Theoretical Notes Baum
Note 113

DNA 3167F

TIME DOMAIN TREATMENT OF MEDIA WITH FREQUENCY-DEPENDENT ELECTRICAL PARAMETERS

Conrad L. Longmire
H. Jerry Longley
Mission Research Corporation
735 State Street
Santa Barbara, California 93101

25 September 1973

Final Report

CONTRACT No. DASA 01-71-C-0105

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Prepared for
Director
DEFENSE NUCLEAR AGENCY
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1. INTRODUCTION

Scott (Ref. 1) has presented results of measurements of the electrical conductivity and dielectric constant, as functions of frequency, of many samples of soil and rock.

In the present note we show how to use the frequency-dependent parameters to formulate a time-domain treatment of electromagnetic problems. The time-domain treatment is useful in finding computer solutions to problems involving electromagnetic pulses in the ground, or in other media.

Our method can be applied to any particular type of ground if the frequency-dependent electrical parameters are known. In many cases, complete data will not be available for the ground of interest. However, Scott found that, subject to some variations, the conductivity and dielectric constant of many samples scale with just one parameter, namely the water content of the samples. Thus, if one knows the conductivity σ or the dielectric constant ε at one frequency, one can estimate both σ and ε as functions of frequency using Scott's "universal" curves. This result is obviously of great importance in practical work. Because of this, we transform Scott's universal curves to the time domain.

For a discussion of situations where large deviations from the universal curves may be expected (in particular, in locations with high rainfall) the reader should refer to Scott's paper.

We also indicate how the time-domain electrical parameters can be measured directly by pulse techniques.

2. TIME-DOMAIN THEORY AND ASSUMPTIONS

We postulate here a general relation between the electric field E and the electric current density j induced by E. We assume that this relation has the following properties:

(a) it is local; i.e., the current density j at a point x in space depends only on the field E at that point;

(b) it is linear; i.e., if j_1 results from E_1 and j_2 results from E_2, then $(j_1 + j_2)$ results from $(E_1 + E_2)$;

(c) it is causal; i.e., j at time t depends only on E at times no later than t;

(d) it is invariant under changes of the time origin.

Of these assumptions, we may expect (a) to be true in an average sense, provided we do not look on the scale of the microscopic or crystalline structure of the medium. We may expect (b) to be true for sufficiently small E applied for sufficiently short times (experimenters should watch out for non-linear effects). Assumption (c) may be expected to be always true. Assumption (d) may be expected to hold unless physical changes are occurring in the medium, due to other effects (e.g., shock waves, drying) on the time scale of the electric fields.

Consistent with the postulates (a) - (d), the most general relation between j and E is

$$j(x,t) = \int_{-\infty}^{t} \dot{E}(x,t')\, K(x,t-t')dt' \quad . \qquad (1)$$

Here $\dot{E}$ is the time derivative of E. We could have written E instead of $\dot{E}$ inside the integral (integration by parts shows the equivalence), but use of $\dot{E}$ will be convenient. The kernel K can depend on x, but since x occurs in Eq. (1) only as a parameter rather than as an essential variable, we shall henceforth suppress it. For the present we think of E and j as scalars rather than vectors.

If $E(t')$ were a step function, $\dot{E}$ would be a delta function. Thus, the kernel, $K(t-t')$, is the current that flows in response to a step function in E. The general form of this current or $K(t-t')$ will be as indicated in Fig. (1). There will be a delta function at $t-t' = 0$,

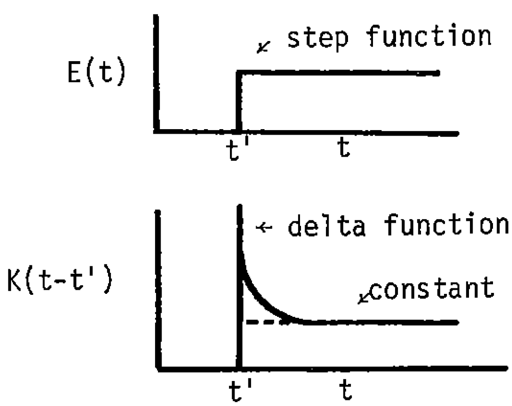

Fig. (1). General form of current that flows in response to a step function in E.

corresponding with the infinite-frequency dielectric constant. For large $t-t'$ the current will approach a constant, corresponding with the d.c. conductivity. At finite times the current will vary in some way.

