Theoretical Notes
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Interaction of External System-Generated EMP With Space Systems

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Abstract

This note treats the problem of calculating the electromagnetic field of photoelectrons emitted from the external surface of a space system. A detailed study of the canonical problem involving a charged particle orbiting a perfectly conducting sphere reveals that sufficiently accurate solutions can be obtained by solving two independent quasi-static boundary-value problems. Validity criteria are established for the quasi-static solutions in terms of the particle's kinetic energy and distance from the sphere. These quasi-static solutions can easily be generalized to arbitrary motions; extensive graphical results are presented for the induced surface currents and charges on the sphere. An integral-equation approach to arbitrary-shaped conductors is briefly discussed. Two initial-boundary-value problems are posed and solved in connection with the question as to how the positive charges redistribute themselves on a sphere after the passage of a short incident photon pulse. The relationship between the findings of this note and the general problem of calculating the external system-generated EMP is also discussed.

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I. Introduction

The threat of System-Generated EMP (SGEMP) to the survivability of space systems in exoatmospheric regions has been recognized for some time [1,2,3]. Past efforts have been limited to the calculations of photoelectric emission currents from various materials [4,5]. The backward scattered photoelectrons give rise to an external electromagnetic pulse, whereas the photoelectrons that are forward scattered into cavities give rise to an internal electromagnetic pulse (or, simply, IEMP). Except for an estimate of the final induced voltage (and hence the final induced net charge) on a space system [6,7], there exist no detailed calculations on the external electromagnetic pulse due to the emitted photoelectrons. The present note gives such a calculation for the case of a space system of spherical shape and discusses some relevant problems within the realm of classical electrodynamics.

The general problem in the study of external SGEMP is that of calculating the electromagnetic field outside a space system for a given incident photon pulse. The self-consistent approach to the general problem involves Maxwell's equations as well as the equations of motion and, hence, is untractable without some approximations. As always, when we are confronted with a problem of such complexity as this one, we first seek an analytically tractable theoretical model (or models), the detailed calculations of which will reveal some essential features of the general problem. We then add more complexities to the model (or models), so that the modified model (or models) resembles more closely the real system. Eventually, due to the complexities we will be forced to relinquish rigorous calculations and be content with approximate estimations. But only this line of approach will create confidence in any conclusions derived from estimations. Accordingly, in this note we will calculate in detail the time-dependent electromagnetic field, especially the induced surface charges and currents, in the neighborhood of a spherical conductor when the charge and current densities of all the emitted electrons outside the conductor are prescribed. The solution of this problem gives what one should expect from Maxwell's equations alone.

In Section II we solve a canonical problem rigorously and quasi-statically. The canonical problem involves a charged particle orbiting a perfectly conducting sphere. We then compare numerically the rigorous and quasi-static solutions and thus establish validity criteria for the quasi-static solution in terms of the particle's speed and distance from the sphere. We then go on in Section III to present extensive graphical results for the induced charges and currents for radial and orbital motions, and hence for arbitrary motion as well because the superposition principle applies. Section IV discusses briefly the integral-equation approach for conductors rather than those of spherical shape. The integral equations involved can be easily solved with standard numerical methods. Two initial boundary value problems are posed and solved in Section V in relation to the question as to how the positive charges redistribute themselves on the surface of a sphere after the passage of a short incident photon pulse. In the final section, Section VI, we summarize the findings of this note and discuss their relations to the general problem. We also point out some natural extensions of the present study that will provide answers to certain aspects of the general problem. Two appendices are included, one dealing with the application of the dyadic Green's function to the problem discussed in this note and the other treating the quasi-magnetostatic problem from the integral-equation point of view. At the end of this note we include a comparison list of symbols and notations for those who are well versed with Baum's work on the problem of a sphere [8].

II. Charged Particle Orbiting a Sphere

The general problem of calculating the electromagnetic field of a charged particle of arbitrary motion in the presence of a perfectly conducting sphere is quite difficult. The canonical problem that can be treated rigorously is the one involving a point charge orbiting around a sphere. Accordingly, we will first solve this problem exactly. Then, we will make a non-relativistic approximation to the exact solution in the hope that the non-relativistic solution can be recognized as the solution to some quasi-static boundary-value problem the solution of which can easily be generalized to arbitrary motions. In order to establish a validity critierion on the approximate solution as a function of the particle's velocity and position a numerical comparison will be made between the approximate solution and the exact one.

To further simplify the geometry of the canonical problem we take the particle's orbit in the equatorial plane of the sphere (Fig.1).

A. General Solution

The problem depicted in Fig.1 can be solved by the standard technique of Debye's potentials u and v. A more formal method utilizing the dyadic Green's function is discussed in Appendix A. The potentials u and v are related to the electromagnetic field by [9,10]

$$\underline{E} = -\nabla\times(\underline{r}\,\frac{1}{c}\,\frac{\partial u}{\partial t} + \underline{r}\times\nabla v)$$

$$Z\underline{H} = -\nabla\times(\underline{r}\times\nabla u - \underline{r}\,\frac{1}{c}\,\frac{\partial v}{\partial t}) \qquad (1)$$

and $\underline{E}$, $\underline{H}$ satisfy the Maxwell equations

$$\nabla\times\underline{E} = -\mu\,\frac{\partial}{\partial t}\,\underline{H}$$

$$\nabla\times\underline{H} = \varepsilon\,\frac{\partial}{\partial t}\,\underline{E} + \underline{J} \qquad (2)$$

$$\nabla\cdot\underline{E} = \rho/\varepsilon$$

$$\nabla\cdot\underline{H} = 0$$

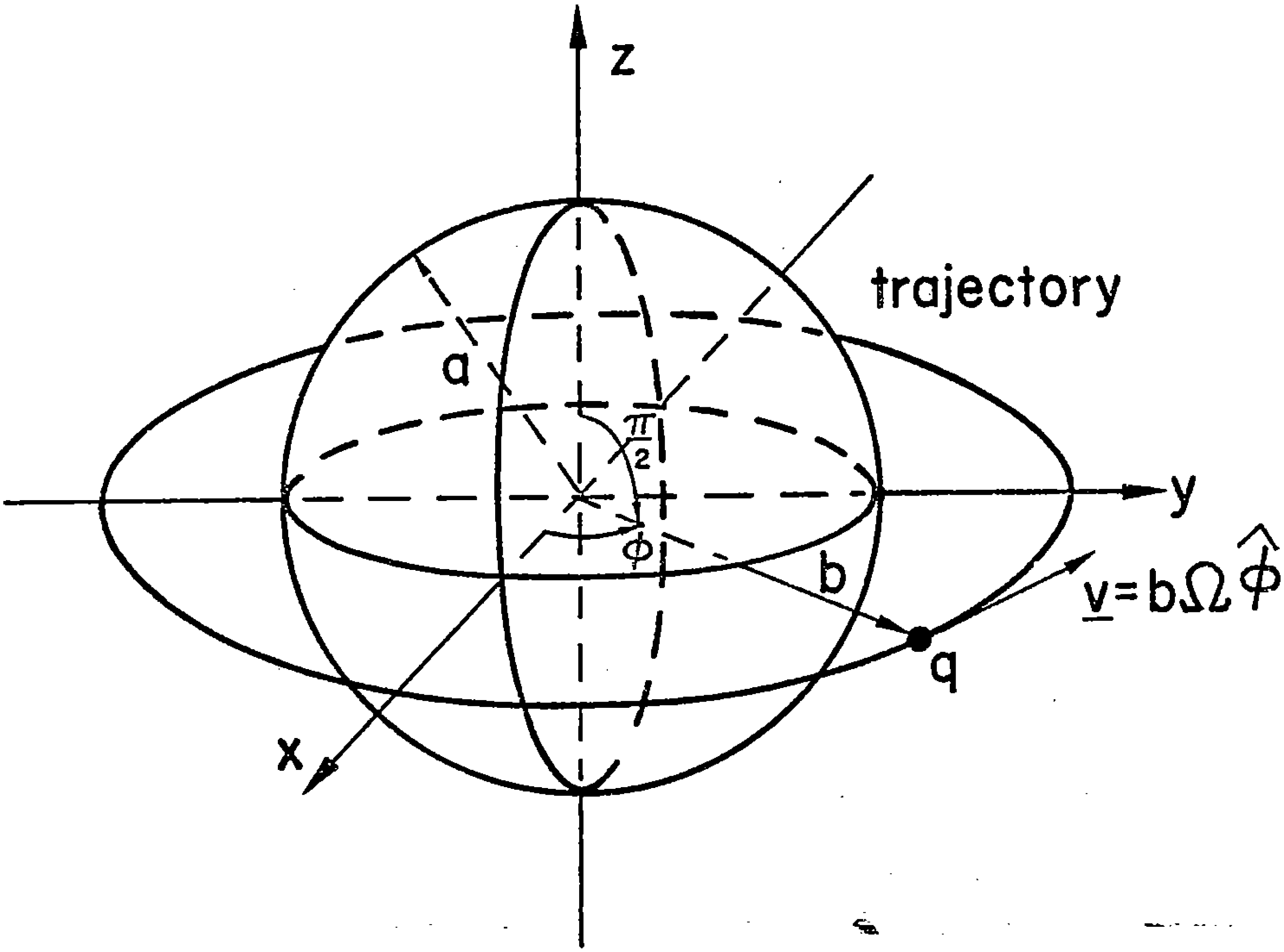

Figure 1. A charged particle orbiting a sphere in the equatorial plane.

The source terms ρ, $\underline{J}$ are given by

$$\rho(\underline{r},t) = \frac{q}{r^2 \sin\theta}\,\delta(r-b)\,\delta(\theta-\frac{\pi}{2})\,\delta(\phi-\Omega t)$$

$$\underline{J}(\underline{r},t) = \frac{q\Omega}{r}\,\delta(r-b)\,\delta(\theta-\frac{\pi}{2})\,\delta(\phi-\Omega t)\hat{\phi} \tag{3}$$

Ω being the angular velocity of the particle and b the radius of the circular orbit. Substitution of (1) into (2) gives

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix} u \\ v \end{pmatrix} = 0, \qquad \text{for } r \neq b \tag{4}$$

Writing (1) out in components we have

$$E_r = \left(\frac{\partial^2}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(rv)$$

$$E_\theta = -\frac{1}{r\sin\theta}\frac{1}{c}\frac{\partial^2(ru)}{\partial t\partial\phi} + \frac{1}{r}\frac{\partial^2(rv)}{\partial r\partial\theta}$$

$$E_\phi = \frac{1}{cr}\frac{\partial^2(ru)}{\partial t\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial^2(rv)}{\partial r\partial\phi}$$

$$ZH_r = \left(\frac{\partial^2}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(ru) \tag{5}$$

$$ZH_\theta = \frac{1}{r}\frac{\partial^2(ru)}{\partial r\partial\theta} + \frac{1}{r\sin\theta}\frac{1}{c}\frac{\partial^2(rv)}{\partial t\partial\phi}$$

$$ZH_\phi = \frac{1}{r\sin\theta}\frac{\partial^2(ru)}{\partial r\partial\phi} - \frac{1}{cr}\frac{\partial^2(rv)}{\partial t\partial\theta}$$

To solve (2) and (3) with the aid of (1) and (4) we first expand

$$\delta(\theta-\frac{\pi}{2})\ \delta(\phi-\Omega t) = \sin\theta \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^{|m|}(0)\ P_\ell^{|m|}(\cos\theta)\ e^{im(\phi-\Omega t)} \quad (6)$$

Then, for the incident field we take, in view of (4) and (6),

$$\begin{pmatrix} u^{inc} \\ v^{inc} \end{pmatrix} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \begin{pmatrix} A_{\ell m} \\ B_{\ell m} \end{pmatrix} j_\ell(k_m r_<)\ h_\ell(k_m r_>)\ P_\ell^{|m|}(\cos\theta)\ e^{im(\phi-\Omega t)} \quad (7)$$

where $k_m = m\Omega/c$, and $r_<(r_>)$ denotes the smaller (larger) of r and b. Here and henceforth, h_ℓ is the spherical Hankel function of the first kind. To find $A_{\ell m}$ and $B_{\ell m}$ we apply the following jump conditions at $r = b$:

$$[E_\theta] = 0, \quad [E_\phi] = 0, \quad [H_r] = 0, \quad [H_\phi] = 0,$$

$$\varepsilon[E_r] = \frac{q}{b^2 \sin\theta}\ \delta(\theta-\frac{\pi}{2})\ \delta(\phi-\Omega t), \quad (8)$$

$$[H_\theta] = \frac{q\Omega}{b}\ \delta(\theta-\frac{\pi}{2})\ \delta(\phi-\Omega t).$$

After some manipulations we arrive at

$$u^{inc} = \frac{iq\Omega^2 b}{4\pi}\ \mu \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{m(2\ell+1)(\ell-|m|)!}{\ell(\ell+1)(\ell+|m|)!} P_\ell^{|m|\,\prime}(0)\ j_\ell(k_m r_<)\ h_\ell(k_m r_>)\ P_\ell^{|m|}(\cos\theta)\ e^{im(\phi-\Omega t)}$$

(9)

$$v^{inc} = \frac{iq\Omega Z}{4\pi} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{m(2\ell+1)(\ell-|m|)!}{\ell(\ell+1)(\ell+|m|)!} P_\ell^{|m|}(0)\ j_\ell(k_m r_<)\ [k_m r_> h_\ell(k_m r_>)]'\ P_\ell^{|m|}(\cos\theta)\ e^{im(\phi-\Omega t)}$$

To find the scattered potentials u^{sc} and v^{sc} we use the outgoing wave condition at infinity and the boundary condition that the total tangential electric field vanishes at the spherical surface $r = a$. Thus,

$$u^{sc} = \frac{-iq\Omega^2 b\mu}{4\pi} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{m(2\ell+1)(\ell-|m|)!}{\ell(\ell+1)(\ell+|m|)!} P_\ell^{|m|\prime}(0) \frac{j_\ell(k_m a)\, h_\ell(k_m b)}{h_\ell(k_m a)} h_\ell(k_m r)\, P_\ell^{|m|}(\cos\theta)\, e^{im(\phi-\Omega t)} \tag{10}$$

$$v^{sc} = \frac{-iq\Omega Z}{4\pi} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{m(2\ell+1)(\ell-|m|)!}{\ell(\ell+1)(\ell+|m|)!} P_\ell^{|m|}(0) \frac{[k_m a j_\ell(k_m a)]'\,[k_m b h_\ell(k_m b)]'}{[k_m a h_\ell(k_m a)]'} h_\ell(k_m r)\, P_\ell^{|m|}(\cos\theta)\, e^{im(\phi-\Omega t)}$$

We now proceed to the calculation of the induced surface charge density σ and current density $\underline{K}$ on the sphere. Using (5), (9), and (10) we obtain

$$\sigma = \varepsilon(E_r^{inc} + E_r^{sc})_{r=a}$$

$$= -\frac{q}{4\pi a^2} - \frac{q}{4\pi a^2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{(2\ell+1)(\ell-|m|)!}{(\ell+|m|)!} P_\ell^{|m|}(0) \frac{[y_m h_\ell(y_m)]'}{[x_m h_\ell(x_m)]'} P_\ell^{|m|}(\cos\theta)\, e^{im(\phi-\Omega t)} \tag{11}$$

where the first term has been added in so that the total induced charge on the sphere is $-q$. Also, $x_m = k_m a$ and $y_m = k_m b$. Let us denote by $\underline{K}'$ the part of $\underline{K}$ due to v (TM-field) and by $\underline{K}''$ the part due to u (TE-field). Using the definition $\underline{K} = \hat{r}\times(\underline{H}^{inc} + \underline{H}^{sc})_{r=a}$ together with (5), (9), and (10) we get

$$K_\theta' = \frac{-iq\Omega \sin\theta}{4\pi a} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{m(2\ell+1)(\ell-|m|)!}{\ell(\ell+1)(\ell+|m|)!} P_\ell^{|m|}(0) \frac{[y_m h_\ell(y_m)]'}{[x_m h_\ell(x_m)]'} P_\ell^{|m|\prime}(\cos\theta)\, e^{im(\phi-\Omega t)}$$

$$K'_\phi = \frac{-q\Omega}{4\pi a \sin\theta} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{m^2(2\ell+1)(\ell-|m|)!}{\ell(\ell+1)(\ell+|m|)!} P_\ell^{|m|}(0) \frac{[y_m h_\ell(y_m)]'}{[x_m h_\ell(x_m)]'}$$

$$P_\ell^{|m|}(\cos\theta)\, e^{im(\phi-\Omega t)}$$

(12)

$$K''_\theta = \frac{-iq\Omega b}{4\pi a^2 \sin\theta} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{m(2\ell+1)(\ell-|m|)!}{\ell(\ell+1)(\ell+|m|)!} P_\ell^{|m|\prime}(0) \frac{h_\ell(y_m)}{h_\ell(x_m)}$$

$$P_\ell^{|m|}(\cos\theta)\, e^{im(\phi-\Omega t)}$$

$$K''_\phi = \frac{-q\Omega b \sin\theta}{4\pi a^2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{(2\ell+1)(\ell-|m|)!}{\ell(\ell+1)(\ell+|m|)!} P_\ell^{|m|\prime}(0) \frac{h_\ell(y_m)}{h_\ell(x_m)}$$

$$P_\ell^{|m|\prime}(\cos\theta)\, e^{im(\phi-\Omega t)}$$

Equations (11) and (12) show that

$$\nabla\cdot\underline{K}' + \frac{\partial\sigma}{\partial t} = 0$$

(13)

$$\nabla\cdot\underline{K}'' = 0$$

Let us now introduce the following normalized functions:

$$\overline{Y}_{\ell m} = \sqrt{\frac{(2\ell+1)(\ell-|m|)!}{4\pi(\ell+|m|)!}}\, Y_{\ell m} = \overline{P}_\ell^{|m|}(\cos\theta)\, e^{im\phi}$$

$$= \sqrt{\frac{(2\ell+1)(\ell-|m|)!}{4\pi(\ell+|m|)!}}\, P_\ell^{|m|}(\cos\theta)\, e^{im\phi}$$

