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THE FIELD OF A MAGNETIC BUBBLE

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ABSTRACT

A perfectly conducting sphere is immersed in a uniform magnetic field. If now the sphere is suddenly expanded or collapsed, it will induce motion of the field lines and hence give rise to a radiated signal. In this note, the resulting electromagnetic field is computed.

Imagine a body, which is a good electrical conductor, immersed in and permeated by a uniform magnetic field. If now this body is suddenly caused to collapse (or to expand) it will tend to drag field lines along with its own motion, compressing (or rarefying) the field inside itself. This leads to an altered magnetic field outside, which in turn induces an electric field.

A rigorous solution to the electromagnetic problem can be obtained if we sufficiently simplify the properties and motion of the body. Accordingly, we take the body to be a perfectly conducting sphere of radius R. R will be a function of time and its initial value we denote by R_o. The collapse or expansion of this sphere is assumed to be uniform. That is to say, if $\rho(t)$ be the radius of any interior point of the sphere at time t and ρ_o be its initial position, then

$$\frac{\rho}{\rho_o} = \frac{R}{R_o} \; . \tag{1}$$

Under this supposition, the interior field will always be uniform and parallel to the initial field, differing only by the compression or expansion ratio. Taking coordinates so that the initial field B_o is along the z-axis, the interior field B_i will be in the same direction and moreover we will have

$$B_i = B_o(R_o/R)^2 . \tag{2}$$

The external field will be the uniform impressed field plus the field of a magnetic dipole, and is most conveniently expressed in polar coordinates. The field components are

$$E_\varphi = (1/r)\left\{\varphi''(t - r/c) + (c/r)\varphi'(t - r/c)\right\} \sin\theta ,$$

$$B_r = -(2c/r^2)\left\{\varphi'(t - r/c) + (c/r)\varphi(t - r/c)\right\} \cos\theta - B_o \cos\theta ,$$

$$B_\varphi = -(1/r)\left\{\varphi''(t - r/c) + (c/r)\varphi'(t - r/c) + (c/r)^2\varphi(t - r/c)\right\} \sin\theta + B_o \sin\theta . \tag{3}$$

In the above expressions θ represents the polar angle, measured from the direction of the B_o field, and r is the radius vector. The function φ is arbitrary and primes denote derivatives with respect to its argument. The units are Gaussian. That Eq. (3) is indeed a rigorous solution to Maxwell's equations can readily be seen by Fourier superposition of elementary spherical wavelets. Alternatively, one can verify the solution by substituting into Maxwell's equations.

Having exact solutions of Maxwell's equations inside and outside the spherical bubble given respectively by Eqs. (2) and (3), one need merely to satisfy the boundary conditions which are

$$B_r = -B_i \cos\theta \qquad \text{at } r = R \tag{4}$$

and

$$E_\varphi = -(v/c)B_\theta \qquad \text{at } r = R\ . \tag{5}$$

The first states that the normal component of the magnetic field is continuous and yields

$$\varphi'(t - R/c) + (c/R)\varphi(t - R/c) = (B_o/2c)(R_o^{\,2} - R^2). \tag{6}$$

The condition of Eq. (5) states that an observer moving with the sphere boundary would see no tangential electric field. This yields an equation which is merely the time derivative of Eq. (6).

The problem is now essentially solved. For knowing R as a function of t, one can solve Eq. (6) and put the result into Eq. (3) to obtain the fields. In many practical cases the velocity of the surface of the bubble will be nonrelativistic and the solution can be written out explicitly. Thus assume that

$$\dot{R} \ll c\ . \tag{7}$$

Then

$$\varphi(t - R/c) = (B_o/2c^2)R(R_o - R) \tag{8}$$

is, to a good approximation, the solution of Eq. (6). Setting this into Eq. (3) yields

$$E_\varphi = \frac{B_o}{2c^2 r}\left\{(R_o^2 - 3R^2)\ddot{R} - 6R(\dot{R})^2 + \frac{c\dot{R}}{r}(R_o^2 - 3R^2)\right\}\sin\theta ,$$

$$B_r = -B_o\left\{1 + \frac{1}{cr^2}\left[(R_o^2 - 3R^2)\dot{R} + \frac{cR}{r}(R_o^2 - R^2)\right]\right\}\cos\theta , \tag{9}$$

$$B_\theta = B_o\left\{1 - \frac{1}{2c^2 r}\left[(R_o^2 - 3R^2)\ddot{R} - 6R(\dot{R})^2 + \frac{c\dot{R}}{r}(R_o^2 - 3R^2) + \frac{c^2R}{r^2}(R_o^2 - R^2)\right]\right\}\sin\theta .$$

Here, of course, R, $\dot{R}$, $\ddot{R}$ are respectively the radius, the velocity, and the acceleration of the boundary all evaluated at the retarded time $t - r/c$; r is the radius of the point of observation.

The hopes of being able to detect the radiation field from such a bubble are quite remote. For let us suppose that τ is the duration of the motion and R_M the maximum radius of the bubble. Then

$$\dot{R} \sim R_M/\tau \qquad \text{and} \qquad \ddot{R} \sim 2R_M/\tau^2$$

whence

$$|E_{rad}| = |B_{rad}| \sim k\, B_o \left(\frac{R_M}{r}\right)\left(\frac{v}{c}\right)^2 , \tag{10}$$

where $v = R_M/\tau$, and is the mean velocity of the expansion, and k is a number between 1 and 10 depending on precisely how the terms combine. It is unlikely that v/c could exceed 10^{-4} nor could R_M/r exceed this figure if we are in the radiation zone. Thus, if B_o represents the field of the earth, the radiation field could hardly exceed a tenth of a microvolt per meter.

On the other hand, the nearby signal is easily detected. For example, imagine a coil of radius r wound in the equatorial plane with the bubble at its center. As we are close, the $1/r^2$ portion of the electric field dominates. Thus, from Eq. (9), we get

$$V = \frac{n\pi B_o}{r}(R_o{}^2 - 3R^2)\left(\frac{\dot{R}}{c}\right) \tag{11}$$

for the voltage induced in a loop of n-turns. Suppose again that B_o is the field of the earth, say 1/3 Gauss. In MKS units, 1/3 Gauss is equivalent to 10^4 volts per meter. If we use $B_o = 10^4$ and measure R, R_o, r in meters, V will be in volts. Because B_o is large, the signal is appreciable unless $\dot{R}/c$ is extremely small.