It is convenient to separate out the infinite-frequency dielectric constant and the d.c. conductivity, rewriting Eq. (1) as

$$j(t) = \sigma_0 E(t) + \frac{\varepsilon_\infty}{4\pi c}\dot{E}(t) + \int_{-\infty}^{t} \dot{E}(t')\, K(t-t')dt' \quad , \qquad (2)$$

where we have used the same symbol for the new kernel, and c is the velocity of light (cgs gaussian units).

The connection with the frequency dependent parameters is found by letting E be proportional to $e^{i\omega t}$. Then one finds

$$j(\omega) \quad = \quad \left[\sigma_0 + \frac{i\omega\varepsilon_\infty}{4\pi c} + i\omega\int_0^\infty e^{-i\omega u}\, K(u)du\right]E(\omega) \quad . \tag{3}$$

Thus the frequency-dependent conductivity is

$$\sigma \quad = \quad \sigma_0 + \mathrm{Re}\left(i\omega\int_0^\infty e^{-i\omega u}\, K(u)du\right) \tag{4}$$

and the frequency-dependent dielectric constant is

$$\varepsilon \quad = \quad \varepsilon_\infty + \frac{4\pi c}{\omega}\,\mathrm{Im}\left(i\omega\int_0^\infty e^{-i\omega u}\, K(u)du\right) \quad . \tag{5}$$

Here Re and Im stand for real and imaginary parts.

Note that the kernel, $K(t-t')$, could be determined experimentally by applying a step-function voltage to a sample and observing the current flow at later times.

3. APPROXIMATION OF K(u)

We shall now assume that $K(u)$ can be approximated as a series of decaying exponentials,

$$K(u) = \sum_n a_n e^{-\beta_n u} \qquad (a_n, \beta_n \text{ positive}) \quad . \tag{6}$$

As we shall see below, this form is advantageous for use in numerical computations, and allows a simple physical interpretation in terms of equivalent electrical circuits.

Inserting this approximation in the time-domain equation (2) we find

$$j(t) = \sigma_0 E(t) + \frac{\varepsilon_\infty}{4\pi c} \dot{E}(t) + \sum_n a_n J_n(t) \tag{7}$$

where

$$J_n(t) = e^{-\beta_n t} \int_{-\infty}^{t} \dot{E}(t') e^{\beta_n t'} dt' \quad . \tag{8}$$

Obviously, the $J_n(t)$ satisfy the differential equation

$$\frac{dJ_n}{dt} + \beta_n J_n = \dot{E}(t) \quad . \tag{9}$$

In numerical calculations, the $J_n(t)$ can be carried forward in time at each step by use of the difference form of Eq. (9). This procedure will be generally much less time-consuming than evaluating the complete integral of Eq. (2) at each time step, and also does not require storage of $\dot{E}$ at all previous times. This advantage derives from the special form of Eq. (6).

Use of this form in Eqs. (4) and (5) leads to the frequency-dependent parameters,

$$\sigma = \sigma_0 + \sum_n a_n \frac{\omega^2}{\beta_n^2+\omega^2} \quad . \tag{10}$$

$$\varepsilon = \varepsilon_\infty + 4\pi c \sum_n a_n \frac{\beta_n}{\beta_n^2+\omega^2} \quad . \tag{11}$$

Later we shall discuss the fitting of Scott's universal curves by these formulae. First, however, we shall establish the plausibility of the approximation Eq. (6) by examining the equivalent electrical circuit.

4. THE EQUIVALENT ELECTRICAL CIRCUIT

Consider the two-terminal a.c. circuit shown in Fig. (2). By standard a.c. circuit theory one finds the current j that flows in response to the applied voltage E,

$$j = \left[\frac{1}{R_0} + i\omega C_\infty + \sum_n \frac{1}{R_n + \frac{1}{i\omega C_n}}\right] E \quad . \tag{12}$$

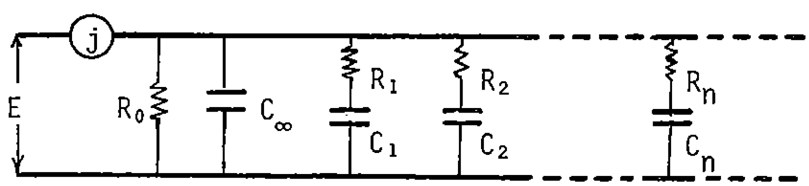

Fig. (2). Equivalent circuit.