(14)

$$\underline{K}'_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}} \left(\frac{\partial\overline{Y}_{\ell m}}{\partial\theta}\,\hat{\theta} + \frac{1}{\sin\theta}\frac{\partial\overline{Y}_{\ell m}}{\partial\phi}\,\hat{\phi} \right)$$

$$\underline{K}''_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}} \left(\frac{-1}{\sin\theta} \frac{\partial \overline{Y}_{\ell m}}{\partial\phi} \hat{\theta} + \frac{\partial \overline{Y}_{\ell m}}{\partial\theta} \hat{\phi} \right)$$

and we have the relations

$$\langle \overline{Y}_{\ell m}, \overline{Y}_{\ell' m'} \rangle = \int \overline{Y}_{\ell m} \overline{Y}^{*}_{\ell' m'} \sin\theta \, d\theta \, d\phi = \delta_{\ell\ell'} \delta_{mm'}$$

$$\langle \underline{K}'_{\ell m}, \underline{K}'_{\ell' m'} \rangle = \langle \underline{K}''_{\ell m}, \underline{K}''_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'}$$

$$\langle \underline{K}'_{\ell m}, \underline{K}''_{\ell' m'} \rangle = 0$$

$$\frac{1}{\sqrt{\ell(\ell+1)}} \nabla_{\Omega} \overline{Y}_{\ell m} = \underline{K}'_{\ell m} \tag{15}$$

$$\nabla_{\Omega} \times \underline{K}'_{\ell m} = \underline{K}''_{\ell m}$$

$$\nabla_{\Omega} \cdot \underline{K}''_{\ell m} = 0$$

$$\hat{r} \times \underline{K}'_{\ell m} = \underline{K}''_{\ell m}$$

where ∇_{Ω} is the surface operator on the unit sphere. In virtue of (14) equations (11) and (12) can be written in the following compact form:

$$\sigma = -\frac{q}{4\pi a^2} - \frac{q}{a^2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{[y_m h_\ell(y_m)]'}{[x_m h_\ell(x_m)]'} \overline{Y}^{*}_{\ell m}(\tfrac{\pi}{2}, \Omega t) \, \overline{Y}_{\ell m}(\theta,\phi) \tag{16a}$$

$$\underline{K}' = -\frac{q\Omega}{a} \hat{\phi} \cdot \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{[y_m h_\ell(y_m)]'}{[x_m h_\ell(x_m)]'} \underline{K}'^{*}_{\ell m}(\tfrac{\pi}{2}, \Omega t) \, \underline{K}'_{\ell m}(\theta,\phi) \tag{16b}$$

$$\underline{K}'' = -\frac{q\Omega}{a}\,\hat{\phi}\cdot\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{y_m h_\ell(y_m)}{x_m h_\ell(x_m)}\,\underline{K}''^{*}_{\ell m}(\tfrac{\pi}{2},\Omega t)\;\underline{K}''_{\ell m}(\theta,\phi) \tag{16c}$$

Equation (16) is the exact solution of the problem posed in Fig.1.

B. Quasi-Static Solution

We now make a non-relativistic approximation to (16) and keep terms up to order $\beta(=\Omega b/c)$, where the particle's velocity is equal to Ωb. After some straightforward small-argument expansions for the spherical Hankel functions, we obtain

$$\sigma = \sigma_s + 0(\beta^2)$$

$$\underline{K}' = \underline{K}'_s + 0(\beta^3)$$

$$\underline{K}'' = \underline{K}''_s + 0(\beta^3)$$

where

$$\sigma_s = -\frac{q}{4\pi a^2} - \frac{q}{a^2}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}(\tfrac{a}{b})^{\ell+1}\,\overline{Y}^{*}_{\ell m}(\tfrac{\pi}{2},\Omega t)\;\overline{Y}_{\ell m}(\theta,\phi) \tag{17a}$$

$$\underline{K}'_s = -\frac{q\Omega}{a}\,\hat{\phi}\cdot\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}(\tfrac{a}{b})^{\ell+1}\,\underline{K}'^{*}_{\ell m}(\tfrac{\pi}{2},\Omega t)\;\underline{K}'_{\ell m}(\theta,\phi) \tag{17b}$$

$$\underline{K}''_s = -\frac{q\Omega}{a}\,\hat{\phi}\cdot\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}(\tfrac{a}{b})^{\ell}\,\underline{K}''^{*}_{\ell m}(\tfrac{\pi}{2},\Omega t)\;\underline{K}''_{\ell m}(\theta,\phi) \tag{17c}$$

Note that

$$\nabla\cdot\underline{K}'_s + \frac{\partial\sigma_s}{\partial t} = 0$$

(18)

$$\nabla\cdot\underline{K}''_s = 0$$

still hold. In the next section we will identify some appropriate electrostatic and magnetostatic boundary-value problems that have solutions given by (17).

C. Numerical Comparison

We wish to establish some validity criteria for expressions (17) for different values of the particle's speed and distance from the sphere, so that when we later generalize them to arbitrary motions we can estimate the accuracy of the corresponding quasi-static solutions. In Figs.2-4 we plot the following relative quantities

$$\frac{\|\sigma-\sigma_s\|}{\|\sigma\|}, \quad \frac{\|\underline{K}'-\underline{K}_s'\|}{\|\underline{K}'\|}, \quad \frac{\|\underline{K}''-\underline{K}_s''\|}{\|\underline{K}''\|}$$

against $\beta(=v/c)$, KE (the particle's relativistic kinetic energy), and $\Omega a/c$ with a/b as a parameter. These quantities are constructed from (16) and (17) with the following definitions:

$$\|\sigma-\sigma_s\|^2 = \int (\sigma-\sigma_s)(\sigma-\sigma_s)^* \sin\theta \, d\theta \, d\phi$$

$$\|\underline{K}'-\underline{K}_s'\|^2 = \int (\underline{K}'-\underline{K}_s')\cdot(\underline{K}'-\underline{K}_s')^* \sin\theta \, d\theta \, d\phi$$

and a similar expression for $\|\underline{K}''-\underline{K}_s''\|$, where the asterisk denotes complex conjugation. These figures show the overall accuracy on the quasi-static solution given by (17).

In Figs.5-6 we plot the exact expressions (16) at the point ($\theta = 90^\circ$, $\phi = \Omega t$) on the sphere against β and KE with b/a as a parameter. These figures show the detailed local accuracy on the quasi-static solution (17).

The following physical explanation may be offered regarding the question as to why the quasi-static solutions agree so well with the exact solutions, as shown in Figs. 2-6. As is well known, a sphere has large damping constants for all the exterior resonant modes that can exist. Let α denote the damping constant of a particular mode, i.e., the amplitude of this mode is proportional

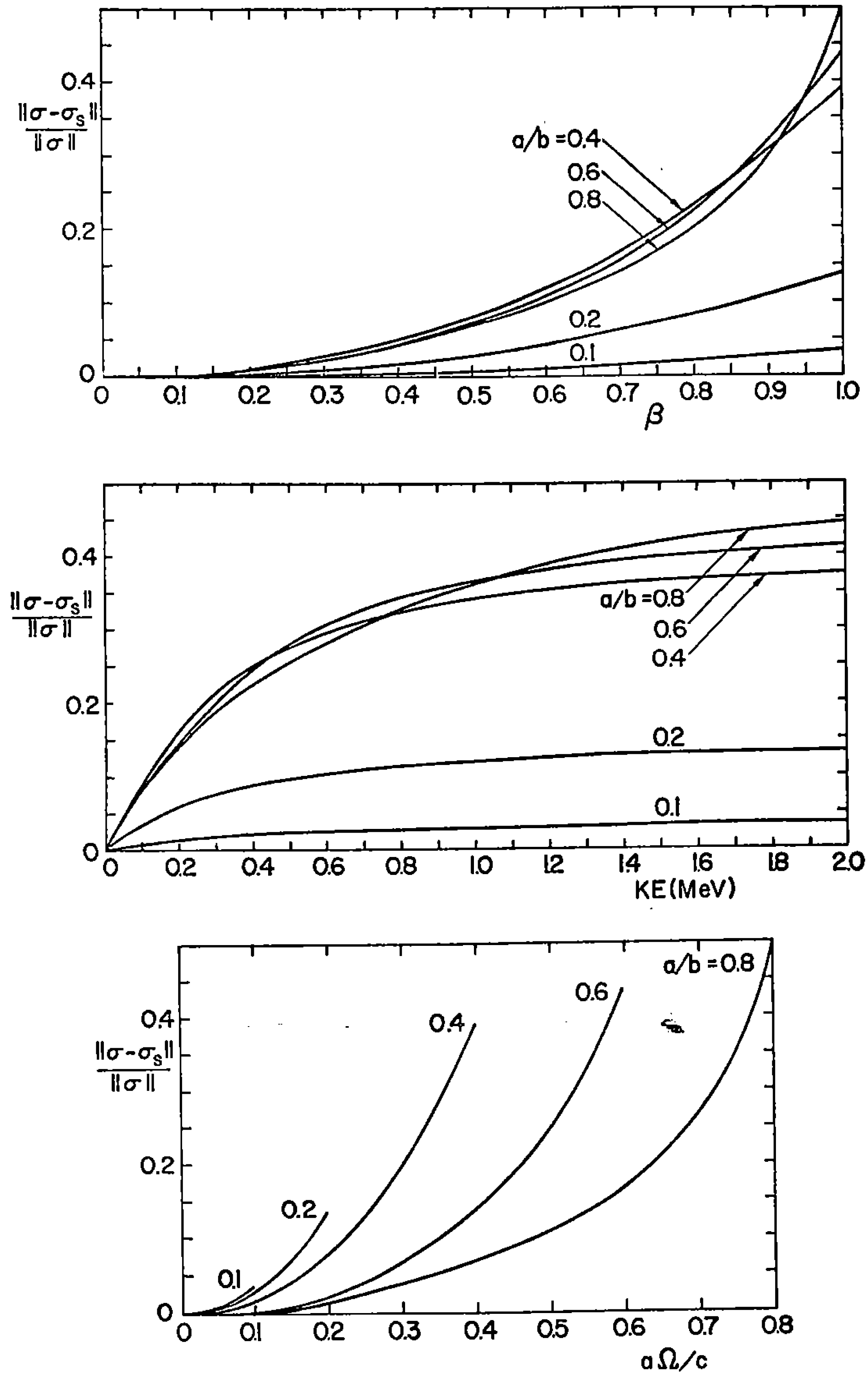

Figure 2. An overall comparison between the exact and quasi-static charge densities.

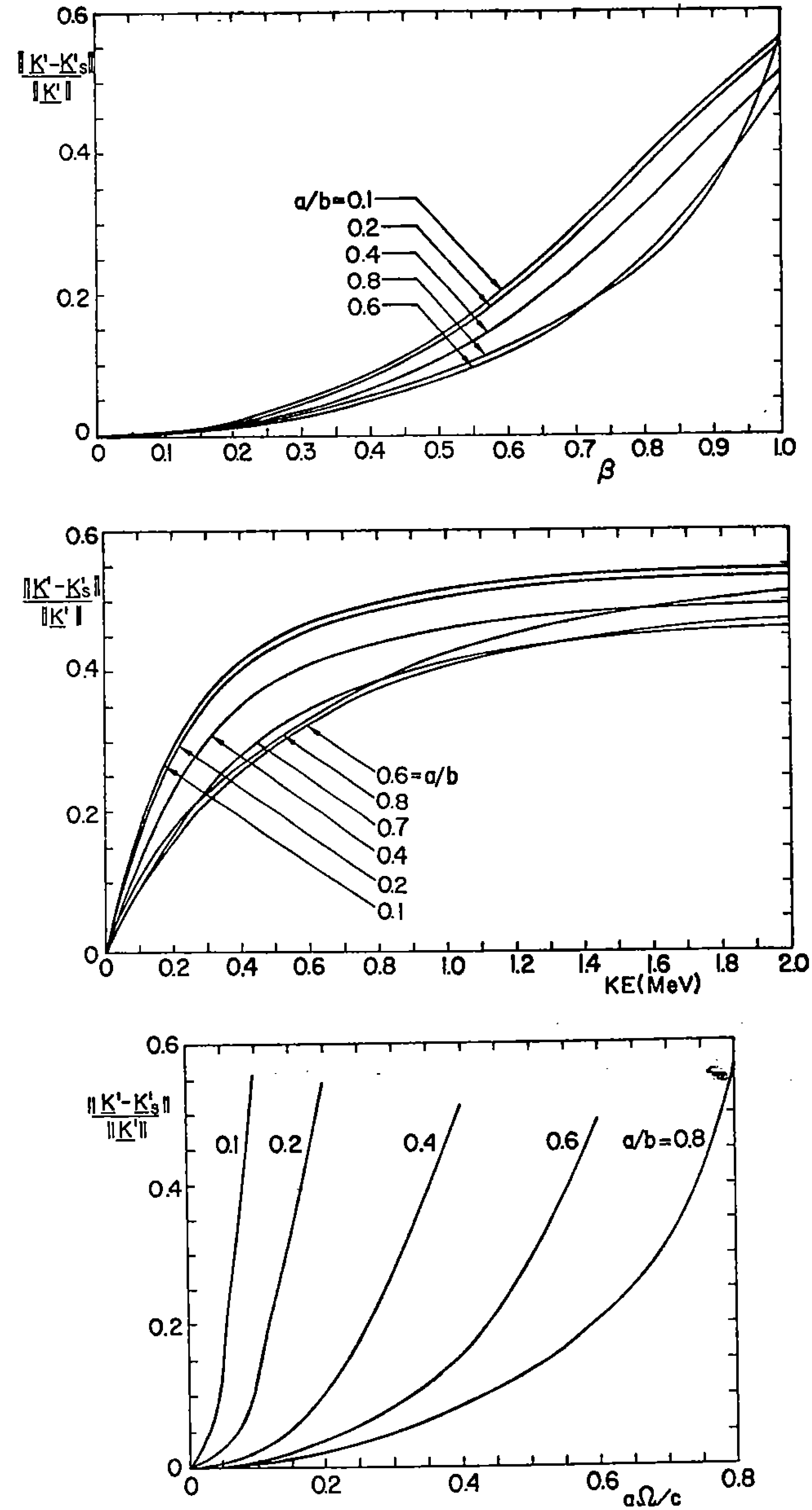

Figure 3. An overall comparison between the exact and quasi-static current densities ($\underline{K}'$ and $\underline{K}'_s$).

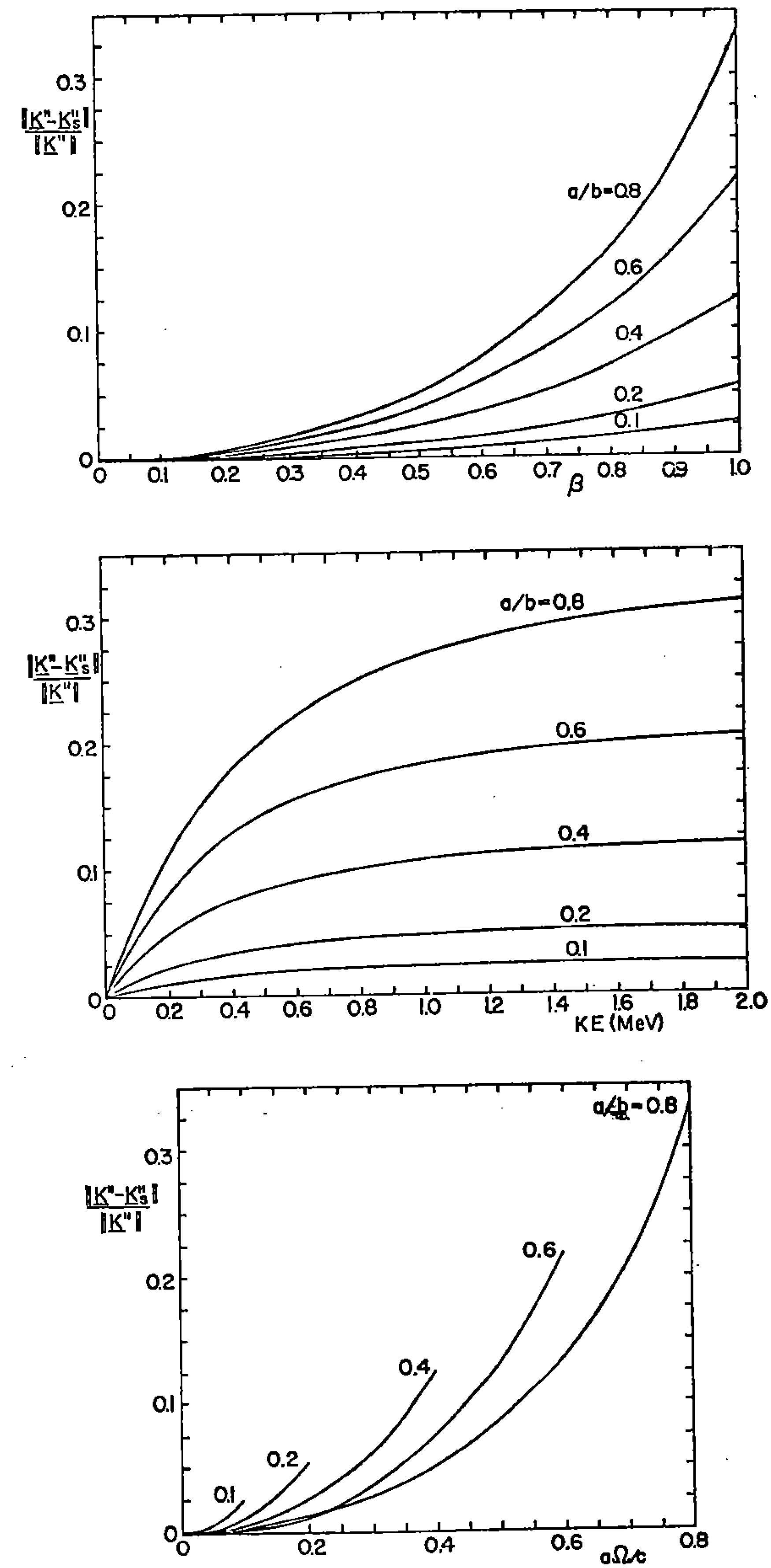

Figure 4. An overall comparison between the exact and quasi-static current densities ($\underline{K}''$ and $\underline{K}''_s$).

