not strictly applicable for $z > 0$. Considering the fact that the solutions (11) and (12) ignore electron turning and finite ground conductivity effects, I don't believe that a more rigorous demonstration is warranted. In any case, we can examine our final E_r and E_z solutions for satisfaction of (4a), (4b), and (5) for $z > 0$.

Our first criterion (4a) is that $|\epsilon \dot{E}_r| \ll |\sigma E_r|$. This criterion is met if both quantities are zero at $z = 0$, and

$$\left|\frac{\sigma \partial E_r}{\partial z}\right| \gg \left|\epsilon \frac{\partial}{\partial t}\left(\frac{\partial E_r}{\partial z}\right)\right| \tag{13}$$

which, in the diffusion approximation, is equivalent to

$$\left|\frac{\sigma \partial B\phi}{\partial t}\right| \gg \left|\epsilon \frac{\partial^2 B\phi}{\partial t^2}\right| \tag{14}$$

Thus, if our $B\phi$ from equation (12) satisfies inequality (14) then it satisfies inequality (13). And, since $|\epsilon \dot{E}_r|$ equals $|\sigma E_r|$ at $z = 0$ (because E_r is always zero at the surface), then the σE_r grows much more rapidly than $\epsilon \dot{E}_r$ with z, and thus the criteria (4a) is satisfied for $z > 0$ and $\epsilon \dot{E}_r$ can be ignored. We can now state the conditions on time and conductivity which insure satisfaction of (4). Let's simplify expression (12) for $B\phi$.

$$B\phi(t) = B_r\sqrt{t - \frac{r}{c}} \tag{15}$$

where

$$B_r = \frac{2\sqrt{\mu}\ J}{\sqrt{\sigma\pi}} \tag{16}$$

then

$$\frac{\partial B\phi}{\partial t} = B_r \frac{1/2}{\sqrt{t - \frac{r}{c}}} = \frac{1}{2}\frac{B\phi}{(t - \frac{r}{c})} \tag{17}$$

and

$$\frac{\partial^2 B\phi}{\partial t} = -\frac{B_r}{4}\frac{1}{\left(\sqrt{t - \frac{r}{c}}\right)^3} = -\frac{1}{4}\frac{B\phi}{(t - \frac{r}{c})^2} \tag{18}$$

and (14) requires that

$$\left|\frac{\sigma}{2}\frac{B\varphi}{(t-\frac{r}{c})}\right| \gg \left|\epsilon\frac{B\varphi}{4(t-\frac{r}{c})^2}\right| \tag{19}$$

or

$$\sigma \gg \frac{\epsilon}{2(t-\frac{r}{c})} = \frac{\epsilon}{2\tau} \cong \frac{4.427(10^{-12})}{\tau} \tag{20}$$

where $\tau = t - \frac{r}{c}$ (the retarded time) .

The second diffusion equation criterion, inequality (4b), is met if (4a) is as will be seen as a substep of the following analysis. The third diffusion approximation criterion, inequality (5), requires that $|\partial E_r/\partial z| \gg |\partial E_z/\partial r|$. The left hand side of this inequality can be expressed as

$$|\partial E_r/\partial z| = |\partial B\varphi/\partial t| \tag{21}$$

in the diffusion approximation. And, if we assume (4b) temporarily,

$$|\epsilon\dot{E}_z| \ll |\sigma E_z| \quad , \tag{22}$$

then we can set

$$E_z = \frac{1}{\mu\sigma r}\frac{\partial(rB\varphi)}{\partial r} \quad , \tag{23}$$

Assuming this allows us to express the right-hand side of inequality (5) in terms of $B\varphi$

$$|\partial E_z/\partial r| = \frac{1}{\mu}\left|\frac{\partial}{\partial r}\left[\frac{1}{\sigma r}\frac{\partial(rB\varphi)}{\partial r}\right]\right| \quad . \tag{24}$$

So our third criterion becomes, in terms of $B\varphi$,

$$|\partial B\varphi/\partial t| \gg \frac{1}{\mu}\left|\frac{\partial}{\partial r}\left[\frac{1}{\sigma r}\frac{\partial rB\varphi}{\partial r}\right]\right| \quad . \tag{25}$$

We know that

$$B\phi = \frac{2\sqrt{\mu}\, J}{\sqrt{\sigma\pi}} \sqrt{t - \frac{r}{c}}$$

for the diffusion approximation solution for a step-function driver. We now need the radial dependence of J and σ. We will assume that they are defined by

$$J = \frac{J_o e^{-r/\lambda}}{4\pi r^2} \qquad \text{for } t > \frac{r}{c} \tag{26}$$

$$\sigma = \frac{\sigma_o e^{-r/\lambda}}{4\pi r^2} \qquad \text{for } t > \frac{r}{c} \tag{27}$$

where λ is an effective attenuation constant for the current conductivity. Then,

$$B\phi = \frac{\sqrt{\mu}\, J_o}{\pi\sqrt{\sigma_o}} \; \frac{e^{-r/2\lambda}}{r} \sqrt{t - \frac{r}{c}} \quad . \tag{28}$$

From this expression,

$$\frac{\partial B\phi}{\partial t} = \frac{1}{2} \frac{B\phi}{(t - \frac{r}{c})} \tag{29}$$

and

$$\frac{\partial (rB\phi)}{\partial r} = \frac{-\sqrt{\mu}\, J_o}{2\pi\sqrt{\sigma_o}} e^{-r/2\lambda} \sqrt{t - \frac{r}{c}} \left[\frac{1}{\lambda} + \frac{1}{c(t - \frac{r}{c})}\right] . \tag{30}$$