Rationalizing the terms inside the summation and letting

$$\beta_n = \frac{1}{R_n C_n} \quad , \tag{13}$$

we find that Eq. (12) can be written as

$$j = \left[\left(\frac{1}{R_0} + \sum_n \frac{1}{R_n} \frac{\omega^2}{\beta_n^2+\omega^2}\right) + i\omega\left(C_\infty + \sum_n \frac{1}{R_n} \frac{\beta_n}{\beta_n^2+\omega^2}\right)\right] E \quad . \tag{14}$$

If a unit volume of the medium is represented by our equivalent circuit, we have the correspondence

$$\sigma = \frac{1}{R_0} + \sum_n \frac{1}{R_n} \frac{\omega^2}{\beta_n^2+\omega^2} , \tag{15}$$

$$\varepsilon = 4\pi c \left(C_\infty + \sum_n \frac{1}{R_n} \frac{\beta_n}{\beta_n^2+\omega^2} \right) . \tag{16}$$

These equations will be completely equivalent to Eqs. (10) and (11). If, in addition to Eq. (13), we let

$$\sigma_0 = \frac{1}{R_0} , \tag{17}$$

$$\varepsilon_\infty = 4\pi c\ C_\infty , \text{ and} \tag{18}$$

$$a_n = \frac{1}{R_n} , \tag{19}$$

then Fig. (2) is indeed an equivalent circuit for the approximation of Eq. (6).

It is plausible that "ground" should be representable by RC networks. In the many fissures, with water present, one would expect to find ionic conduction, accounting for the resistors in the equivalent circuit. A fissure which terminates will be capacitively coupled to other fissures which begin nearby. Unless the fissures have helical paths, one would not expect to find inductances beyond the free-space inductance.

While the RC network of Fig. (2) is special in that all paths are in parallel without cross coupling, it can be shown that any two-terminal RC network can be replaced by an RC network of this type having identical external characteristics. The proof of this theorem is outlined in Appendix A, for those who wish to pursue it.

5. A FIT TO SCOTT'S UNIVERSAL CURVES

We now turn to the fitting of Scott's universal curves by forms of the type of Eqs. (10) and (11). For purposes of numerical computing, we desire to keep the number of terms in the summation to a minimum.

To see how many terms may be required, consider the following example. Scott's ε, for some frequencies, is approximately proportional to ω^{-1}, so that $\omega\varepsilon$ is roughly constant. We therefore need to discuss how one fits a constant function by a series of functions of the form

$$g(\omega\ \beta_n) = \frac{\beta_n\omega}{\beta_n^2+\omega^2} \tag{20}$$

In Fig. (3) we have superimposed two functions of this type, one with $\beta_n = 10$ and one with $\beta_n = 100$, both with amplitude factors unity. The sum of the two functions is approximately constant between $\omega = 10$ and $\omega = 100$. Thus by superimposing such functions one decade apart in β_n, one can approximate a constant function. By spacing the functions closer in frequency (logarithmically), one could do a better job, but this accuracy is good enough for our purposes. In general it is desirable to keep small the number of terms in the fit.

Therefore we decided to fit Scott's universal curves using one term per decade in frequency. In Fig. (4) the solid curves are Scott's universal curves for ε. We fitted these curves first, using $\beta_n = 2\pi\ (10^2, 10^3, 10^4, 10^5)$, with an adjustable a_n (Eq. (11)) to go with each of these four β_n's. By also adjusting ε_∞ we could fit the curves exactly at five points, which we took to be $\omega = 2\pi\ (10^2, 10^3, 10^4, 10^5, 10^6)$. These points are the triangles in Fig. (4). The values of ε_∞ and the a_n so determined for each water content are given in Table 1.