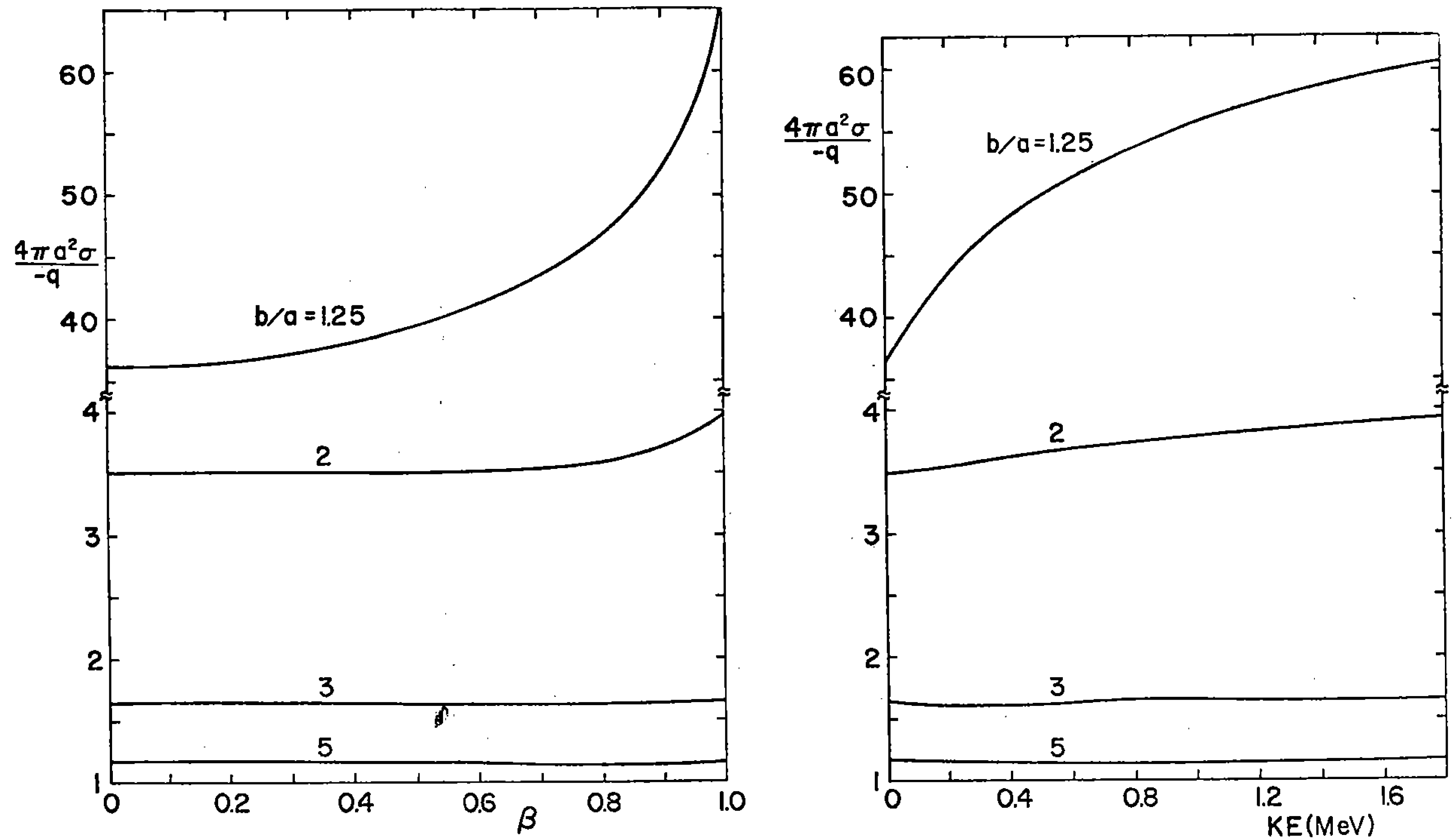

Figure 5. The exact charge density at $\theta = 90^o$ and $\phi = \Omega t$.

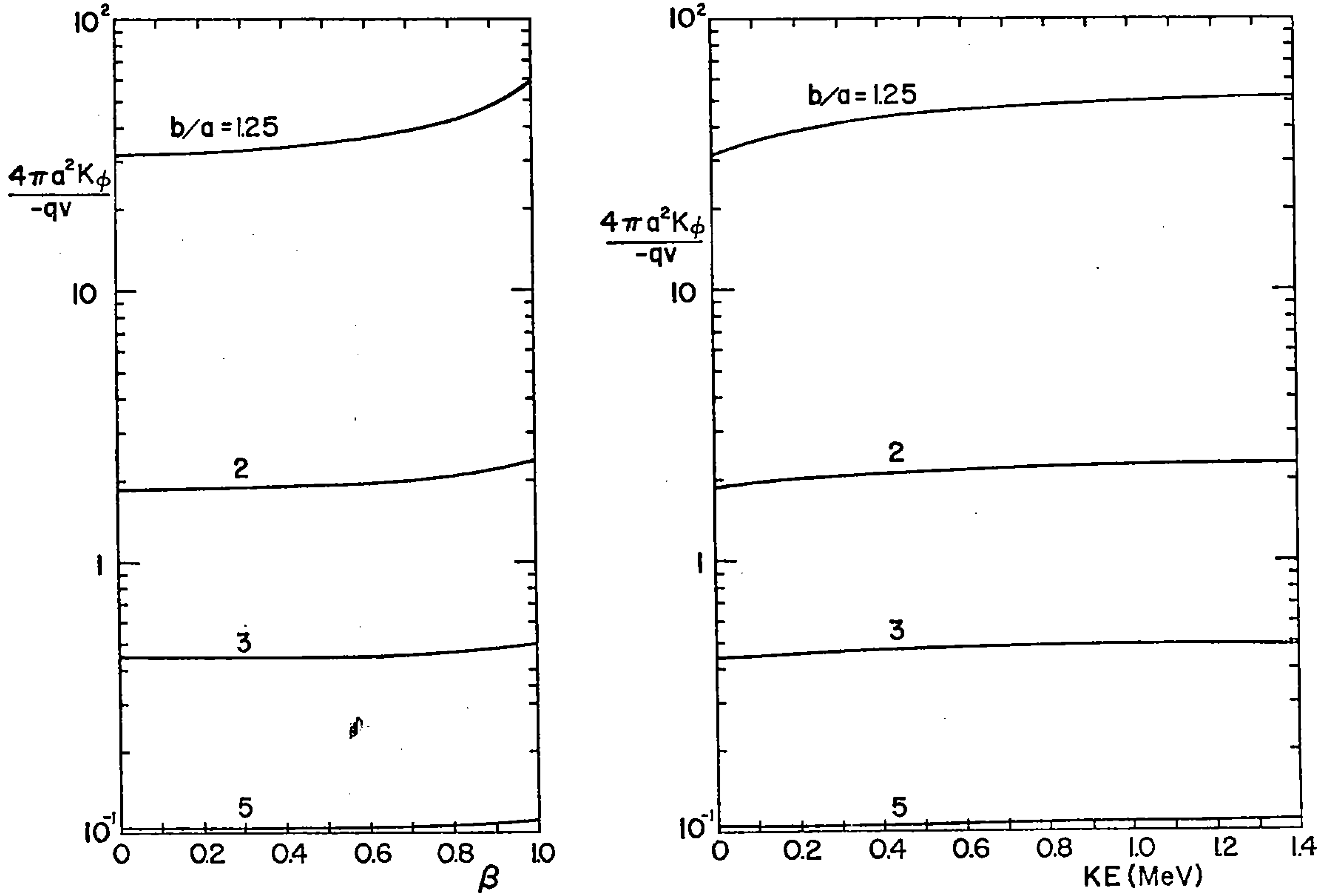

Figure 6. The exact current density at $\theta = 90^o$ and $\phi = \Omega t$.

to $\exp(-\alpha\, ct/a)$. In a time $t = a/(c\alpha)$, the angular distance $\Delta\phi$ that the charge has traveled is given by

$$\Delta\phi = \frac{1}{\alpha}\frac{v}{c}\frac{a}{b}$$

Since v/c and a/b are never greater than unity, the product $(v/c)(a/b)$ can be quite small. From Ref.[8] we also know that $\alpha \geq 1$ for all the TE-modes, whereas $\alpha \geq \frac{1}{2}$ for all the TM-modes. Hence, any resonances on the sphere will be damped out before the charged particle has traveled an appreciable distance, implying that the quasi-static approximation should be quite good.

III. Charged Particle With Arbitrary Motion in the Presence of a Sphere

In the preceding section we saw that the quasi-static solution is sufficiently accurate for $b/a \geq 2$ regardless of the value of β. However, for $1 < b/a < 2$ the accuracy depends strongly on the values of b/a and β. In this section we will first identify an appropriate electrostatic problem for obtaining (17a) and (17b) and an appropriate magnetostatic problem for obtaining (17c). Then, we will generalize and solve these two static problems for arbitrary motions as shown in Fig.7. Finally, explicit numerical results will be presented for radial as well as orbital motion.

A. Quasi-Electrostatic Problem

By the method of images it can easily be seen that σ_s given by (17a) is the solution of the following quasi-electrostatic problem:

$$\nabla^2 V = -\rho/\varepsilon$$

$$\rho = \frac{q\ \delta(r-b)\ \delta(\theta-\frac{\pi}{2})\ \delta(\phi-\Omega t)}{r^2 \sin\theta}$$

$$V = \text{constant on S (the spherical surface } r=a) \qquad (19)$$

$$-\int_S \frac{\partial V}{\partial n}\, dS = -q/\varepsilon$$

Here and in the following, the subscript s in σ_s, $\underline{K}'_s$, and $\underline{K}''_s$ is omitted. With a knowledge of σ one can determine $\underline{K}'$ from the continuity equation (18), since $\underline{K}'$ is derivable from the surface gradient of the Debye potential v, i.e.,

$$\underline{K}' = \frac{1}{c}\frac{\partial}{\partial t}\nabla_s(av)$$

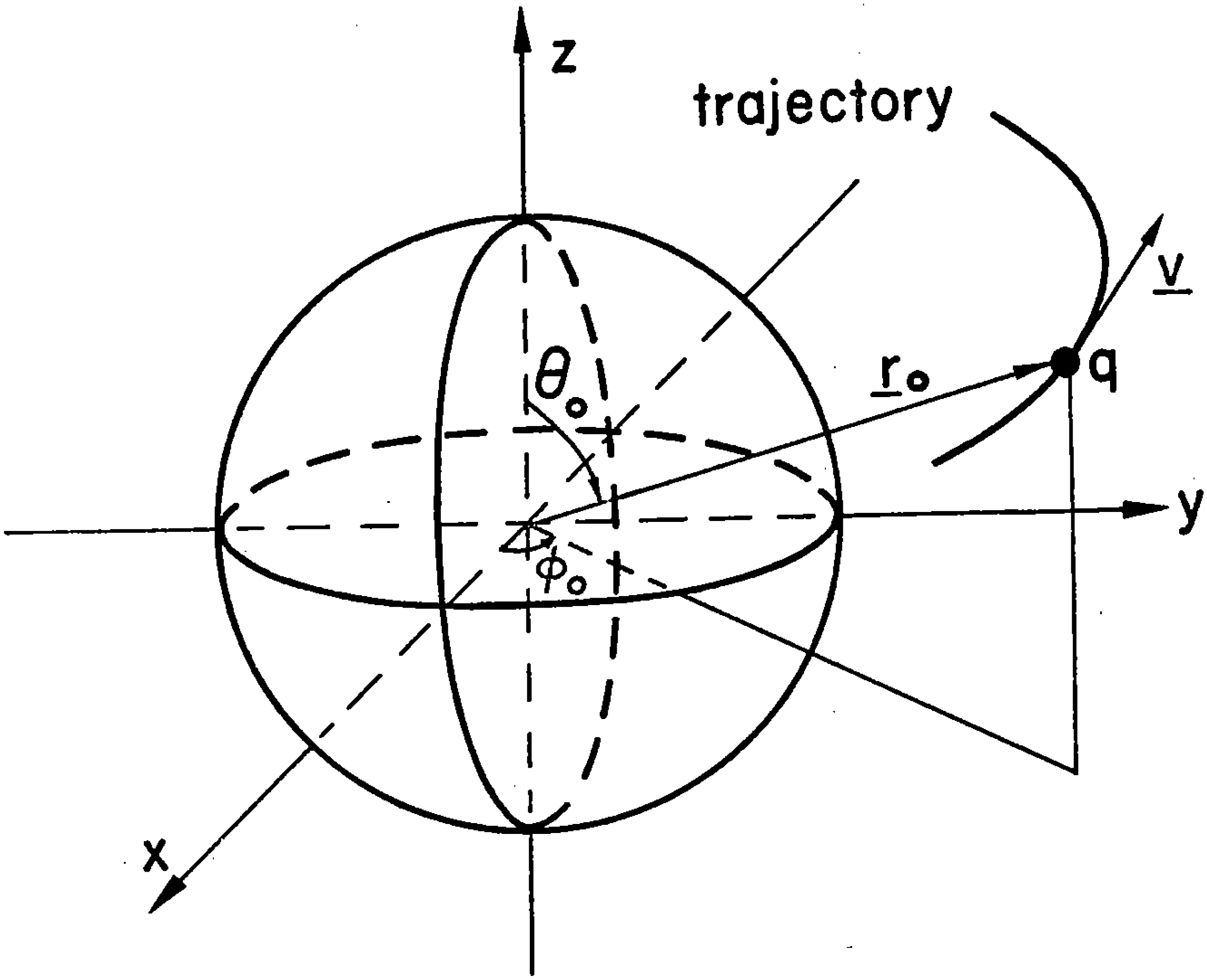

Figure 7. A charged particle with arbitrary trajectory in the presence of a sphere.

which follows from (5). Introducing $\chi = va/c$ we have the relationships

$$\underline{K}' = \frac{\partial}{\partial t} \nabla_s \chi, \quad \nabla_s^2 \chi = -\sigma \tag{20}$$

which enables us to determine $\underline{K}'$ from σ. Solving (20) we find $\underline{K}'$ to be identical to (17b).

For arbitrary motions as shown in Fig.7 we have

$$\rho = \frac{q\ \delta(r-r_o)\ \delta(\theta-\theta_o)\ \delta(\phi-\phi_o)}{r^2 \sin\theta}$$

where $r_o = r_o(t)$, $\theta_o = \theta_o(t)$ and $\phi_o = \phi_o(t)$. Solving (19) and (20) with this expression for ρ we immediately get

$$\begin{aligned} \sigma &= -\frac{q}{4\pi a^2} - \frac{q}{a^2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{a}{r_o}\right)^{\ell+1} \overline{Y}^*_{\ell m}(\theta_o,\phi_o)\ \overline{Y}_{\ell m}(\theta,\phi) \\ \underline{K}' &= -\frac{q}{r_o^2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{a}{r_o}\right)^{\ell} \left[\underline{v}\cdot\underline{K}'^*_{\ell m}(\theta_o,\phi_o) + v_r \sqrt{\frac{\ell+1}{\ell}}\ \overline{Y}^*_{\ell m}(\theta_o,\phi_o)\right] \underline{K}'_{\ell m}(\theta,\phi) \end{aligned} \tag{21}$$

where $\underline{v} = \dot{\underline{r}}_o$ and $v_r = \underline{v}\cdot\hat{r}$.

Several points are now in order regarding (21). Equations (21) become simpler in form when they are referred to the instantaneous spherical coordinate system (θ,ϕ) with the charged particle lying on the polar axis. With respect to this coordinate system $\underline{K}'$ has only a θ-component. One can also choose the coordinate system in which the instantaneous position of the particle is at $r_o = b$, $\theta_o = \pi/2$, $\phi_o = 0$. In this system the term $\underline{v}\cdot\underline{K}'^*_{\ell m}$ reduces exactly to (16b) obtained previously for the orbiting particle problem. By superposition, then, one can obtain $\underline{K}'$ for an arbitrarily moving particle by adding to equation

(17b) the term in (21) corresponding to the radial motion.

B. Quasi-Magnetostatic Problem

Before finding the solenoidal part of the current $\underline{K}''$ for a charged particle with arbitrary motion we will first show that equation (16c) is the solution to the following quasi-magnetostatic problem:

$$\nabla \times \underline{H} = \underline{K}_{ot}\ \delta(r-b)$$

$$\nabla \cdot \underline{H} = 0 \tag{22}$$

$$\hat{n} \cdot \underline{H} = 0, \quad \text{for } r=a$$

$$\underline{J} \equiv \underline{K}_o\ \delta(r-b) = q\Omega b \hat{\phi}\ \frac{\delta(r-b)\ \delta(\theta-\frac{\pi}{2})\ \delta(\phi-\Omega t)}{r^2 \sin\theta}$$

where $\underline{K}_{ot}$ is the solenoidal part of the source surface current density $\underline{K}_o$, i.e.,

$$\nabla_s \cdot \underline{K}_{ot} = 0$$

which ensures that

$$\nabla \cdot [\underline{K}_{ot}(\theta,\phi)\ \delta(r-b)] = 0$$

and, hence, the first equation of (22) is valid <u>everywhere</u>. To find $\underline{K}_{ot}$ from $\underline{K}_o$ we make use of the orthonormal vector functions $\underline{K}''_{\ell m}$ and obtain

$$\underline{K}_{ot} = \sum_{\ell,m} \langle \underline{K}_o, \underline{K}''_{\ell m} \rangle\ \underline{K}''_{\ell m}. \tag{23}$$

To solve (22) with (23) we first scalarize the problem by writing $\mu\underline{H} = \nabla\times\underline{A}$ and $\underline{A} = \nabla\times(\underline{r}W)$ where W satisfies $\nabla^2 W = 0$, $r\neq b$. In this connection one can refer to Smythe [11]. Actually, W is the static limit of u used in Section II. Note that the boundary condition $\hat{n}\cdot\underline{H} = 0$ at $r=a$ implies that W = constant at $r=a$. For definiteness, one can set this constant equal to zero, since a constant only introduces an additional H_r component that is independent of θ,ϕ. But this isotropic term H_r vanishes everywhere due to the requirement

$$\int r^2 \, H_r(r,\theta,\phi) \sin\theta \, d\theta \, d\phi = 0$$

which follows from $\nabla\cdot\underline{H} = 0$ and $\hat{n}\cdot\underline{H} = 0$ for $r=a$.