Then, one can express the right-hand side of inequality (25) as

$$\frac{1}{\mu}\frac{\partial}{\partial r}\left(\frac{1}{\sigma r}\frac{\partial rB\phi}{\partial r}\right) = \frac{1}{\mu}\frac{\partial}{\partial r}\left[\frac{-2\sqrt{\mu}\, r\, J_o\, e^{+r/2\lambda}\sqrt{t - \frac{r}{c}}}{\sigma_o^{3/2}} \left(\frac{1}{\lambda} + \frac{1}{c(t - \frac{r}{c})}\right)\right] \tag{31}$$

or, finally,

$$\frac{1}{\mu}\frac{\partial}{\partial r}\left[\frac{1}{\sigma r}\quad\frac{\partial rB\phi}{\partial r}\right] = -\frac{B\phi}{\mu r\sigma}\left[\frac{1}{2\lambda} + \frac{r}{(2\lambda)^2} + \frac{1}{2c\tau} + \frac{r}{4c^2\tau^2}\right] \tag{32}$$

where $\tau = t - \frac{r}{c}$.

So, our inequality (5) becomes

$$\left|\frac{1}{2}\frac{B\phi}{\tau}\right| \gg \left|\frac{B\phi}{\mu\sigma r}\left[\frac{1}{2\lambda} + \frac{r}{(2\lambda)^2} + \frac{1}{2c\tau} + \frac{r}{(2c\tau)^2}\right]\right| \quad , \tag{33}$$

or, in terms of σ

$$\sigma \gg \frac{2\tau}{\mu r}\left[\frac{1}{2\lambda} + \frac{r}{(2\lambda)^2} + \frac{1}{2c\tau} + \frac{r}{(2c\tau)^2}\right] \tag{34}$$

This conductivity requirement is a function of time, radius, and the spatial attenuation constant of the current and conductivity.

Back at inequality (22) we assumed satisfaction of criterion (4b) which required that $|\epsilon\dot{E}_z| \ll |\sigma E_z|$. We have not yet justified this assumption. If this assumption is true, then, as was stated in equation (23),

$$E_z = \frac{1}{\mu\sigma r}\frac{\partial(rB\phi)}{\partial r} \quad , \tag{35}$$

Thus, the right-hand side of inequality (22) can be expressed in terms of $B\phi$

$$|\sigma E_z| = \left|\frac{1}{\mu r}\frac{\partial rB\phi}{\partial r}\right| \tag{36}$$

or

$$|\sigma E_z| = \left|\frac{-\sqrt{\mu}\, J_o\, e^{-r/2\lambda}\sqrt{t - \frac{r}{c}}}{\mu r\, 2\pi\sqrt{\sigma_o}}\left[\frac{1}{\lambda} + \frac{1}{c(t - \frac{r}{c})}\right]\right| \tag{37}$$

The left-hand side is expressible as

$$|\epsilon \dot{E}_z| = \left| \frac{\epsilon \sqrt{\mu}\, J_c\, e^{-r/2\lambda}}{\mu r \sigma\, 2\pi \sqrt{\sigma_c}} \frac{\partial}{\partial t}\left[\frac{\sqrt{t - \frac{r}{c}}}{\lambda} + \frac{1}{c\sqrt{t - \frac{r}{c}}}\right]\right|$$

$$= \left| \frac{\epsilon\, J_o\, e^{-r/2\lambda}}{2\pi\, r\sigma \sqrt{\mu\sigma_o}} \left[\frac{1}{2\lambda\sqrt{t - \frac{r}{c}}} - \frac{1}{2c(t - \frac{r}{c})^{3/2}}\right]\right| \tag{38}$$

Substituting (37) and (38) into (22) yields the conditions for satisfaction of the second criterion, (4b).

$$\left| \frac{\epsilon\, J_o\, e^{-r/2\lambda}}{2\pi\, r\sigma \sqrt{\mu\sigma_o}} \frac{\sqrt{t - \frac{r}{c}}}{2(t - \frac{r}{c})} \left[\frac{1}{\lambda} - \frac{1}{c(t - \frac{r}{c})}\right]\right|$$

$$\ll \left| \frac{\sqrt{\mu}\, J_o\, e^{-r/2\lambda} \sqrt{t - \frac{r}{c}}}{\mu r\, 2\pi \sqrt{\sigma_c}} \left[\frac{1}{\lambda} + \frac{1}{c(t - \frac{r}{c})}\right]\right| \tag{39}$$

or

$$\left| \frac{\epsilon}{2\sigma(t - \frac{r}{c})} \left[\frac{1}{\lambda} - \frac{1}{c(t - \frac{r}{c})}\right]\right| \ll \left|\left[\frac{1}{\lambda} + \frac{1}{c(t - \frac{r}{c})}\right]\right| \tag{40}$$

and finally

$$\sigma \gg \frac{\epsilon}{2T} \left|\frac{(1/\lambda) - (1/cT)}{(1/\lambda) + (1/cT)}\right| \quad . \tag{41}$$

But this condition is never more restrictive than our condition on σ due to our first criterion (see inequality 20). Thus, we have only two unique conductivity requirements, (20) and (34), which insure that the diffusion equation (10) describes the EMP fields and these fields are defined by (11), (6), and (7) for the step function driver. We have assumed that the magnetic field given by (11) behaves similarly to that of (12) for points near the ground.

If a more rigorous analysis is necessary, one should be able to determine the conductivity requirements based on equation (11) rather than (12). Also this analysis could be extended to other driving forms such as an exponentially varying J and σ.