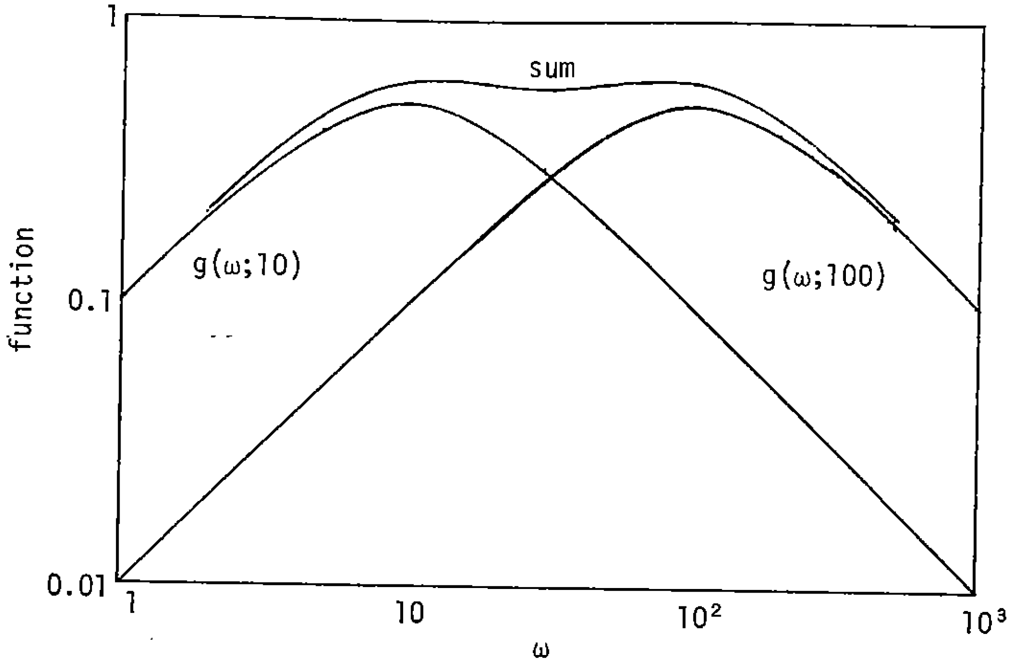

Fig. (3). Approximation of a constant by the sum of two terms of the form of Eq. (20).