With all these remarks in mind one can easily show that, indeed, equation (22) has the solution (16c). It is interesting to point out that the series with $m = 0$ in (16c) corresponds to the current induced by a loop around a sphere.

For an arbitrarily oriented current element we simply replace $\Omega b\hat{\phi}$ by $\underline{v}$, and the coordinates $(b,\pi/2,\Omega t)$ of the orbiting charge by its instantaneous position (r_o,θ_o,ϕ_o) in (22), i.e.,

$$\underline{J} = q\underline{v} \, \frac{\delta(r-r_o) \; \delta(\theta-\theta_o) \; \delta(\phi-\phi_o)}{r^2 \sin\theta}$$

Solving (22) with this current source by the above technique we obtain

$$\underline{K}'' = \frac{-q}{r_o^2} \, \underline{v}\cdot \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{a}{r_o}\right)^{\ell-1} \underline{K}''^{*}_{\ell m}(\theta_o,\phi_o) \; \underline{K}''_{\ell m}(\theta,\phi) \tag{24}$$

In Appendix B we use an integral-equation approach to arrive at (24).

C. Numerical Results for Radial and Orbital Motion

In the case where the charged particle is moving radially along the polar axis $\theta_o = 0$ at a constant velocity v we have from (21) and (24), with r_o as the instantaneous radial distance of the charge q from the center of the sphere,

$$\sigma = -\frac{q}{4\pi a^2} - \frac{q}{4\pi a^2}\sum_{\ell=1}^{\infty}(2\ell+1)\left(\frac{a}{r_o}\right)^{\ell+1} P_\ell(\cos\theta)$$

$$\underline{K}' = \frac{qv}{4\pi a^2}\sum_{\ell=1}^{\infty}\frac{2\ell+1}{\ell}\left(\frac{a}{r_o}\right)^{\ell+2}\sin\theta\, P_\ell'(\cos\theta)\hat{\theta} \qquad (25)$$

$$\underline{K}'' = 0.$$

Equations (25) are plotted in Figs.8-9. The induced surface charge and current densities due to an orbiting charge as given by equations (17) are plotted in Figs.10-14. By suitably combining the curves for radial and orbital motion as presented in Figs.8-10 one obtains the instantaneous induced charges and currents on a sphere due to a charge with arbitrary motion in a plane passing through the center of the sphere.

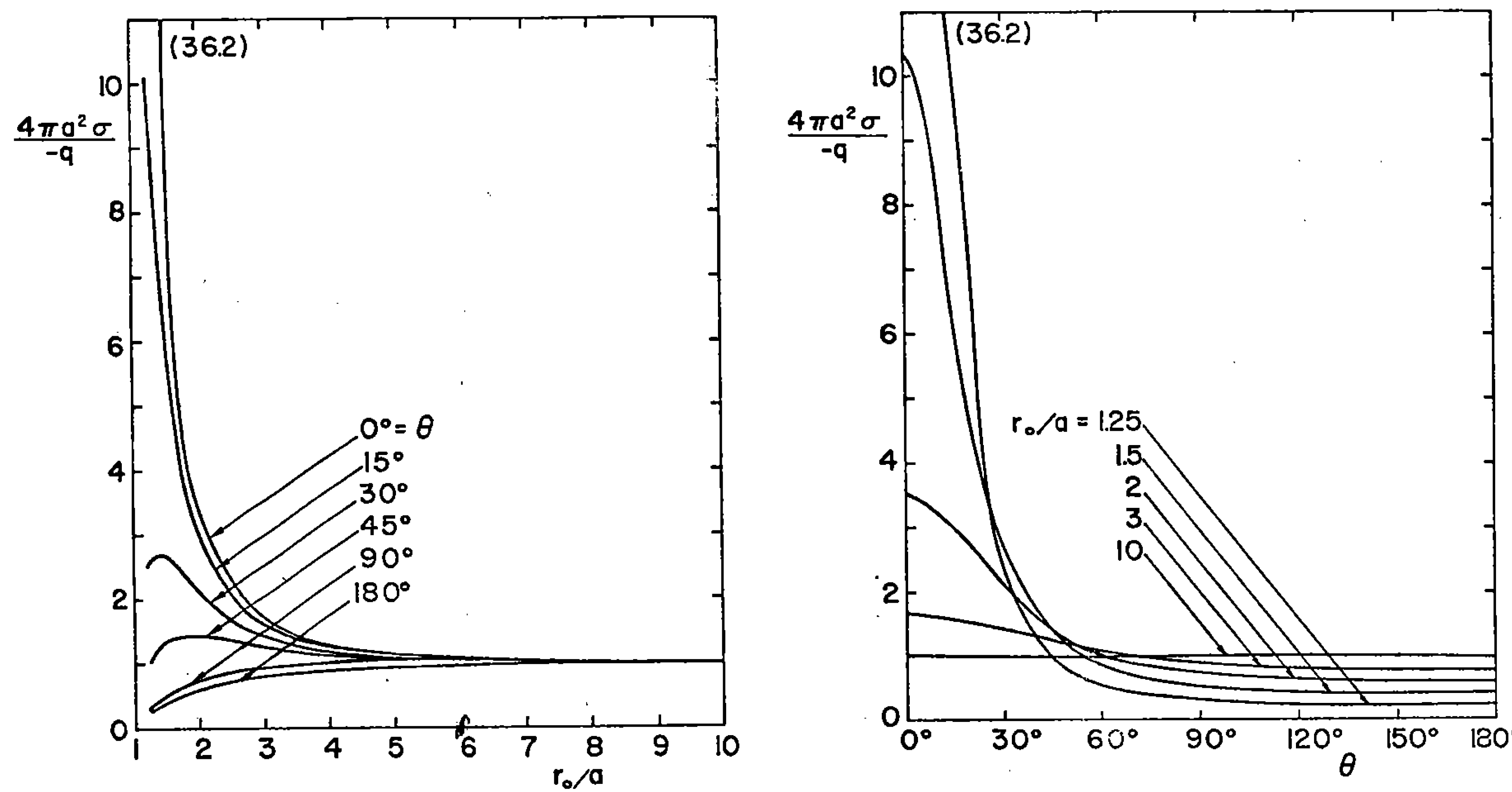

Figure 8. Quasi-static charge density induced by a radially moving charge.

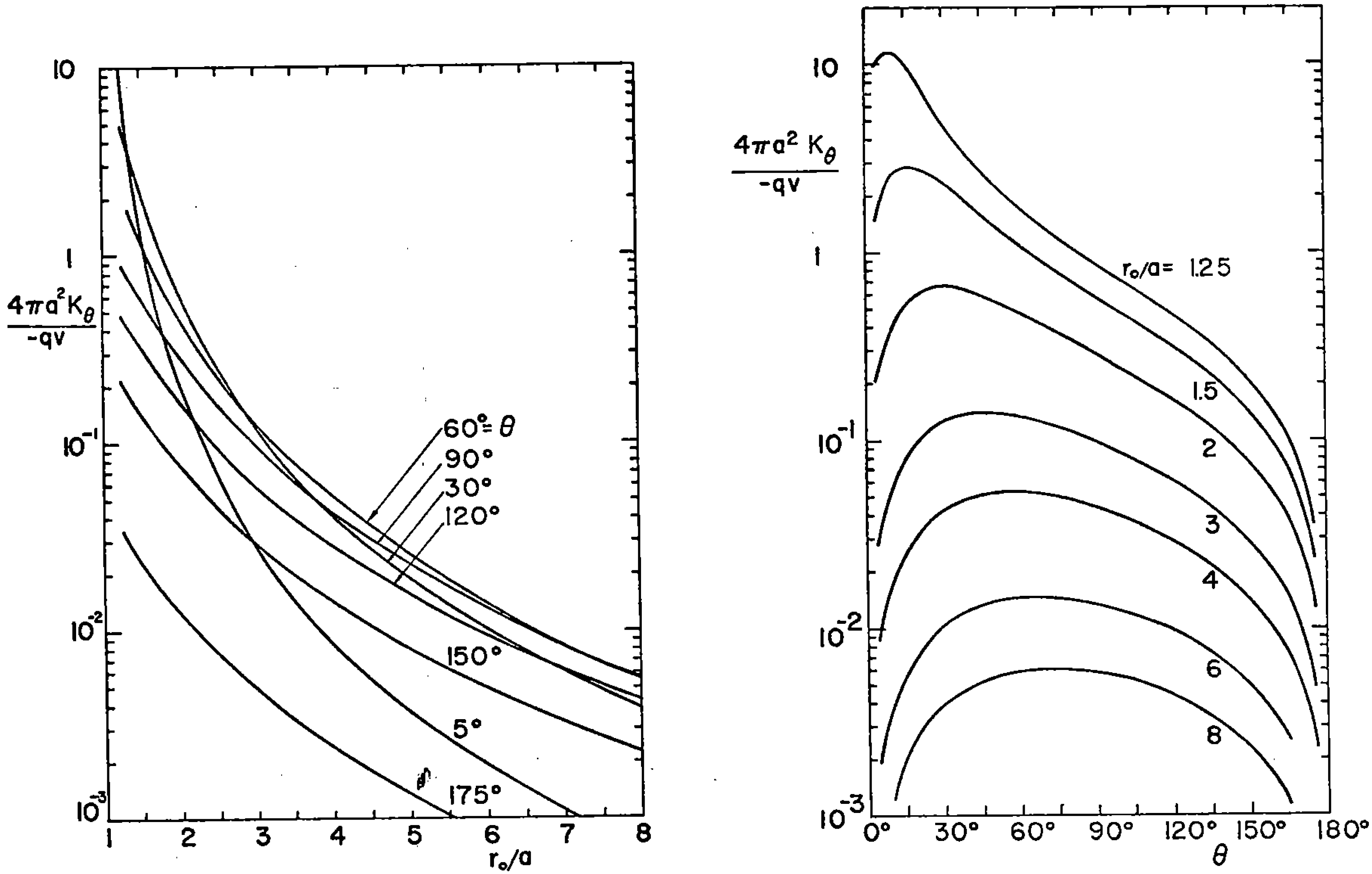

Figure 9. Quasi-static current density induced by a radially moving charge.

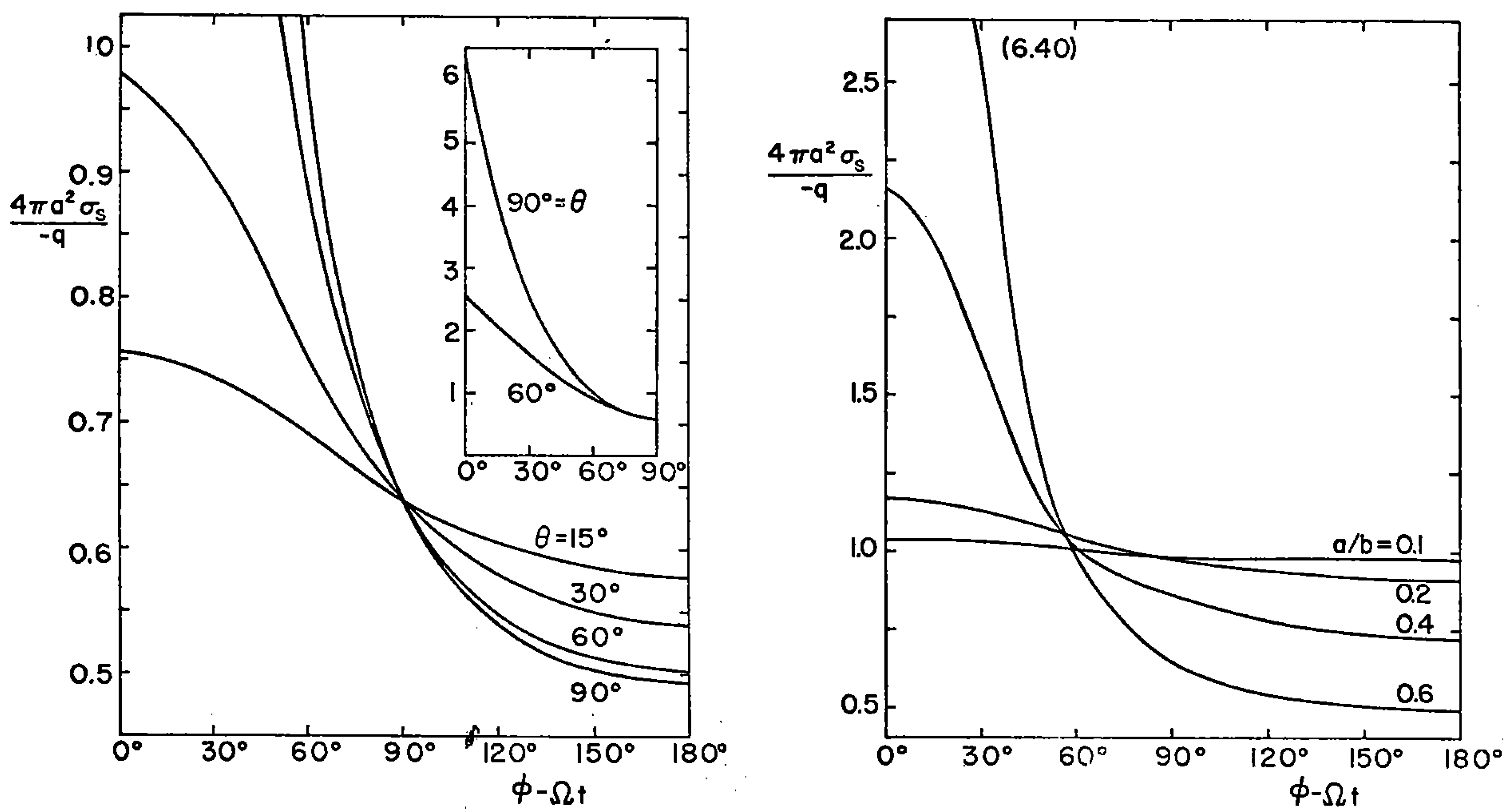

Figure 10. Quasi-static charge density induced by an orbiting charge.

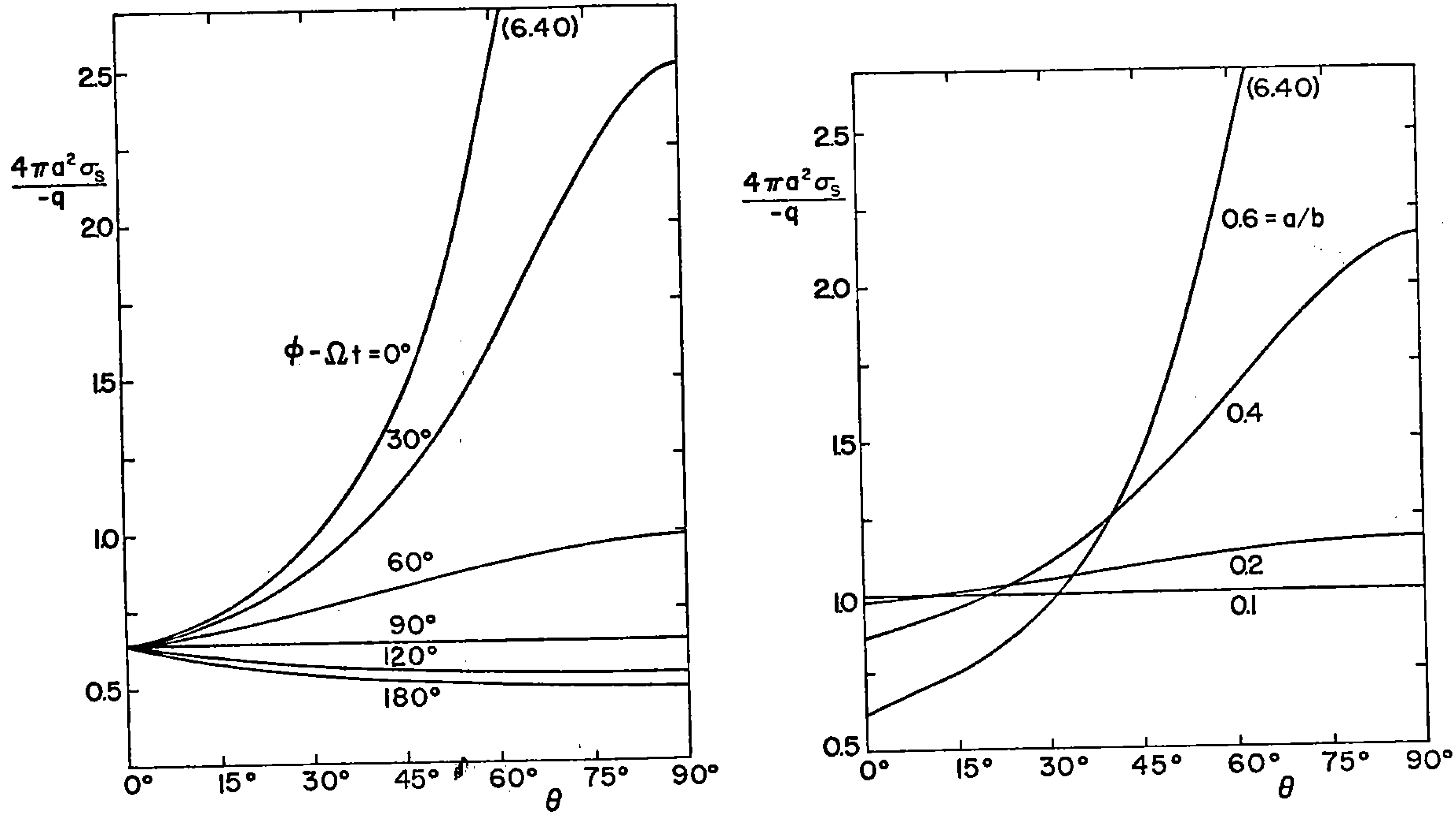

Figure 11. Quasi-static charge density induced by an orbiting charge.