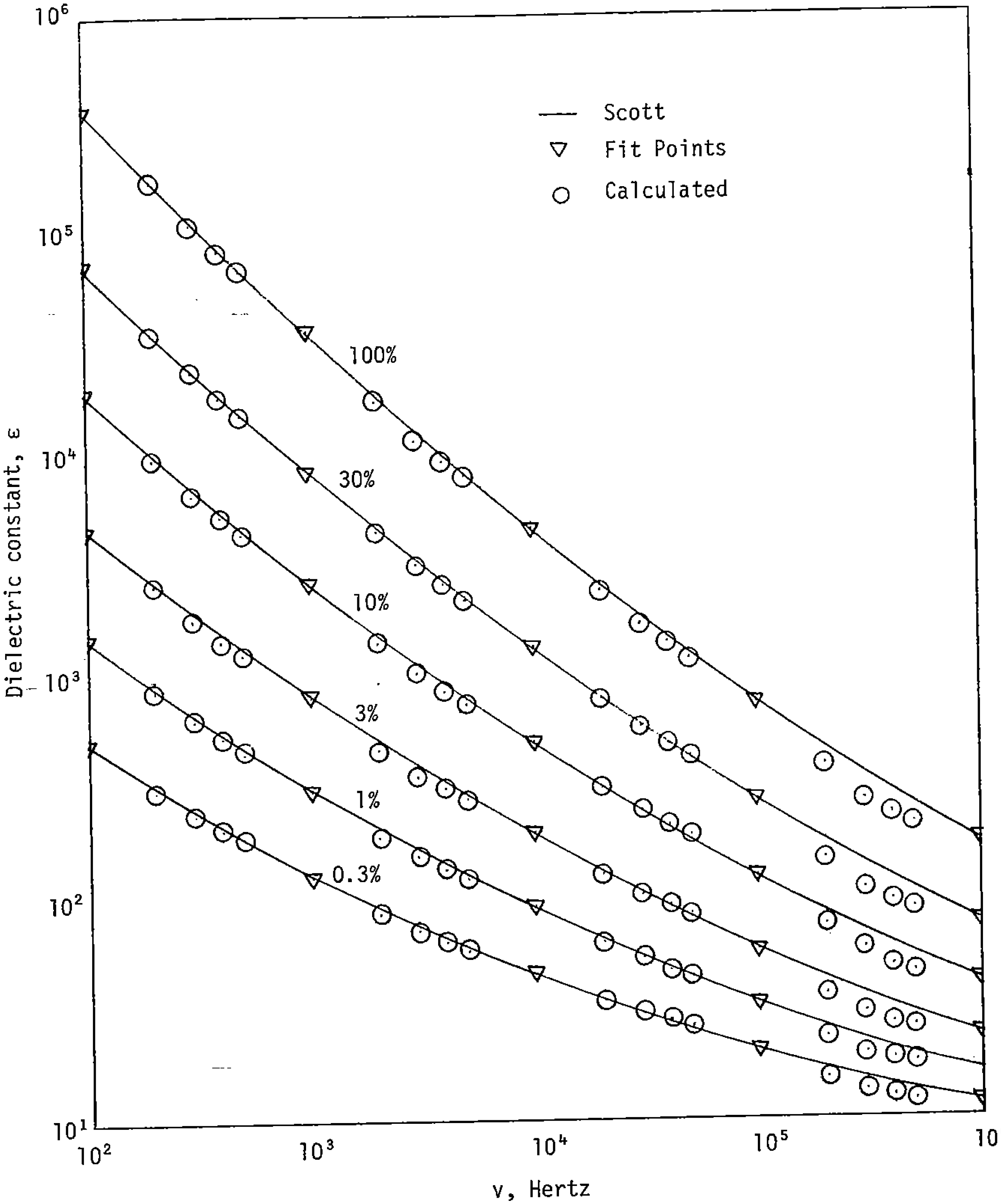

Fig. (4). Dielectric constant versus frequency for various volume percent of water.

Table I. Fit Parameters for Ground Conductivity and Dielectric Constant, from Scott's Universal Curves. Fit is from $\omega = 2\pi \times 10^2$ to $2\pi \times 10^6$.

water, volume %	ε_∞	a_1	a_2	a_3	a_4	σ_0
0.3	11.8	1.08(-6)	2.12(-6)	5.10(-6)	2.95(-5)	1.25(-5)
1	16.7	3.86(-6)	5.25(-6)	1.38(-5)	5.15(-5)	7.25(-5)
3	23.3	1.24(-5)	1.49(-5)	3.35(-5)	1.12(-4)	3.75(-4)
10	39.3	5.33(-5)	5.48(-5)	8.88(-5)	2.84(-4)	2.15(-3)
30	74.1	2.07(-4)	1.82(-4)	2.77(-4)	6.27(-4)	1.20(-2)
100	169	1.12(-3)	8.04(-4)	1.01(-3)	1.79(-3)	8.26(-2)

$$\sigma(\mathrm{cm}^{-1}) = \sigma_0 + \sum_{n=1}^{4} a_n \frac{\omega^2}{\beta_n^2+\omega^2} = 0.3\,\sigma\ (\mathrm{mho/meter})$$

$$\varepsilon = \varepsilon_\infty + 4\pi c \sum_{n=1}^{4} a_n \frac{\beta_n}{\beta_n^2+\omega^2} \qquad (\varepsilon_{\mathrm{vacuum}} = 1)$$

$$\beta_n = 2\pi 10^{1+n}$$

Note: 1.08(-6) = 1.08×10^{-6}

Using the fit, we then calculated ε at other frequencies, and the results are represented by the circles in Fig. (4). In general, the fit droops a little below Scott's curves between the fitted points, especially in the upper frequency decade. The fit could be improved easily by increasing the number of terms, but we were satisfied with the present results.

We then had left one adjustable parameter, σ_0, to fit the conductivity. In Fig. (5), the solid curves are Scott's universal curves for σ. By adjusting σ_0, and using the β_n and a_n determined above, we obtained the fit represented by the circles in Fig. (5). The values of σ_0 so obtained are also listed in Table 1.

The satisfactory quality of the fit indicates that Scott's universal curves are at least approximately consistent with the RC network model.

For readers who may wish to use the conductivity in mho/meter, note that

$$\sigma(\text{mho/meter}) = \sigma(\text{cm}^{-1})/0.3 \quad . \tag{21}$$

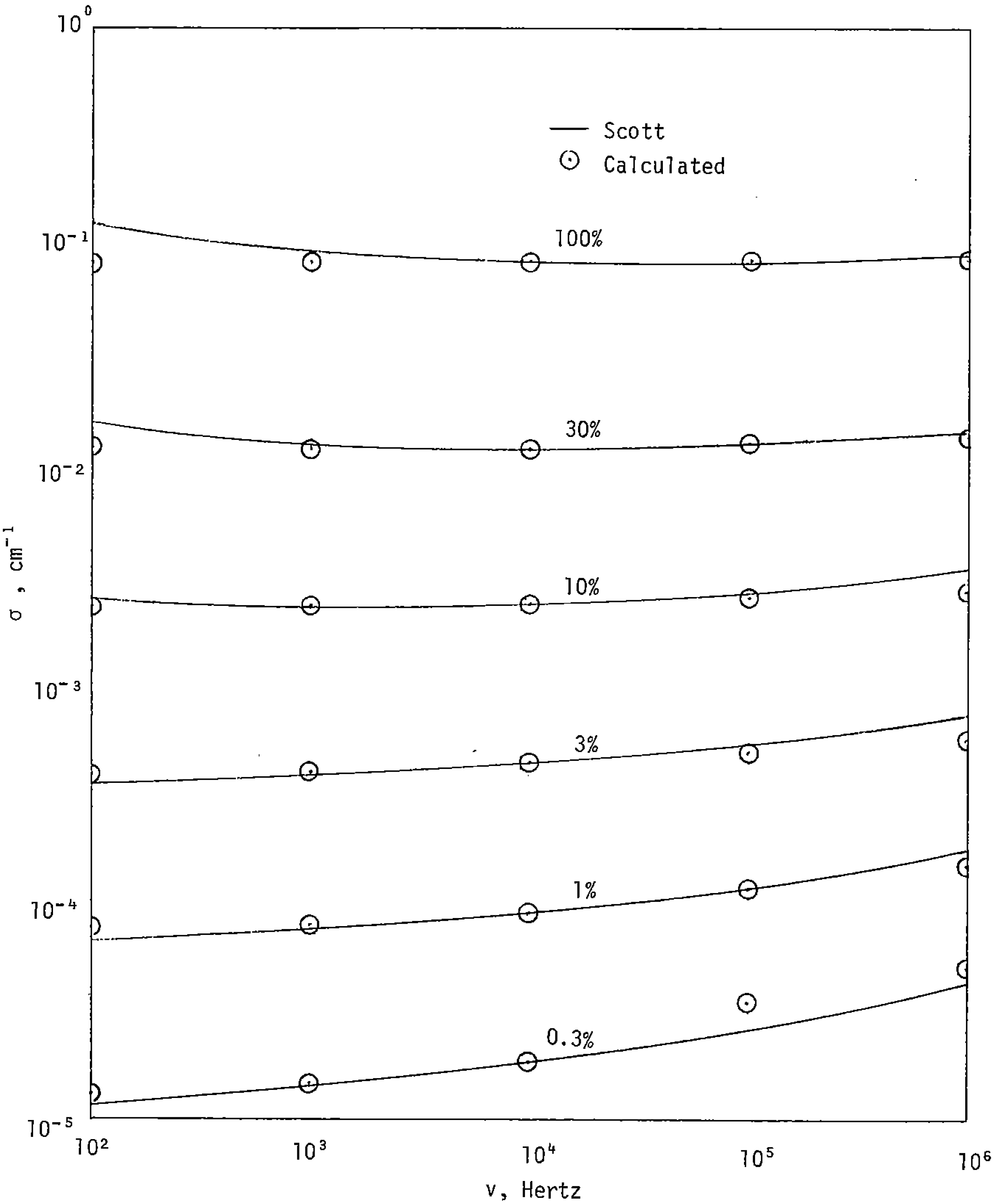

Fig. (5). Conductivity versus frequency for various volume percent of water.