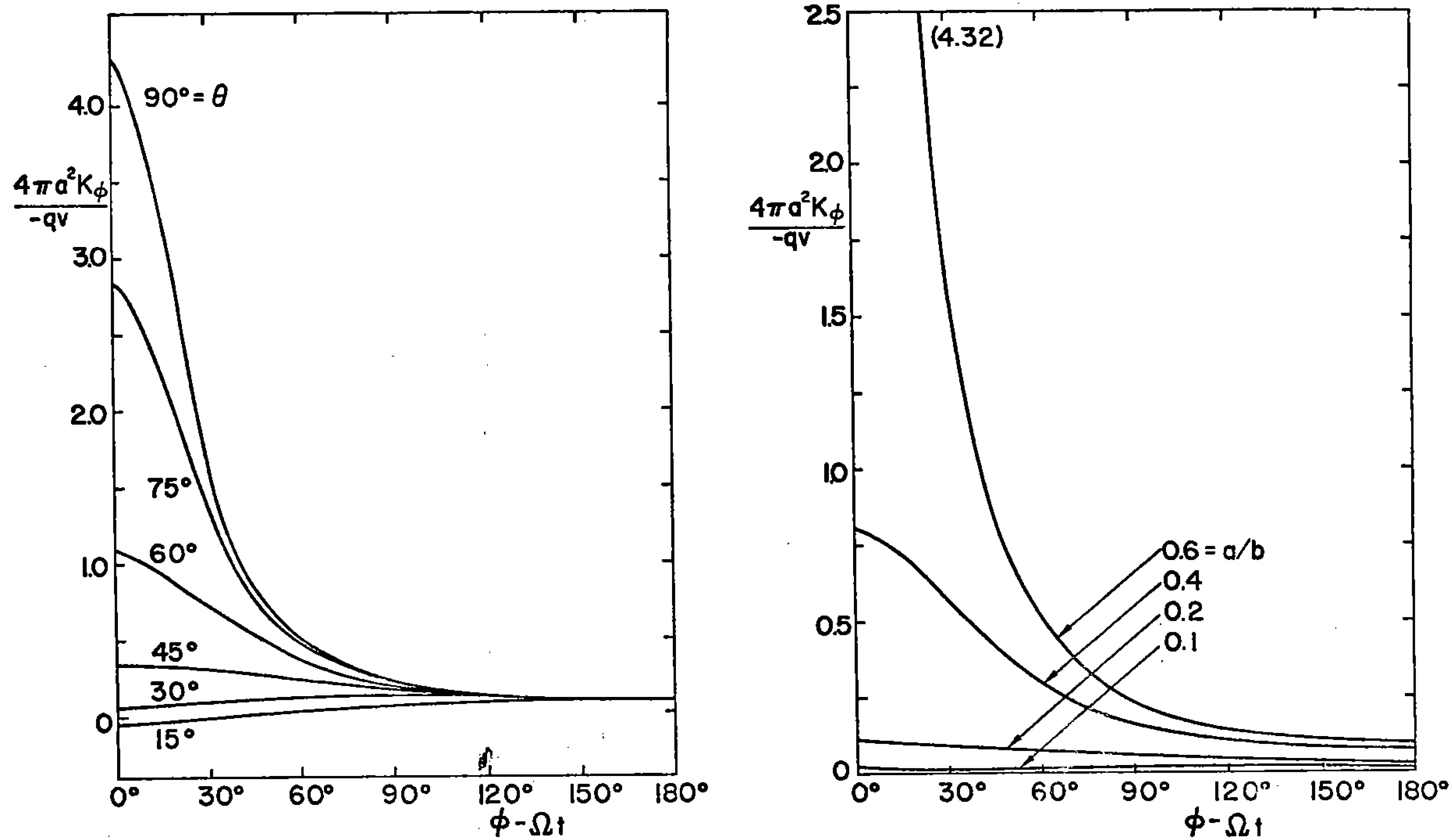

Figure 12. Quasi-static current density induced by an orbiting charge.

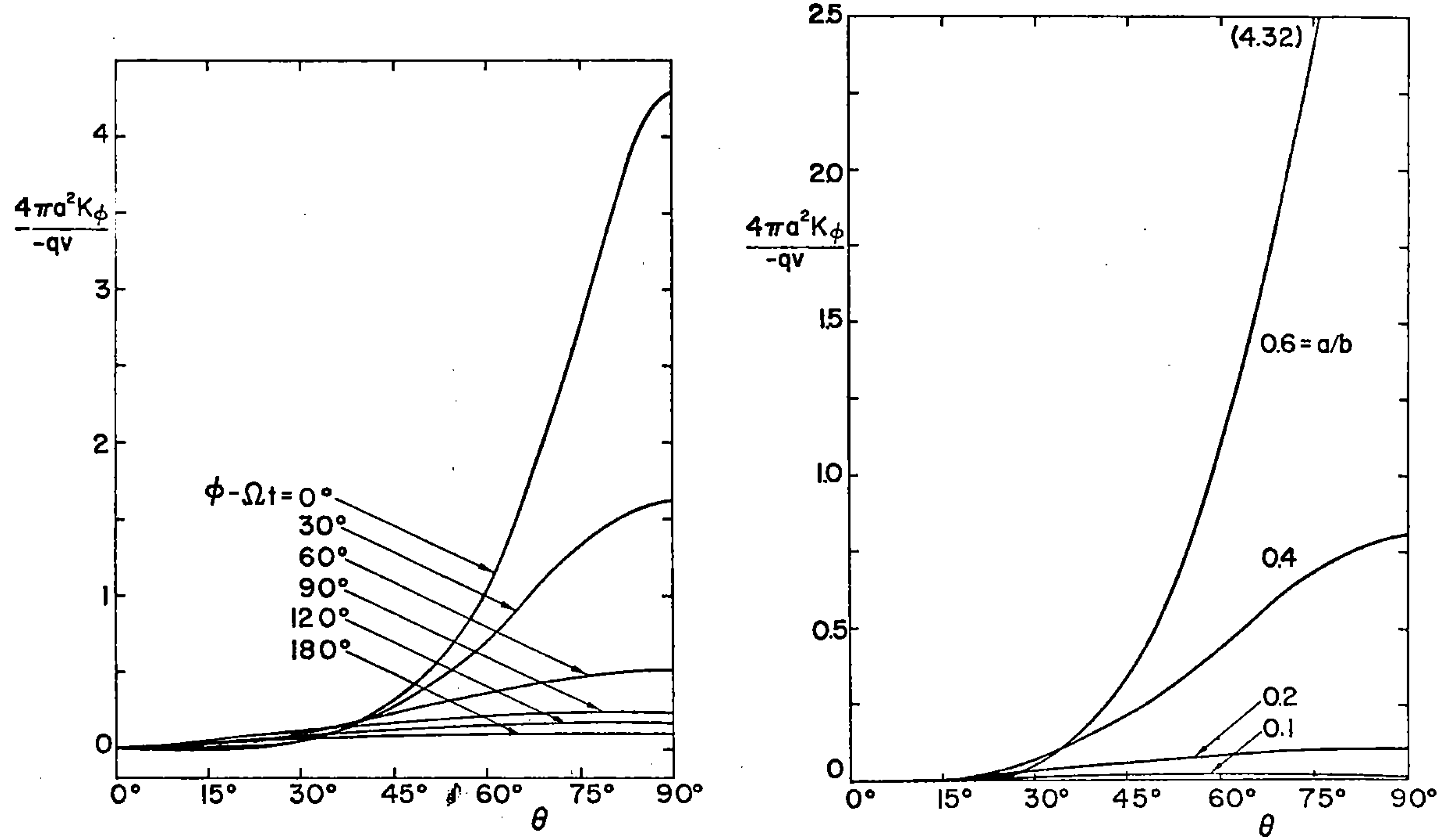

Figure 13. Quasi-static current density induced by an orbiting charge.

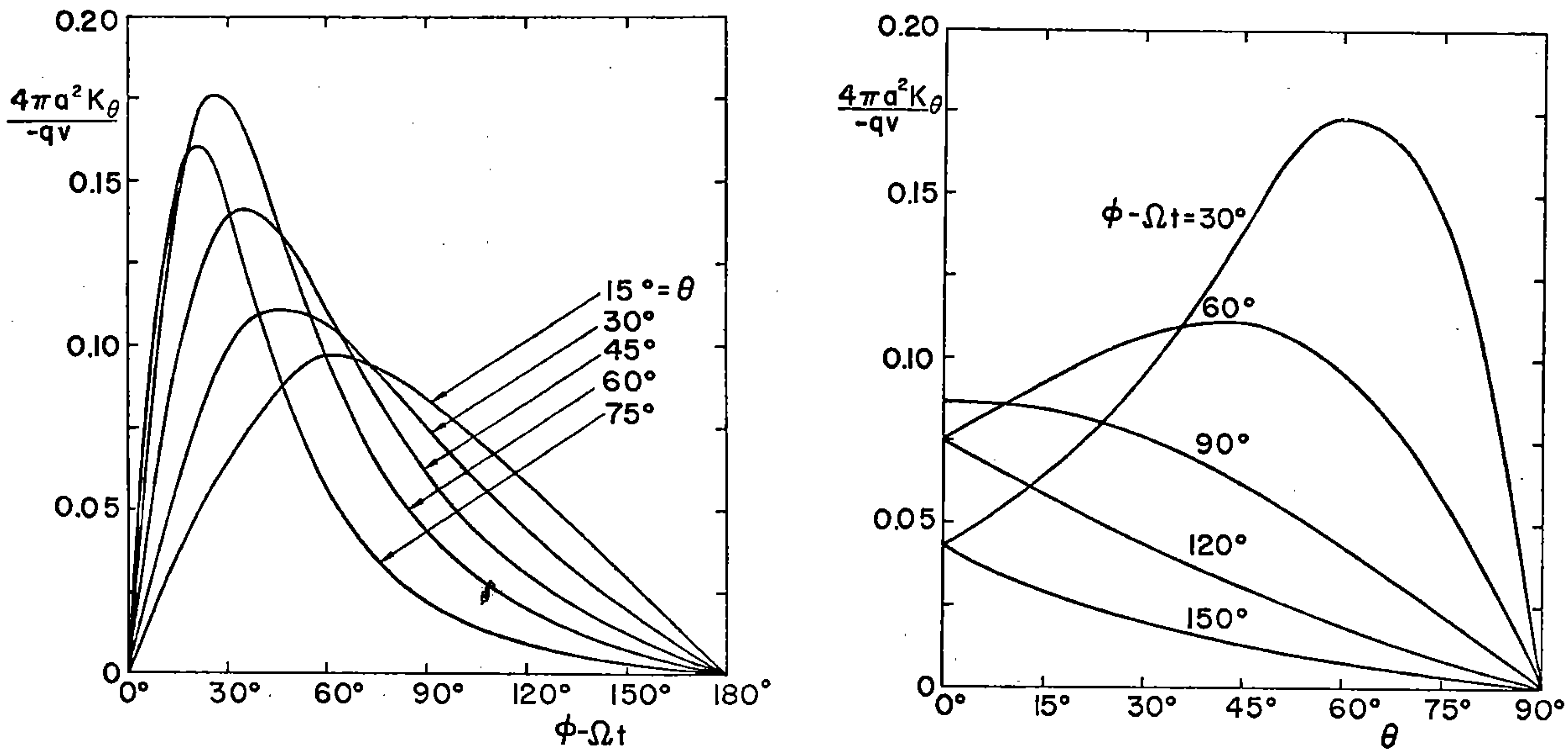

Figure 14. Quasi-static current density induced by an orbiting charge.

IV. A Moving Charged Particle in the Presence of an Arbitrarily Shaped Conductor

In this section we will discuss methods to generalize the results in Section III for a sphere to an arbitrarily shaped conductor.

A. Quasi-Electrostatic Problem

Let us first consider the quasi-electrostatic problem. Mathematically, this problem can be stated as follows:

$$\nabla^2\phi = -\frac{q}{\varepsilon}\,\delta(\underline{r}-\underline{r}_o) \tag{26a}$$

with

$$\phi = \phi_o = \text{constant on S (the surface of the conductor)} \tag{26b}$$

and

$$\int_S \sigma dS = -\int_S \varepsilon \frac{\partial\phi}{\partial n}\, dS = -q. \tag{26c}$$

Equations (26) can, of course, be reduced to the following integral equation

$$\phi_o = \phi^{inc} + \int_S G\sigma dS \tag{26d}$$

and the value of ϕ_o is determined by (26c). To find $\underline{K}'$ from σ we write $\underline{K}' = \nabla_s\chi$ and insert it into the continuity equation

$$\nabla_s\cdot\underline{K}' + \frac{\partial\sigma}{\partial t} = 0$$

to obtain

$$\nabla_s^2 \chi = -\frac{\partial \sigma}{\partial t} \tag{27}$$

the particular solution of which will lead to the desired solution. The general method to solve (27) is yet to be found, but for surfaces which allow separation of variables for the surface Laplacian, χ can be obtained straightforwardly from (27). Fortunately, one can get the total current directly from the magnetostatic problem without first finding $\underline{K}'$.

B. Quasi-Magnetostatic Problem

The total quasi-static surface current $\underline{K}$ $(=\underline{K}' + \underline{K}'')$ can be found by solving the following well-known integral equation appropriate for the magnetostatic problem

$$\frac{1}{2}\underline{K} - \int \underline{n}\times(\nabla G\times\underline{K})dS' = \underline{K}^{inc} \tag{28}$$

where

$$G = 1/(4\pi R), \quad R = |\underline{r}-\underline{r}'|$$

and $\underline{K}^{inc}$ can be obtained from the Biot-Savart formula as

$$\underline{K}^{inc} = \underline{n}\times\underline{H}^{inc} = q\,\frac{\underline{n}\times(\underline{v}\times\underline{R}_o)}{4\pi R_o^3}, \quad \underline{R}_o = \underline{r}-\underline{r}_o \tag{29}$$

and $\underline{r}_o$ is the position vector of the current element $q\underline{v}$.

For an arbitrary-shape body, equation (28) can be solved numerically. In those cases where $\nabla_s\cdot\underline{K}^{inc} = 0$ (for example, when $\underline{H}^{inc}$ is a uniform magnetic field or is the field due to a magnetic dipole) one can show that $\nabla_s\cdot\underline{K} = 0$. Taking the surface divergence of (28) and calling $\underline{K}$ by $\underline{K}''$ one gets

$$\frac{1}{2}\nabla_s\cdot\underline{K}'' + \int \frac{\partial G}{\partial n}\nabla_s'\cdot\underline{K}''dS' = \nabla_s\cdot\underline{K}^{inc} \tag{30}$$

where we have used

$$\nabla_s \cdot [\underline{n} \times \int \nabla G \times \underline{K}'' dS'] = \underline{n} \cdot \nabla \int \nabla G \cdot \underline{K}'' dS' - \underline{n} \cdot \int \nabla^2 G \, \underline{K}'' dS'$$

$$= \int \frac{\partial G}{\partial n} \nabla_s' \cdot \underline{K}'' dS' .$$

The uniqueness of the solution of (30) implies that $\nabla_s \cdot \underline{K}'' = 0$ when $\nabla_s \cdot \underline{K}^{inc} = 0$. Hence one can write

$$\underline{K}'' = \nabla_s \times (\underline{n}\psi) \tag{31}$$

Let us see if we can derive from (28) an integral equation for ψ. Post-multiplying (28) vectorially by $\underline{n}$ and then substituting (31) in the resulting equation, we get

$$\frac{1}{2}\psi - \int \frac{\partial G}{\partial n'} \psi dS' = \psi^{inc} \tag{32}$$

where $\underline{n} \times \nabla_s \psi^{inc} = \underline{K}^{inc}$. In deriving (32) we have made use of the following vector calculus:

$$\int \nabla G \times [\nabla' \times \underline{n}' \psi(\underline{r}')] dS' = \nabla \times \int \{\nabla_s' \times [\underline{n}' G \psi(\underline{r}')] - \nabla' G \times \underline{n}' \psi(\underline{r}')\} dS'$$

$$= -\nabla \times \int \nabla' G \times \underline{n}' \psi(\underline{r}') dS' = \int (\underline{n}' \cdot \nabla) \nabla G \psi(\underline{r}') dS' - \int \underline{n}' \nabla^2 G \psi(\underline{r}') dS'$$

$$= -\nabla \int \frac{\partial G}{\partial n'} \psi(\underline{r}') dS'$$

In general, it is not possible to find ψ^{inc} from $\underline{H}^{inc}$. However, for some special cases where the geometry of the problem has a high degree of symmetry one may make a judicious guess to write down ψ^{inc} by inspection. In this sense (28) is more fundamental.

Equations (26), (27), (28) and (29) constitute the complete formulation of the problem involving a moving charge in the presence of an arbitrary-shaped conductor.

V. Two Initial-Value Problems

In this section we will pose and solve, within the realm of classical electrodynamics, two initial-value problems which may relate to the situation (Fig.15) where a plane-wave pulse of photon flux impinges upon a metallic sphere. The problems we propose to solve will shed some light on the question as to how the positive charges redistribute themselves on the surface of the sphere after the passage of the photon pulse, with the assumption that the electrons outside the sphere have negligible effect on the redistribution process of the positive charges. The first step towards solving the problems is to specify a set of sufficient initial data on the sphere, so that the problems will become mathematically well posed, i.e., the existence of a unique solution of Maxwell's equations will be guaranteed. Obviously, a specification of the surface charge density alone at one instant will not suffice. But a unique solution is ensured if one specifies the tangential electric field on the surface of the sphere for all times together with the total net charge at one instant (the law of charge conservation demands the total net charge be constant for all times). Therefore, we will start with a set of sufficient data on the sphere and see how much information one can deduce by specifying the initial charge distribution alone. This charge distribution is readily available from the electron emission function (or the so-called photoelectric emission current function), since by integrating this function (multiplied by the electronic charge) over all electron energies and the duration and energy spectrum of the photon pulse one will obtain the initial charge distribution on the sphere [4,5].

Referring to Fig.15 let us take as our first initial-value problem, the incident photon pulse to be of the form $\delta(t + z/c)$. The tangential electric field $\underline{E}_t$ on the surface of the sphere can be described as

$$\underline{E}_t(\theta,\phi,t) = \underline{E}_o(\theta,\phi)\ \delta(t + \frac{a}{c}\cos\theta) \qquad (33)^*$$

and there is no electromagnetic field before the photon pulse strikes the sphere.

* It will lately become evident that the specification of $\underline{E}_t$ by (33) can be regarded as an artifice if one is interested only in how much information one can deduce by just specifying the initial charge distribution on the sphere.

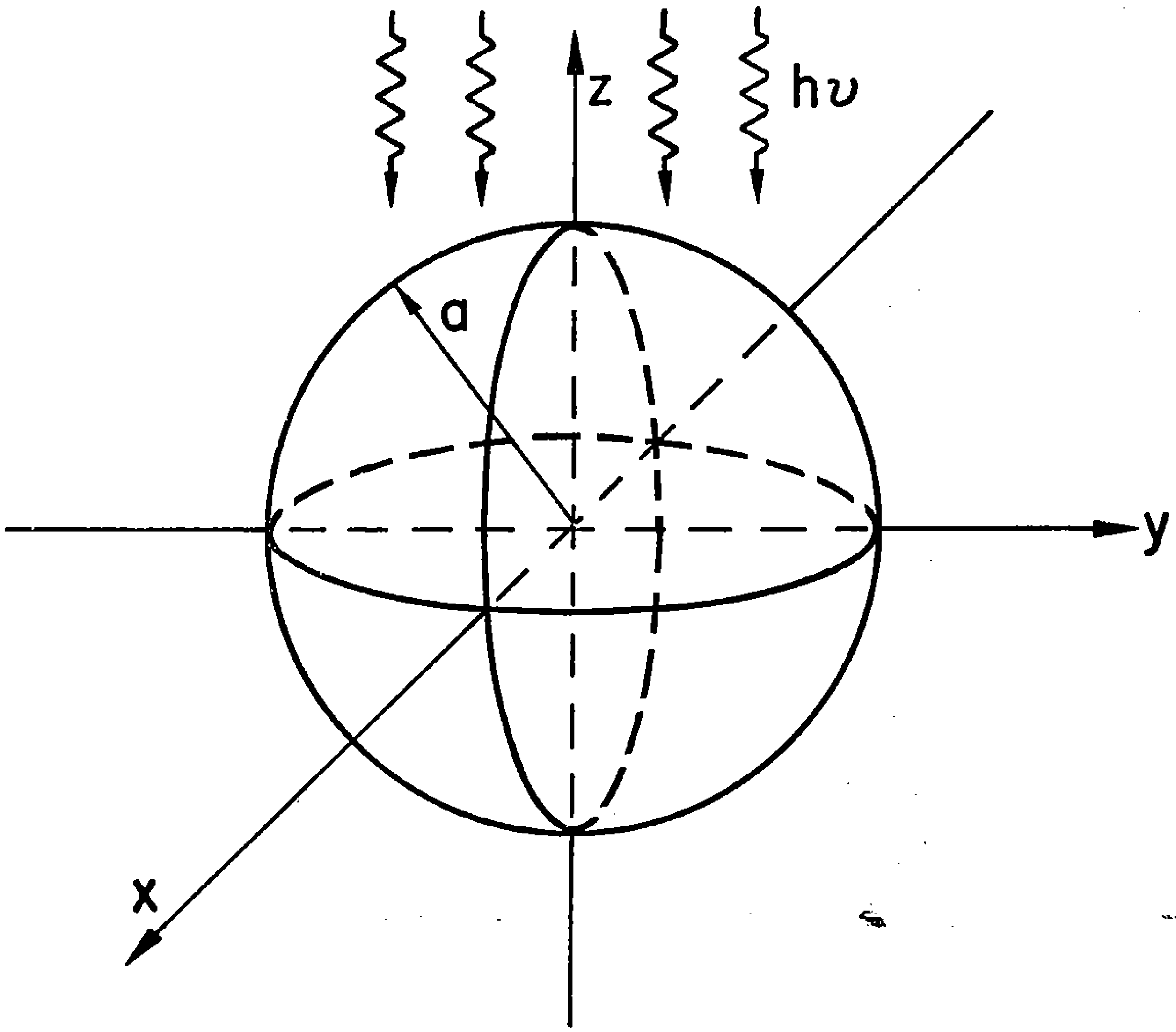

Figure 15. A sphere struck by a photon pulse.

Physically, (33) means that the spherical surface S is perfectly conducting at every point except at the instant when the photons strike the point in question. Given $\underline{E}_t$ on S for all times and the radiation condition at infinity the field is uniquely determined everywhere to within a static field which is related to the total net charge on the sphere. Using the procedure in Section II one can immediately write down the Debye potentials u and v in the frequency domain

$$
\begin{aligned}
u(r,\theta,\phi,\omega) &= \frac{i}{k} \sum_{\ell,m} \frac{A''_{\ell m}(k)}{\sqrt{\ell(\ell+1)}} \frac{h_\ell(kr)}{h_\ell(ka)} \overline{Y}_{\ell m}(\theta,\phi) \\
v(r,\theta,\phi,\omega) &= a \sum_{\ell,m} \frac{A'_{\ell m}(k)}{\sqrt{\ell(\ell+1)}} \frac{h_\ell(kr)}{[ka\ h_\ell(ka)]'} \overline{Y}_{\ell m}(\theta,\phi)
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
A'_{\ell m}(k) &= \int \exp(-ika\cos\theta)[e'_{\ell m}\, \underline{K}'_{\ell m}(\theta,\phi) + e''_{\ell m}\, \underline{K}''_{\ell m}(\theta,\phi)]\cdot \underline{K}'^{*}_{\ell m}(\theta,\phi) \sin\theta\, d\theta\, d\phi \\
e'_{\ell m} &= \int \underline{E}_o(\theta,\phi)\cdot \underline{K}'^{*}_{\ell m}(\theta,\phi) \sin\theta\, d\theta\, d\phi
\end{aligned}
\tag{35}
$$

and, similarly, for $A''_{\ell m}$ and $e''_{\ell m}$. With u and v known the currents $\underline{K}'$, $\underline{K}''$ and the charge density σ are directly obtainable by straightforward differentiations. By performing an inverse Fourier transform, or by applying the Singularity Expansion Method, one gets

$$
\begin{aligned}
\sigma(\theta,\phi,t) = \frac{Q}{4\pi a^2} - \frac{i\varepsilon}{a} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{n} \sqrt{\ell(\ell+1)}\, B_{\ell mn}(t) \frac{h_\ell(k_{\ell n}a)}{[k_{\ell n}a h_\ell(k_{\ell n}a)]'} \\
\overline{Y}_{\ell m}(\theta,\phi)\, e^{-i\omega_{\ell n}t}\, U(t + \frac{a}{c}\cos\theta)
\end{aligned}
\tag{36}
$$

where $\omega_{\ell n} = ck_{\ell n}$, $k_{\ell n}$ satisfies $[k_{\ell n}ah_\ell(k_{\ell n}a)]' = 0$ and

$$B_{\ell mn}(t) = \int \underline{E}_o(\theta,\phi)\cdot\underline{K}'^{*}_{\ell m}(\theta,\phi)\; U(t + \frac{a}{c}\cos\theta)\; e^{-ik_{\ell n} a \cos\theta} \sin\theta \, d\theta \, d\phi .$$

The total net charge on the sphere is denoted by Q in (36). The expressions for $\underline{K}'$ and $\underline{K}''$ can be obtained with the same technique and we will not write them down here.

Before concluding this first problem, let us take $\underline{E}_t = \underline{E}_o \; \delta(t)$ which applies to times larger than the transit time across the sphere. Then, evaluating (36) at $t = 0+$ we get without much effort,

$$\sigma(\theta,\phi,0+) = \frac{Q}{4\pi a^2} - \frac{\varepsilon c}{a} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\ell(\ell+1)}\; e'_{\ell m}\; \overline{Y}_{\ell m}(\theta,\phi) \tag{37}$$

Thus, we see that in this case, specification of the intial charge distribution gives sufficient data for determining the TM fields only, while fields of the TE-type $(e''_{\ell m})$ remain undetermined.

We now consider a different initial-value problem where a static charge distribution σ_o is given on the surface, S, of a sphere for time $t < 0$:

$$\sigma_o(\theta,\phi) = \sum_{\ell,m} \sigma_{\ell m}\; \overline{Y}_{\ell m}(\theta,\phi) \tag{38}$$

At $t = 0$ the surface of the sphere is made a perfectly conducting surface. We wish to find the subsequent current and charge distributions on the sphere with radius a. In the following we will omit the term $\ell = 0$, $m = 0$ in (38) from our consideration, since the static electric field corresponding to this term does not charge for $t > 0$.

The general expansion of the tangential electric field $\underline{E}_t$ at $r = a$ can be written as

$$\underline{E}_t(\theta,\phi,t) = \sum_{\ell,m} e'_{\ell m}(t)\; \underline{K}'_{\ell m}(\theta,\phi) + \sum_{\ell,m} e''_{\ell m}(t)\; \underline{K}''_{\ell m}(\theta,\phi) \tag{39}$$

for all t. To determine $e'_{\ell m}(t)$ and $e''_{\ell m}(t)$ we note that

$$\underline{E}_t(\theta,\phi) = 0, \quad \text{for } t > 0 \tag{40}$$

and hence $e'_{\ell m}(t) = e''_{\ell m}(t) = 0$, for $t > 0$. Moreover, we have

$$\underline{E}_t(\theta,\phi,t) = \underline{E}_o(\theta,\phi), \quad \text{for } t < 0 \tag{41}$$

where

$$\varepsilon\, \underline{E}_o(\theta,\phi) = -\nabla_s \int G(\theta,\phi;\theta',\phi')\; \sigma_o(\theta',\phi')dS$$

and G is the static Green's function which, at $r = a$, is given by

$$G(\theta,\phi;\theta',\phi') = \frac{1}{a} \sum_{\ell,m} \frac{1}{2\ell+1}\, \overline{Y}_{\ell m}(\theta,\phi)\; \overline{Y}^*_{\ell m}(\theta',\phi')$$

so that

$$\int G\, \sigma_o\, dS = a \sum_{\ell,m} \frac{\sigma_{\ell m}}{2\ell+1}\, \overline{Y}_{\ell m}(\theta,\phi)$$

and

$$\underline{E}_o(\theta,\phi) = \sum_{\ell,m} e_{\ell m}\, \underline{K}'_{\ell m}(\theta,\phi) \tag{42}$$

where

$$e_{\ell m} = -\frac{\sqrt{\ell(\ell+1)}}{2\ell+1}\, \frac{\sigma_{\ell m}}{\epsilon}$$

By performing an inverse Fourier transform we can determine v and hence all field components for all times and everywhere.

Let us calculate E_r and determine its discontinuity across $r = a$. From (45) we have

$$\tilde{E}_r = \begin{cases} \sum_{\ell,m} \tilde{A}_{\ell m} \frac{\ell(\ell+1)}{r} h_\ell(kr)\, \overline{Y}_{\ell m}(\theta,\phi), & r > a \\ \sum_{\ell,m} \tilde{B}_{\ell m} \frac{\ell(\ell+1)}{r} j_\ell(kr)\, \overline{Y}_{\ell m}(\theta,\phi), & r < a \end{cases} \tag{47}$$

Hence,

$$\tilde{E}_r(a+,\omega) - \tilde{E}_r(a-,\omega) = \frac{1}{\omega\varepsilon} \sum_{\ell,m} \frac{\ell(\ell+1)}{(2\ell+1)ka} \frac{1}{[ka\, h_\ell(ka)]'} \frac{1}{[ka\, j_\ell(ka)]'} \sigma_{\ell m} \overline{Y}_{\ell m}(\theta,\phi)$$

where $a+(a-)$ denotes the exterior (interior) side of the spherical surface S. Performing an inverse Fourier transform we get

$$\varepsilon[E_r(a+,t) - E_r(a-,t)]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{k} \sum_{\ell,m} \left\{ \frac{\ell(\ell+1)}{(2\ell+1)ka} \frac{1}{[ka\, h_\ell(ka)]'} \frac{1}{[ka\, j_\ell(ka)]'} \right\} \sigma_{\ell m} \overline{Y}_{\ell m}(\theta,\phi)\, e^{-i\omega t} dk \tag{48}$$

The path of integration is along the real axis with downward indentation at $\omega = 0$ and upward indentation at all other poles on the real axis.

From the small-argument expansions of the spherical Bessel functions one can see that

$$\lim_{ka \to 0} \left\{ \frac{\ell(\ell+1)}{(2\ell+1)ka} \frac{1}{[ka\, h_\ell(ka)]'} \frac{1}{[ka\, j_\ell(ka)]'} \right\} = 1$$

so that by Cauchy's integral theorem equation (48) gives, with the integration

This expression can be viewed as the charge density on the interior side of S. We note that expression (51) contains only pure sinusoidal oscillations for $t > 0$, since all the zeros of $[ka\ j_\ell(ka)]'$ are real.

We now continue with the calculation of the currents on S. On the exterior part of S, the surface current density $\underline{K}(a+,t)$ is

$$\underline{K}(a+,t) = \hat{r}\times\underline{H}(a+,t) = Z^{-1}\ c^{-1} \sum_{\ell,m} \sqrt{\ell(\ell+1)}\ \frac{dA_{\ell m}}{dt}(t)\,\underline{K}'_{\ell m}(\theta,\phi)$$

and from (42) and (46) we have

$$\tilde{\underline{K}}(a+,\omega) = a \sum_{\ell,m} \frac{\sqrt{\ell(\ell+1)}}{2\ell+1}\ \frac{1}{[ka\ h_\ell(ka)]'}\ \sigma_{\ell m}\ \underline{K}'_{\ell m}(\theta,\phi) \tag{52}$$

An inverse Fourier transform gives

$$\underline{K}(a+,t) = \begin{cases} 0, & t < 0 \\ ac \displaystyle\sum_{\substack{\ell,m,n \\ \text{ext}}} \frac{\sqrt{\ell(\ell+1)}}{2\ell+1}\ \underset{k_{\ell n}}{\text{Res}} \left\{ \frac{1}{[ika\ h_\ell(ka)]'} \right\} \sigma_{\ell m}\ \underline{K}'_{\ell m}(\theta,\phi)\ e^{-i\omega_{\ell m} t}, & t > 0 \end{cases}$$

and similarly, on the interior side of S we have

$$\underline{K}(a-,t) = \begin{cases} 0, & t < 0 \\ -ac \displaystyle\sum_{\substack{\ell,m,n \\ \text{int}}} \frac{\sqrt{\ell(\ell+1)}}{2\ell+1}\ \underset{k_{\ell n}}{\text{Res}} \left\{ \frac{1}{[ika\ j_\ell(ka)]'} \right\} \sigma_{\ell m}\ \underline{K}'_{\ell m}(\theta,\phi)\ e^{-i\omega_{\ell m} t}, & t > 0 \end{cases} \tag{53}$$

VI. Conclusions and Suggestions

The present note treats certain aspects of the general problem of calculating the external system-generated EMP due to photoelectrons emitted from the surface of a conductor. Specifically, the problem posed and solved in this note is that of calculating the induced surface currents and charges, and hence the electromagnetic field, on a perfect conductor with prescribed trajectories of the photoelectrons. It is found that under certain general conditions on the electron's speed and distance from the conductor, the solution of a quasi-electrostatic problem and the solution of a quasi-magnetostatic problem are sufficiently accurate. Validity criteria on the quasi-static solutions are established by solving a canonical problem rigorously as well as quasi-statically. The canonical problem involves an electron orbiting a perfectly conducting sphere. The rigorous solutions for the induced surface currents and charges on the sphere are compared numerically with the quasi-static solutions, thus establishing validity criteria in terms of the electron's kinetic energy and distance from the sphere. The numerical comparison shows that the accuracy of the quasi-static solutions is more sensitive to the change in the electron's distance from the conductor than the electron's speed. When the electron is more than two radii away from the sphere, the quasi-static solutions are extremely accurate regardless of the electron's speed.

The quasi-static solutions can be generalized to a charged particle with arbitrary motion in the presence of a sphere. Extensive numerical results are presented graphically for the induced currents and charges on the sphere for radial as well as orbital motion of the charged particle. By superposition, results for arbitrary motion can then be obtained from these two special motions.

A discussion based on the integral-equation approach is given for solving the two quasi-static problems involving arbitrary-shaped conductors. The resulting integral equations can be easily solved on an electronic computer for a conductor of any given shape.

Two initial-boundary-value problems are discussed within the realm of classical electrodynamics. These two problems are related to the question as to how the positive charges redistribute themselves on the surface of the sphere after the passage of an incident plane-wave photon pulse, with the assumption that the photoelectrons outside the sphere have negligible effect on the redistribution process of the positive charges. It is found that in some cases a specification of the initital charge distribution together with the total net charge is sufficient for determining the subsequent charge distribution on a sphere, while in other cases such a specification does not constitute a sufficent set of initial data.