6. AN ALTERNATIVE TYPE OF FIT

Although the fit described above is most convenient for the purpose we had in mind, there is an alternative fit that is theoretically interesting. The reader may observe that Scott's universal curves for ε have the property that two curves for different water content can be made to coincide (very nearly) by shifting one of them horizontally in Fig. (4), i.e., by simply scaling the frequency. In fact, within a few percent, all of the curves can be represented by one function

$$\varepsilon = \varepsilon(\omega/W^{1.3}) \quad , \tag{22}$$

where W is the fractional water content by volume.

Note that Eq. (16) can also be written as

$$\varepsilon = 4\pi c\left(C_\infty + \sum_n \frac{C_n}{1+\left(\frac{\omega}{\beta_n}\right)^2}\right) \quad . \tag{23}$$

We therefore see that, in order that a change of water content should correspond only to scaling the frequency, the capacitances C_n must be independent of water content, while

$$\frac{1}{R_n C_n} = \beta_n \sim W^{1.3} \quad ,$$

or

$$\frac{1}{R_n} \sim W^{1.3} \quad . \tag{24}$$

These results imply that the fissure geometry (C_n) does not change, while the conductance in the fissures increases with water content.

The conductivity σ is also reasonably well fitted on this model, although it turns out that σ_0 increases slightly faster with W than do the $1/R_n$ from Eq. (24). It would be interesting to know whether the data from Scott's many samples could be fitted just as well by functions of the form

$$\varepsilon = \varepsilon(\omega/g(W)) \quad , \tag{25}$$

$$\sigma = g(W)\sigma_1(\omega/g(W)) \quad . \tag{26}$$

for some suitably chosen function $g(W)$ of the water content.

7. NUMERICAL METHOD FOR MAXWELL'S EQUATIONS

With the current density from Eq. (7), Maxwell's equations become

$$\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = -\nabla\times\vec{E} \quad , \tag{27}$$

$$\frac{\varepsilon_\infty}{c}\frac{\partial \vec{E}}{\partial t} = \nabla\times\vec{B} - 4\pi\sigma_0\vec{E} - 4\pi \sum_n a_n J_n(t) \quad . \tag{28}$$

One carries $\vec{E}$ and $\vec{B}$ forward in time using these equations, whereas the quantities $J_n(t)$ are carried forward in time using Eq. (9).

If the medium is stratified, the quantities ε_∞ , σ_0 , a_n (and β_n) may be different in different directions.

APPENDIX A

NETWORK THEOREMS

Consider an arbitrary two-terminal RC network. Represent it in loop form, as in Fig. (6). Here each box represents a resistance and a

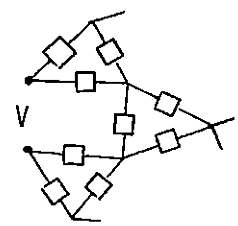

Fig. (6). Two-terminal network.

capacitance in series, although some R's may be zero and some C's may be infinite. All loops may have elements in common with any other loop. Let loop 1 be the loop with the external terminals, to which the voltage V is applied, and let the loop currents be I_k. Then the loop equations (voltage drop around each loop equals zero) are

$$
\begin{aligned}
Z_{11}I_1 + Z_{12}I_2 + \dots \quad Z_{1n}I_n &= V \,, \\
Z_{21}I_1 + Z_{22}I_2 + \dots \quad Z_{2n}I_n &= 0 \,, \\
\dots \quad \dots \quad & \dots \\
Z_{n_1}I_1 + Z_{n_2}I_2 + \dots \quad Z_{nn}I_n &= 0 \,.
\end{aligned}
\tag{29}
$$

The solution for I_1 is

$$I = V \frac{\det(Z'_{jk})}{\det(Z_{jk})} \tag{30}$$

where det stands for the determinant of the indicated matrix, Z_{jk} is the full matrix of loop impedances, and Z'_{jk} is the matrix obtained by leaving off the first row and the first column.