So far we have been recapitulating the results of the present study. Let us now discuss how this work is related to the general problem where a plane-wave photon pulse strikes a sphere and causes electrons to emit from the sphere. To be sure, if the the photoelectron charge density ρ and current density $\underline{J}$ are known outside the sphere, then one can use some of the results in this work as the Green's function of the problem to compute the field. From equations (21) and (24) one can immediately write down

$$\sigma(\theta,\phi,t) = -\frac{Q(t)}{4\pi a^2} - \sum_{\ell,m}{}' \left\{ \int \rho(\underline{r}',t)\left(\frac{a}{r'}\right)^{\ell-1} \overline{Y}^*_{\ell m}(\theta',\phi')\, d\Omega' dr' \right\} \overline{Y}_{\ell m}(\theta,\phi)$$

$$\underline{K}'(\theta,\phi,t) = -\sum_{\ell,m}{}' \left\{ \int \left[\left(\frac{a}{r'}\right)^{\ell} \underline{J}(\underline{r}',t)\cdot\underline{K}'^*_{\ell m}(\theta',\phi') + \sqrt{\frac{\ell+1}{\ell}} \left(\frac{a}{r'}\right)^{\ell} J_r(\underline{r}',t)\, \overline{Y}^*_{\ell m}(\theta',\phi') \right] d\Omega' dr' \right\} \underline{K}'_{\ell m}(\theta,\phi) \tag{54}$$

$$\underline{K}''(\theta,\phi,t) = -\sum_{\ell,m}{}' \left\{ \int \left(\frac{a}{r'}\right)^{\ell-1} \underline{J}(\underline{r}',t)\cdot\underline{K}''^*_{\ell m}(\theta',\phi')\, d\Omega' dr' \right\} \underline{K}''_{\ell m}(\theta,\phi)$$

where $d\Omega' = \sin\theta' d\theta' d\phi'$, the $'$ on the summation sign denotes omission of the $\ell = m = 0$ term, and the total charge Q is

$$Q(t) = \int \rho(\underline{r}',t)\, r'^2 dr' d\Omega'$$

The domain of integration in the above integrals is, of course, the region outside the sphere. Expanding ρ and $\underline{J}$ one has

$$\rho = \sum_{\ell,m}{}' \; a_{\ell m}(r,t)\; \overline{Y}_{\ell m}(\theta,\phi)$$

$$\underline{J} = \sum_{\ell,m}{}' \left[b_{\ell m}(r,t)\; \underline{K}'_{\ell m}(\theta,\phi) + c_{\ell m}(r,t)\; \underline{K}''_{\ell m}(\theta,\phi) + \hat{r}\; d_{\ell m}(r,t)\; \overline{Y}_{\ell m}(\theta,\phi) \right] \tag{55}$$

The expansion coefficients $a_{\ell m}$, $b_{\ell m}$, $c_{\ell m}$, and $d_{\ell m}$ are of course determined uniquely in terms of ρ and $\underline{J}$ due to the orthonormal conditions (15). Substitution of (55) in (54) gives the following rather simple expressions:

$$\sigma(\theta,\phi,t) = -\frac{Q(t)}{4\pi a^2} - \sum_{\ell,m}{}' \left\{ \int_a^\infty \left(\frac{a}{r'}\right)^{\ell-1} a_{\ell m}(r',t)dr' \right\} \overline{Y}_{\ell m}(\theta,\phi)$$

$$\underline{K}'(\theta,\phi,t) = -\sum_{\ell,m}{}' \left\{ \int_a^\infty \left[b_{\ell m}(r',t) + \sqrt{\frac{\ell+1}{\ell}}\; d_{\ell m}(r',t)\right]\left(\frac{a}{r'}\right)^{\ell} dr' \right\} \underline{K}'_{\ell m}(\theta,\phi) \tag{56}$$

$$\underline{K}''(\theta,\phi,t) = -\sum_{\ell,m}{}' \left\{ \int_a^\infty \left(\frac{a}{r'}\right)^{\ell-1} c_{\ell m}(r',t)dr' \right\} \underline{K}''_{\ell m}(\theta,\phi)$$

If the quasi-static approximation is not made, the coupling coefficients inside the curly brackets of (56) become quite complicated. To illustrate this point let us write down the rigorous expression for σ only:

$$\sigma(\theta,\phi,t) = -\frac{Q(t)}{4\pi a^2} + \sum_{\ell,m}{}' \left\{ \sqrt{\ell(\ell+1)}\; \frac{1}{2\pi} \int_C \frac{e^{-i\omega t} d\omega}{\left\{ i\omega a^2 [ka\; h_\ell(ka)]' \right\}} \right.$$

$$\left. \int_a^\infty \left\{ [kr\; h_\ell(kr)]'\; \tilde{b}_{\ell m}(r,\omega) - \sqrt{\ell(\ell+1)}\; h_\ell(kr)\; \tilde{d}_{\ell m}(r,\omega) \right\} r dr \right\} \overline{Y}_{\ell m}(\theta,\phi) \tag{57}$$

where $\tilde{b}_{\ell m}(\omega)$ and $\tilde{d}_{\ell m}(\omega)$ are, respectively, the Fourier transforms of $b_{\ell m}(t)$ and $d_{\ell m}(t)$, and the contour C for the Fourier integral in (57) is chosen to lie above all singularities of the integrand in the entire ω-plane. We leave the derivations of $\underline{K}'$ and $\underline{K}''$ to the interested reader and simply state that the dyadic Green's function discussed in Appendix A would be very useful in the derivations. With considerable efforts one can show that equation (57) indeed reduces to the first equation of (56) under the quasi-static (or low-frequency) approximation, and again we leave the proof to the interested reader. Let us point out in passing that equation (57) is in a form consistent with the theory of Singularity Expansion Method applied to a sphere, as has been expounded in [8].

Let us return to equations (56) and note that the coupling coefficient for $\underline{K}'$ can be expressed in terms of $(\partial/\partial t)\, a_{\ell m}(r,t)$ alone because of the charge continuity equation. In the case where there is rotational symmetry (i.e., no ϕ variations) in the problem under consideration, equations (56) reduce to the following extremely simple form:

$$\sigma(\theta,t) = -\frac{Q(t)}{4\pi a^2} - \sum_{\ell=1}^{\infty} \left\{ \int_a^{\infty} \left(\frac{a}{r}\right)^{\ell-1} a_\ell(r,t)\,dr \right\} \overline{P}_\ell(\cos\theta)$$

$$\underline{K}'(\theta,t) = -\hat{\theta} \sum_{\ell=1}^{\infty} \left\{ \frac{a}{\ell(\ell+1)} \int_a^{\infty} \left(\frac{a}{r}\right)^{\ell-1} \dot{a}_\ell(r,t)\,dr \right\} \frac{d\overline{P}_\ell}{d\theta} \tag{58}$$

$$\underline{K}''(\theta,t) \equiv 0$$

where $a_\ell(r,t)$ is the expansion coefficient for $\rho(r,\theta,t)$, $\dot{a}_\ell = (\partial/\partial t)a_\ell$, and $\overline{P}_\ell$ is the normalized Legendre polynomial defined in (14). It is interesting to note that a knowledge of a_ℓ alone completely determines the induced surface charges and currents on the sphere. In general, a_ℓ is unknown. What is known in the problem of practical interest is the photoelectric emission current at the "surface" of the sphere for a given incident photon pulse. This emission current, which can be taken to be ϕ-independent, describes the number of emitted electrons per unit area, per unit time, per unit electron exit energy, and per unit exit angle with respect to the normal at the surface [4,5]. Given the emission current as the only input datum the determination of a_ℓ will involve a self-consistent approach to the solution of Maxwell's equations and the equations of motion. Although such an approach is not too straightforward, the importance of the problem warrants an immediate investigation along this line. Also, one may think of undertaking a numerical study of $\underline{\sigma}$ and $\underline{K}'$ based on (58) by judiciously choosing different appropriate forms for a_ℓ. Another undertaking is the determination of the complete trajectory of each emitted photoelectron under the assumption that the only significant force each electron experiences is the Coulomb force between the electron and the positive charges on the sphere.

Another area for future study directly related to the present note includes the investigation of the effects of various appendages attached to a sphere on our present calculations for the induced surface currents and charges and the investigation of extending the techniques in this note to conductors of other shapes. For very slender structures quasi-static calculations may not be accurate enough and other workable approaches should be explored before one embarks upon using the full-fledged Maxwell's equations.

Appendix A

On the Solenoidal and Irrotational Parts of the Dyadic Green's Function of a Sphere

In finding the time-harmonic electromagnetic field of a distributed current source density $\underline{J}(\underline{r},\omega)$ it is customary to first seek the dyadic Green's function $\underline{\underline{G}}$ and then to construct the field by integration. We are interested in finding a $\underline{\underline{G}}$ which satisfies the boundary condition

$$\underline{n}\times\underline{\underline{G}} = 0 \tag{A1}$$

on the surface of a sphere of radius a, the differential equation

$$\nabla\times\nabla\times\underline{\underline{G}} - k^2\underline{\underline{G}} = \underline{\underline{I}}\,\delta(\underline{r}-\underline{r}'), \quad r \geq a, \quad r' > a \tag{A2}$$

and the radiation condition at infinity.

Naturally, one splits $\underline{\underline{G}} = \underline{\underline{G}}^o + \underline{\underline{G}}^s$, $\underline{\underline{G}}^o$ being the free-space part and $\underline{\underline{G}}^s$ the scattered part. Obviously, most of our effort in solving the problem just posed lies in finding a representation of $\underline{\underline{G}}^o$ which will be appropriate for the spherical coordinate system and possesses a correct singularity at $\underline{r} = \underline{r}'$.

It is well known that [9], with $G = \exp(ik|\underline{r}-\underline{r}'|)/(4\pi|\underline{r}-\underline{r}'|)$,

$$\underline{\underline{G}}^o(\underline{r},\underline{r}') = (\underline{\underline{I}} - \frac{1}{k^2}\nabla\nabla')\,G(\underline{r},\underline{r}') \tag{A3}$$

satisfies (A2) and the radiation condition at infinity. Making use of the result [12,13]

$$\underline{\underline{I}}\,G(\underline{r},\underline{r}') = ik \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} [\underline{M}^1_{\ell,m}(\underline{r}_<)\,\underline{M}^3_{\ell,-m}(\underline{r}_>) + \underline{N}^1_{\ell m}(\underline{r}_<)\,\underline{N}^3_{\ell,-m}(\underline{r}_>)$$

$$+ \underline{L}^1_{\ell m}(\underline{r}_<)\,\underline{L}^3_{\ell,-m}(\underline{r}_>)] + \underline{L}^1_{oo}(\underline{r}_<)\,\underline{L}^3_{oo}(\underline{r}_>) \tag{A4}$$

and

$$\frac{1}{k^2} \nabla\nabla' \, G(\underline{r},\underline{r}') = ik \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \underline{L}^1_{\ell m}(\underline{r}_<) \, \underline{L}^3_{\ell,-m}(\underline{r}_>) \tag{A5}$$

one obtains from (A3)

$$\underline{\underline{G}}^o(\underline{r},\underline{r}) = ik \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} [\underline{M}^1_{\ell m}(\underline{r}_<) \, \underline{M}^3_{\ell,-m}(\underline{r}_>) + \underline{N}^1_{\ell m}(\underline{r}_<) \, \underline{N}^3_{\ell,-m}(\underline{r}_>)] \tag{A6}$$

Here, $\underline{r}_> = r_> \hat{r}$ ($\underline{r}_< = r_< \hat{r}$) and $r_>(r_<)$ denotes the larger (smaller) of r,r'. The vector spherical wave functions in (A4)-(A6) are defined as [8,18]

$$\underline{L}_{\ell m}(\underline{r}) = \frac{1}{k} \nabla[z_\ell(kr) \, \overline{Y}_{\ell m}(\theta,\phi)] = \hat{r} \, z'_\ell(kr) \, \overline{Y}_{\ell m}(\theta,\phi) + \sqrt{\ell(\ell+1)} \, \frac{z_\ell(kr)}{kr} \, \underline{K}'_{\ell m}(\theta,\phi)$$

$$\underline{M}_{\ell m}(\underline{r}) = \frac{1}{\sqrt{\ell(\ell+1)}} \nabla\times[\underline{r} \, z_\ell(kr) \, \overline{Y}_{\ell m}(\theta,\phi)] = -z_\ell(kr) \, \underline{K}''_{\ell m}(\theta,\phi) \tag{A7}$$

$$\underline{N}_{\ell m}(\underline{r}) = \frac{1}{k} \nabla\times\underline{M}_{\ell m}(\underline{r}) = \hat{r} \sqrt{\ell(\ell+1)} \, \frac{z_\ell(kr)}{kr} \, \overline{Y}_{\ell m}(\theta,\phi) + \frac{1}{kr} [kr \, z_\ell(kr)]' \, \underline{K}'_{\ell m}(\theta,\phi)$$

where z_ℓ is equal to j_ℓ or h_ℓ according as the superscript is 1 or 3. The representation (A6) of $\underline{\underline{G}}^o$ in terms of $\underline{M}$ and $\underline{N}$ has also been given by Jones [14]. His method of obtaining $\underline{\underline{G}}^o$ is conventional in that he finds the electromagnetic field of an electric dipole by matching the field quantities across the spherical surface containing the point dipole. It is interesting to note that the $\underline{L}$ function does not enter into the representation (A6). From the series representation (A6) one can show that $\nabla\cdot\underline{\underline{G}}^o = -k^2 \nabla\delta(\underline{r}-\underline{r}')$ which is in agreement with the differential equation (A2).

To find $\underline{\underline{G}}^s$ one simply invokes the boundary condition (A1) to obtain, in view of (A6),

$$\underline{\underline{G}}^s(\underline{r},\underline{r}') = -ik \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left[\frac{j_\ell(ka)}{h_\ell(ka)} \underline{M}^3_{\ell m}(\underline{r}) \underline{M}^3_{\ell,-m}(\underline{r}') \right.$$

$$\left. \frac{[ka\, j_\ell(ka)]'}{[ka\, h_\ell(ka)]'} \underline{N}^3_{\ell m}(\underline{r}) \underline{N}^3_{\ell,-m}(\underline{r}') \right] . \tag{A8}$$

An observation of the coefficients in the series expansion for $\underline{\underline{G}}^s$ reveals that $\underline{M}$ is associated with fields of TE-type and $\underline{N}$ with fields of TM-type.

We have just succeeded in obtaining a representation of the dyadic Green's function $\underline{\underline{G}}$ for a sphere. We now go on to investigate how to use $\underline{\underline{G}}$ to get the field of a distributed $\underline{J}$ outside the sphere. Since $\underline{\underline{G}}$ behaves as $|\underline{r} - \underline{r}'|^{-3}$ when $\underline{r}' \to \underline{r}$, the integral

$$\int \underline{\underline{G}}(\underline{r},\underline{r}')\cdot\underline{J}(\underline{r}')dV'$$

has to be interpreted in a principal-value sense. To this end we start from first principles and apply Green's theorem to

$$\nabla\times\nabla\times\underline{E} - k^2\underline{E} = i\omega\mu\underline{J}$$

$$\nabla\times\nabla\times\underline{\underline{G}} - k^2\underline{\underline{G}} = 0, \quad \underline{r} \neq \underline{r}' \tag{A9}$$

and obtain

$$\int_{S+S_\varepsilon+S_\infty} [-\underline{\underline{G}}\cdot(\underline{n}'\times\nabla'\times\underline{E}) + (\nabla'\times\underline{\underline{G}})\cdot(\underline{E}\times\underline{n}')]dS' = i\omega\mu \int_V \underline{\underline{G}}\cdot\underline{J}dV \tag{A10}$$

where the volume V is bounded by the spherical surfaces $S + S_\varepsilon + S_\infty$ (see Figure A1). Using the radiation condition at infinity on S_∞ and the boundary conditions on S one can discard the integrals over S_∞ and S. The integral over S_ε in the limit as $\varepsilon \to 0$ gives [15]

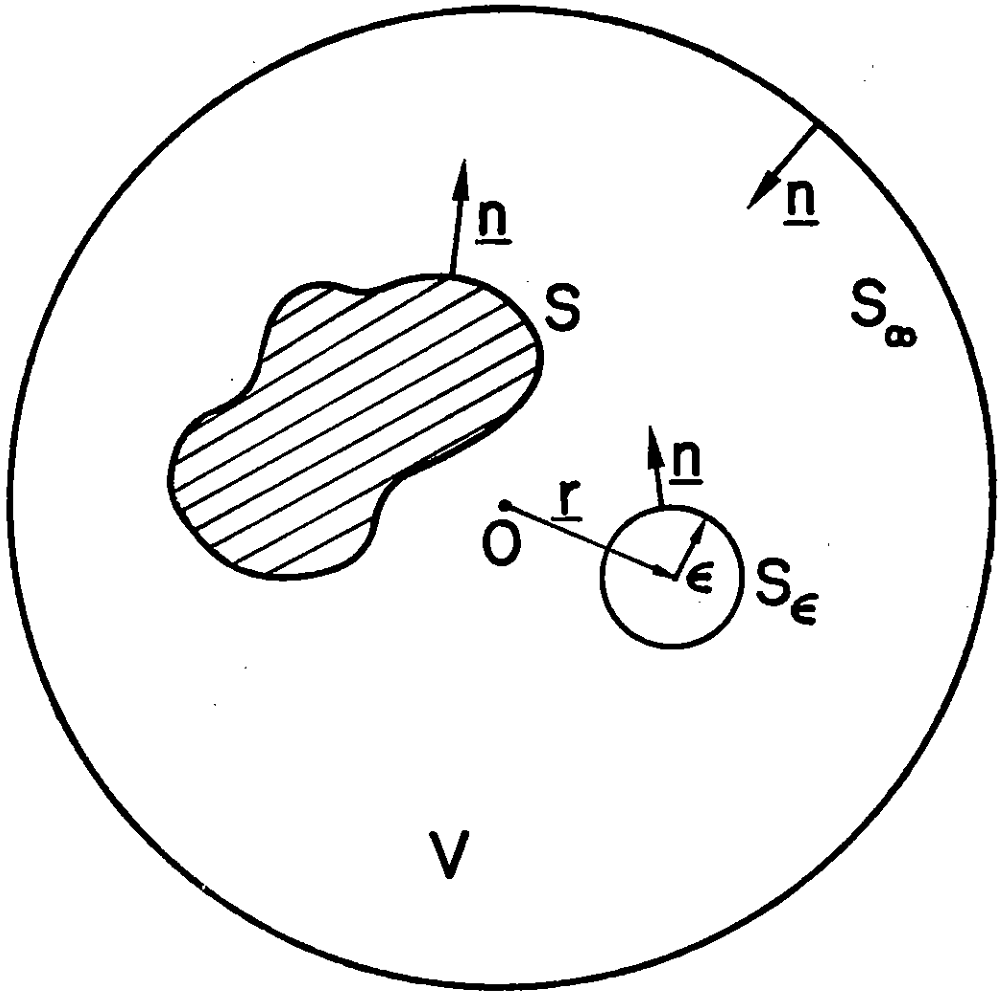

Figure A1. Domains of integration for Green's theorem.

$$\lim_{\varepsilon\to 0} \int_{S_\varepsilon} - \underline{\underline{G}}\cdot(\underline{n}'\times\nabla'\times\underline{E})dS' = \frac{2}{3}\,\underline{E}(\underline{r})$$

$$\lim_{\varepsilon\to 0} \int_{S_\varepsilon} (\nabla'\times\underline{\underline{G}})\cdot(\underline{E}\times\underline{n}')dS' = \frac{1}{3k^2}\,\nabla\times\nabla\times\underline{E} = \frac{1}{3}\,\underline{E}(\underline{r}) - \frac{1}{3i\omega\varepsilon}\,\underline{J}(\underline{r})$$

Inserting these limiting values in (A10) one obtains

$$\underline{E}(\underline{r}) = i\omega\mu \;⨍\; \underline{\underline{G}}(\underline{r},\underline{r}')\cdot\underline{J}(\underline{r}')dV' + \frac{\underline{J}}{3i\omega\varepsilon} \tag{A11}$$

The principal-value integral is a well defined function of $\underline{r}$ when $\underline{J}$ is a continuous function.