The matrix Z_{jk} is symmetrical, since $Z_{jk} = Z_{kj}$ is the impedance in common with the j'th and k'th loops. Furthermore, each Z_{jk} is of the form

$$Z_{jk} = \pm\left[R_{jk} + \frac{1}{i\omega}\left(\frac{1}{C}\right)_{jk}\right] , \tag{31}$$

the plus or minus sign depending on whether the current I_j and I_k pass through the impedance in the same or opposite directions. Thus both matrices Z and Z' can be made real by replacing $i\omega$ by

$$s = i\omega . \tag{32}$$

Thus we are considering voltages (and currents) of the form

$$V \sim e^{st} . \tag{33}$$

For convenience, multiply numerator and denominator in Eq. (30) by s^n . We then have

$$I_1 = sV \frac{\det(T'_{jk})}{\det(T_{jk})} , \tag{34}$$

where

$$T_{jk} = s\, R_{jk} + K_{jk} \qquad \left(K_{jk} \equiv \left(\frac{1}{C}\right)_{jk}\right) . \tag{35}$$

The matrices R_{jk} and K_{jk} are real and symmetric.

To solve for the current that flows in response to a step function in V , we need the transient or free solutions of the system with the input terminal, shorted. These are the solutions of the equations

$$\sum_k T_{jk} I_k \quad = \quad 0 \quad , \tag{36}$$

which exist only for those values of s for which

$$\det(T_{jk}) \quad = \quad 0 \quad . \tag{37}$$

From Eq. (35), this determinant is a polynomial of the n'th order in s , so that there are n values of s which satisfy Eq. (37).

That these n values are all real may be proved as follows. Write out Eq. (36), and its complex conjugate,

$$\sum_k (s\, R_{jk} + K_{jk}) I_k \quad = \quad 0 \quad , \tag{38}$$

$$\sum_k (\bar{s}\, R_{jk} + K_{jk}) \bar{I}_k \quad = \quad 0 \quad . \tag{39}$$

Multiply Eq. (38) by $\bar{I}_j$, Eq. (39) by I_j , subtract the latter product from the former, and sum over j . From the symmetry of R_{jk} and K_{jk} , one then finds

$$(s-\bar{s}) \sum_{j,k} R_{jk}\, \bar{I}_j I_k \quad = \quad 0 \quad . \tag{40}$$

The double sum here is the power being dissipated in all the resistors of the network; it is positive definite. Therefore $s = \bar{s}$ and s is real.

In fact, all the allowed values of s are negative, since power can only be dissipated, not generated, in an RC circuit, and currents of the form of Eq. (33) must decay, not grow with time.

We now return to Eq. (34). We know that $\det(T_{jk})$ is an n'th order polynomial in s with n real, negative roots. Let the roots be

$$s_m = -\beta_m \quad , \tag{41}$$

where the β_m are all positive. Then Eq. (34) can be written

$$I_1 = sV \frac{\det(T'_{jk})}{\det(R_{jk})[(s+\beta_1)(s+\beta_2) \ \ldots \ (s+\beta_n)]} \quad . \tag{42}$$

Now, $\det(T'_{jk})$ is an (n-1)'th order polynomial in s . Therefore, using partial fractions, we can write

$$\frac{\det(T'_{jk})}{(s+\beta_1)(s+\beta_2) \ \ldots \ (s+\beta_n)} = \frac{b_1}{s+\beta_1} + \frac{b_2}{s+\beta_2} + \ldots \frac{b_n}{s+\beta_n} \quad , \tag{43}$$

where the b_n are constants. Using this expansion in Eq. (42), and replacing s by $i\omega$, we have

$$I = V \frac{a_1}{1 + \frac{\beta_1}{i\omega}} + \frac{a_2}{1 + \frac{\beta_2}{i\omega}} + \ldots + \frac{a_n}{1 + \frac{\beta_n}{i\omega}} \quad , \tag{44}$$

where the a_m are equal to $b_m/\det(R_{jk})$. This form is equivalent to Eq. (12), for the equivalent circuit of Fig. (2), provided all the a_m are positive. The positiveness of the a_m again follows from the positive-definiteness of the power dissipation; $a_m = 1/R_m$ where the R_m are the resistances of the equivalent parallel circuit.

REFERENCE

1. Scott, James H., United States Geological Survey, Technical Letter, Special Project-16, May 26, 1966, "Electrical and Magnetic Properties of Rock and Soil;" also appears as Note VIII in DASA 1882-1, "Electromagnetic Pulse Theoretical Notes, Vol. I, May 1967.