Let us now verify that (A11) implies $\nabla\cdot\underline{E} = \rho/\varepsilon$, ρ being the volume charge density. Taking the divergence of (A11) and noting that the part due to the scattered dyadic Green's function has zero divergence one has

$$\begin{aligned}\nabla\cdot\underline{E} &= i\omega\mu\nabla\cdot ⨍\; \underline{\underline{G}}^{o}\cdot\underline{J}dV' + \rho/3\varepsilon \\ &= i\omega\mu\nabla\cdot ⨍\; [\,\underline{\underline{I}} - \frac{1}{k^2}\,\nabla\nabla' G]\cdot\underline{J}dV' + \rho/3\varepsilon\end{aligned} \tag{A12}$$

Using [16]

$$\begin{aligned}⨍\; [(k^2\underline{\underline{I}}-\nabla\nabla')G]\cdot\underline{J}dV' &= -⨍\; (\underline{\underline{I}}\nabla^2 G-\nabla\nabla G)\cdot\underline{J}dV' \\ &= ⨍\; \nabla\times\nabla\times(G\underline{J})dV' = \nabla\times\nabla\times ⨍\; G\underline{J}dV' - \frac{2}{3}\,\underline{J}\end{aligned} \tag{A13}$$

one can immediately see from (A12) and (A13) that $\nabla\cdot\underline{E} = \rho/\varepsilon$. The principal-value integral in the last expression of (A13) can, of course, be interpreted as an ordinary integral.

We now proceed to the splitting of $\underline{\underline{G}}$ into its solenoidal (longitudinal) part $\underline{\underline{G}}_{\ell}$ and its irrotational (transverse) part $\underline{\underline{G}}_{t}$. As we will see in the following, this splitting is advantageous because $\underline{\underline{G}}_{\ell}$ and $\underline{\underline{G}}_{t}$ have distinct features from both physical and mathematical viewpoints.

The governing equations for $\underline{\underline{G}}_\ell$ and $\underline{\underline{G}}_t$ are given by

$$\underline{\underline{G}} = \underline{\underline{G}}_\ell + \underline{\underline{G}}_t$$

$$\nabla\times\underline{\underline{G}}_\ell = 0, \quad \text{outside S} \tag{A14a}$$

$$\nabla\cdot\underline{\underline{G}}_\ell = \nabla\cdot\underline{\underline{G}} = \frac{-1}{k^2}\nabla\delta(\underline{r}-\underline{r}'), \quad \text{outside S} \tag{A14b}$$

$$\underline{n}\times\underline{\underline{G}}_\ell = 0, \quad \text{on S} \tag{A14c}$$

$$\nabla\times\underline{\underline{G}}_t = \nabla\times\underline{\underline{G}}, \quad \text{outside S} \tag{A15a}$$

$$\nabla\cdot\underline{\underline{G}}_t = 0, \quad \text{outside S} \tag{A15b}$$

$$\underline{n}\times\underline{\underline{G}}_t = 0, \quad \text{on S} \tag{A15c}$$

together with appropriate conditions at infinity. Equation (A14b) follows from the differential equation (A2) for $\underline{\underline{G}}$.

To solve (A14) for $\underline{\underline{G}}_\ell$ we set, with $G_s = (4\pi|\underline{r}-\underline{r}'|)^{-1}$,

$$\underline{\underline{G}}_\ell = -\frac{1}{k^2}\nabla\nabla' G_s \tag{A16}$$

which is in compliance with (A14a) and the reciprocity condition with respect to $\underline{r}$ and $\underline{r}'$. Then, one can easily see from (A14b,c) and (A16) that

$$\nabla^2 G_s = -\delta(\underline{r}-\underline{r}'), \quad \text{outside S}$$

$$G_s = 0, \quad \text{on S} \tag{A17}$$

the solution of which is

$$G_s(\underline{r},\underline{r}') = \frac{1}{4\pi|\underline{r}-\underline{r}'|} - \frac{1}{4\pi|\underline{r}-\underline{r}''|} \tag{A18}$$

where

$$\underline{r}'' = \frac{a^2}{r'^2}\,\underline{r}'.$$

It should be noted that the boundary condition G_s = constant on S also satisfies (A14c), and the value of this constant is dictated through $\underline{E}$ by the total net charge on the sphere.

We have just shown how $\underline{\underline{G}}_\ell$ can be constructed from the solution of Poisson's equation (A17). To find $\underline{\underline{G}}_t$, we simply subtract $\underline{\underline{G}}_\ell$ from $\underline{\underline{G}}$ which has been given in (A6) and (A8). There are two points worth mentioning regarding $\underline{\underline{G}}_t$: (1) $\underline{\underline{G}}_t$ is indeed divergenceless, since from (A3) and (A14b)

$$\nabla\cdot\underline{\underline{G}}_t = \nabla\cdot\underline{\underline{G}} - \nabla\cdot\underline{\underline{G}}_\ell = \nabla G - \frac{1}{k^2}\,(k^2\nabla G + \nabla\delta) + \frac{1}{k^2}\,\nabla\delta = 0,$$

and (2) as $\underline{r} \to \underline{r}'$ $|\underline{\underline{G}}_t| \sim |\underline{r}-\underline{r}'|^{-1}$ whereas $|\underline{\underline{G}}_\ell| \sim |\underline{r}-\underline{r}'|^{-3}$; therefore, the highest singular term in $\underline{\underline{G}}$ has been separated out and given to $\underline{\underline{G}}_\ell$ which can easily be handled by solving an electrostatic problem.

In concluding, we wish to point out that although an explicit series representation of the time-harmonic dyadic Green's function has been obtained for a perfectly conducting sphere, it is difficult to use it to derive quantitatively useful information on the time-dependent problem discussed in the text in which the moving electrons emitted from the surface are the source functions of the fields.

Appendix B

Magnetostatic Problem of a Sphere - An Integral Equation Approach

We will solve the magnetostatic problem posed in Section III using an integral equation approach. The reason for doing this is twofold: (1) the decomposition of the current-source term into solenoidal and irrotational parts becomes unnecessary in this approach; and (2) this approach brings into play the general method of solving an integral equation from the solution of the associated eigenvalue problem.

We begin with the integral equation

$$\frac{1}{2}\underline{K} - \int_S \underline{n}\times(\nabla G\times\underline{K})\,dS' = \underline{K}^{inc} \tag{B1}$$

Expanding $\underline{K}$ in terms of $\{\underline{K}'_{\ell m}\}$ and $\{\underline{K}''_{\ell m}\}$ and substituting

$$\underline{K} = \sum_{\ell,m} (a'_{\ell m}\,\underline{K}'_{\ell m} + a''_{\ell m}\,\underline{K}''_{\ell m}) \tag{B2}$$

into (B1) we get

$$\sum_{\ell,m} (\lambda'_\ell\, a'_{\ell m}\,\underline{K}'_{\ell m} + \lambda''_\ell\, a''_{\ell m}\,\underline{K}''_{\ell m}) = \underline{K}^{inc}$$

where

$$\lambda'_\ell = \frac{\ell}{2\ell+1}\ , \qquad \lambda'' = \frac{\ell+1}{2\ell+1}$$

In the following we only show how to evaluate $a''_{\ell m}$ explicitly, since the evaluation of $a'_{\ell m}$ follows identically the same line. We have

$$a''_{\ell m} = \frac{2\ell+1}{\ell+1}\int \underline{K}^{inc}\cdot\underline{K}''^{*}_{\ell m}\sin\theta\,d\theta\,d\phi. \tag{B3}$$

To evaluate the integral in (B3) we note that

$$\underline{K}^{inc}\cdot\underline{K}''^{*}_{\ell m} = \underline{v}\cdot(\underline{K}'^{*}_{\ell m}\times\nabla G) = -\underline{v}\cdot\underline{K}''^{*}_{\ell m}\frac{\partial G}{\partial r} + \nabla_s\cdot(G\,\underline{v}\times\underline{K}'^{*}_{\ell m}) + G\,\underline{v}\cdot\nabla_s\times\underline{K}'^{*}_{\ell m} \tag{B4}$$

where $G(\underline{r},\underline{r}') = (4\pi|\underline{r}-\underline{r}'|)^{-1}$. Formulas in Weatherburn [17] give with $a\nabla_s = \nabla_\Omega$

$$\nabla_\Omega\times\underline{K}'_{\ell m} = \underline{K}''_{\ell m}$$

$$\nabla_\Omega\cdot\underline{K}''_{\ell m} = 0 \tag{B5}$$

and

$$\int \nabla_s\cdot(G\,\underline{v}\times\underline{K}'^{*}_{\ell m})\sin\theta\,d\theta\,d\phi = -2\int G\,\underline{v}\cdot\underline{K}''^{*}_{\ell m}\sin\theta\,d\theta\,d\phi. \tag{B6}$$

Equations (B4) and (B5) enable us to get the following expression for the integral in (B3):

$$\int \underline{K}^{inc}\cdot\underline{K}''^{*}_{\ell m}\sin\theta\,d\theta\,d\phi = -\underline{v}\cdot\int\frac{\partial(rG)}{\partial r}\underline{K}''^{*}_{\ell m}\sin\theta\,d\theta\,d\phi$$

$$= -\sum_{\ell',m'}\frac{\ell'+1}{2\ell'+1}\frac{r^{\ell'-1}}{r_o^{\ell'+1}}\overline{Y}^{*}_{\ell'm'}(\theta_o,\phi_o)\int\underline{v}\cdot\underline{K}''^{*}_{\ell m}(\theta,\phi)\,Y_{\ell'm'}(\theta,\phi)\sin\theta\,d\theta\,d\phi \tag{B7}$$

We now evaluate the integral in (B7) for three different current orientations.

(i) <u>Radial Orientation</u>

Without loss of generality we choose $\underline{v} = v\hat{z}$ at $\theta_o = 0$. For this special case we get, for (B7),

$$\overline{Y}_{\ell' m'}(0,\phi_o) \int \underline{v}\cdot\underline{K}''^{*}_{\ell m}(\theta,\phi)\ \overline{Y}_{\ell' m'}(\theta,\phi)\ \sin\theta\ d\theta\ d\phi$$

$$= \frac{-imv}{\sqrt{\ell(\ell+1)}}\ \overline{Y}_{\ell' m'}(0,\phi_o) \int \overline{Y}^{*}_{\ell m}(\theta,\phi)\ \overline{Y}_{\ell' m'}(\theta,\phi)\ \sin\theta\ d\theta\ d\phi = 0, \qquad \text{(B8)}$$

since $\overline{Y}_{\ell m}(0,\phi_o) = 0$ for $1 \le m \le \ell$. Thus, a radially oriented current element will not give rise to a solenoidal surface current density on the sphere.

(ii) ϕ-Orientation

We choose $\underline{v} = v\hat{y}$, $\theta_o = \frac{\pi}{2}$, $\phi_o = 0$ and get, for (B7),

$$\overline{Y}^{*}_{\ell' m'}(\tfrac{\pi}{2},0) \int \underline{v}\cdot\underline{K}''^{*}_{\ell m}(\theta,\phi)\ \overline{Y}_{\ell' m'}(\theta,\phi)\ \sin\theta\ d\theta\ d\phi$$

$$= \frac{v}{\sqrt{\ell(\ell+1)}}\ \overline{Y}^{*}_{\ell' m'}(\tfrac{\pi}{2},0)$$

$$\int \left(\sin\theta\ \cos\phi\ \frac{\partial \overline{Y}^{*}_{\ell m}}{\partial\theta} - \cos\theta\ \sin\phi\ \frac{\partial \overline{Y}^{*}_{\ell m}}{\partial\phi}\right) \overline{Y}_{\ell' m'}(\theta,\phi)\ \sin\theta\ d\theta\ d\phi$$

$$= \frac{v}{2}\sqrt{\frac{(\ell-m')!(\ell+m)!}{(\ell+m')!(\ell-m)!}}\left[\delta_{m-1,m'} - (\ell+m+1)(\ell-m)\ \delta_{m+1,m'}\right]\frac{\delta_{\ell\ell'}}{\sqrt{\ell(\ell+1)}}\ \overline{Y}^{*}_{\ell' m'}(\tfrac{\pi}{2},0)$$

$$= \frac{v}{\sqrt{\ell(\ell+1)}}\ \frac{\partial \overline{Y}_{\ell' m'}}{\partial\theta}(\tfrac{\pi}{2},0)\ \delta_{\ell\ell'}\delta_{mm'} = \underline{v}\cdot\underline{K}''^{*}_{\ell' m'}(\tfrac{\pi}{2},0)\ \delta_{\ell\ell'}\delta_{mm'} \qquad \text{(B9)}$$

Using this result in (B2), (B3) and (B7) one then obtains an expression for $\underline{K}$ identical to the one in (17c).

(iii) θ-Orientation

From symmetry considerations one can obtain the solution for this case directly from (ii) by a rotation of the coordinate axes.

Combining the results in (i), (ii) and (iii) one arrives at the result (24) in Section III.

Symbols and Notations

In the following we summarize the different notations and symbols used in this note and those used in IN88 [8]. These differences are only formal and are due to the fact that (1) Laplace transform notations are used in IN88, whereas Fourier transform notations are used in this note, and (2) the normalizations of the spherical harmonics differ in the two notes.

This note	IN88
$-i\omega$	s
$-ik = -i\omega/c$	$\gamma = s/c$
$\frac{1}{2\pi}\int_{C_\omega} f(-ik)\, e^{-i\omega t} d\omega$	$\frac{1}{2\pi i}\int_{C_s} f(\gamma)\, e^{st} ds$
(C_ω parallel to the real axis in the complex ω-plane)	(C_s parallel to the imaginary axis in the complex s-plane)
ℓ	n
$i^{-\ell} j_\ell(kr)$	$i_n(\gamma r)$
$i^{\ell+2}\, h_\ell(kr)$	$k_n(\gamma r)$
$\hat{r}\, n_{\ell m}\, \overline{Y}_{\ell m}(\theta,\phi)$	$\vec{P}_{n,m,\sigma}(\theta,\phi)$
$\sqrt{\ell(\ell+1)}\, n_{\ell m}\, \underline{K}'_{\ell m}(\theta,\phi)$	$\vec{Q}_{n,m,\sigma}(\theta,\phi)$
$-\sqrt{\ell(\ell+1)}\, n_{\ell m}\, \underline{K}''_{\ell m}(\theta,\phi)$	$\vec{R}_{n,m,\sigma}(\theta,\phi)$

(This note)	(IN88)
$n_{\ell m}\, \underline{L}_{\ell m}(\underline{r})$	$\vec{L}_{n,m,\sigma}(\gamma\vec{r})$
$\sqrt{\ell(\ell+1)}\; n_{\ell m}\, \underline{M}_{\ell m}(\underline{r})$	$\vec{M}_{n,m,\sigma}(\gamma\vec{r})$
$-\sqrt{\ell(\ell+1)}\; n_{\ell m}\, \underline{N}_{\ell m}(\underline{r})$	$\vec{N}_{n,m,\sigma}(\gamma\vec{r})$
$(0 \le \ell,\ -\ell \le m \le \ell)$	$(0 \le n,\ 0 \le m \le n,\ \sigma = e,o)$

The normalization factor $n_{\ell m}$ is defined as

$$n_{\ell m} = \sqrt{\frac{4\pi(\ell+|m|)!}{(2\ell+1)(\ell-|m|)!}}$$

For those who are familiar with Baum's work simply replace the symbols of this note in the left column by the corresponding ones in the right column.

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