Theoretical Notes
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Some Considerations Concerning Analytic EMP Criteria Waveforms

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Abstract

This note considers time and frequency domain characteristics of some simple analytic waveforms used for approximating and bounding important characteristics of EMP waveforms. In time domain these are the rise, peak, and complete time integral. In frequency domain these correspond (in reverse order) to approximate low, intermediate, and high frequency regimes. The first example consists (in time domain) of the difference of two exponentials times a unit step function; it has a discontinuous slope at t=0. The second example consists of the reciprocal of the sum of two exponentials; it has continuous derivatives of all orders with respect to time for all time.

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I. Introduction

The roughly decade and a half of intensive research into the characteristics of that nuclear environment known as the electromagnetic pulse (EMP) has led to a large variety of specific waveform calculations, supported by measurements where practical. However, from the viewpoint of investigating the interaction of this environment with complex systems of interest some simplification is needed. Instead of considering every possible example of an EMP environment, and its interaction with say an aircraft, one would prefer to have one or some small number of canonical EMP environments. Such canonical environments would presumably contain the important features of the possible EMP environment in some across-the-board sense. The canonical environments might be constructed using averages, extreme values (upper and lower bounds), and perhaps statistical distributions of pertinent EMP environmental parameters.

Having established canonical environments, these can in turn become the basis for criteria EMP environments, i.e., environments which can be specified for system design purposes. Such environments can be used for both analysis and test of the response of systems of interest. Such EMP criteria are a very important consideration in the design of EMP simulators since such simulators may be intended to test certain types, sizes, etc. of systems to such criteria or "threat" conditions.

The EMP environment can be a quite complicated thing, especially in nuclear source regions where the nuclear radiation produces a source current density and a nonlinear conductivity. As discussed in a previous note [1] there are four basic types of electromagnetic field quantities:

$$\begin{array}{ll} \vec{J}_t \left(= \vec{J}_c + \sigma\vec{E} + \epsilon\frac{\partial}{\partial t}\vec{E}\right) & \text{current density } (\mathrm{Am}^{-2}) \\ \vec{H} & \text{magnetic field } (\mathrm{Am}^{-1}) \\ \frac{\partial\vec{B}}{\partial t} & \text{voltage density } (\mathrm{Vm}^{-2}) \\ \vec{E} & \text{electric field } (\mathrm{Vm}^{-1}) \end{array} \tag{1.1}$$

In principle one may wish to specify waveforms, or more generally spatio-temporal distributions, for all of these. However, sometimes simplifications can be introduced.

A case of particular concern is what is referred to as the high altitude EMP. Not including the reflection from the earth surface, the resulting EMP below the source region (in the upper atmosphere produced by an exoatmospheric nuclear detonation) is usually approximated as an expanding spherical wave. Over a sufficiently small region of space (which might contain a system of interest) this incident wave can be approximated as a plane wave of the form

$$\vec{E}(t) = E_2 f_2\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)\vec{1}_2 + E_3 f_3\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)\vec{1}_3$$

$$Z_o\vec{H}(t) = -E_3 f_3\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)\vec{1}_2 + E_2 f_2\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)\vec{1}_3 \tag{1.2}$$

$$Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \quad \text{(free space wave impedance)}$$

where the orthogonal unit vectors are related as

$$\vec{1}_1 \times \vec{1}_2 = \vec{1}_3 \;,\; \vec{1}_2 \times \vec{1}_3 = \vec{1}_1 \;,\; \vec{1}_3 \times \vec{1}_1 = \vec{1}_2$$

$$\vec{1}_1 \quad = \text{direction of propagation} \tag{1.3}$$

$$\vec{1}_2, \vec{1}_3 = \text{orthogonal polarizations}$$

The reflected wave from the earth surface can be similarly approximated provided, for observation near the earth surface, the surface is approximately flat and any variation in the soil or water properties is only in the direction normal to the surface.

The scalar functions f_2 and f_3 are examples of what we usually mean by waveforms (here as functions of retarded time). Considering only one component of the electric field let us write a scalar equation as

$$E(t) = E_o f(t) \tag{1.4}$$

with E_o of units Vm^{-1} so that $f(t)$ is appropriately normalized. Here it is an arbitrary time variable which may be typically interpreted in an appropriately retarded fashion.

In this note we consider unipolar waveforms increasing from zero to a maximum and then decreasing back to zero. For this purpose we assume

$$\begin{gathered} f(t) \geq 0 \quad , \quad -\infty \leq t \leq +\infty \\ f_{max} \equiv \max_t f(t) > 0 \end{gathered} \tag{1.5}$$

Note we have

$$E_{max} \equiv \max_t E(t) = E_o f_{max} \tag{1.6}$$

for scaling purposes.

Since frequency domain aspects of waveforms are important (being the dominant aspect for highly resonant (narrow band) system response cases) then we define the two-sided Laplace transform

$$\begin{gathered} \tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t)e^{-st}\, dt \\ f(t) = \frac{1}{2\pi j}\int_{\Omega_o - j\infty}^{\Omega_o + j\infty} \tilde{f}(s)e^{st}\, ds \\ s \equiv \Omega + j\omega \end{gathered} \tag{1.7}$$

where s is referred to as the complex frequency. Here Ω_o is chosen to lie in a strip of convergence for the defining integral giving $\tilde{f}(s)$ in the s plane. Outside this strip $\tilde{f}(s)$ is defined by analytic continuation. For the usual frequency domain considerations the Laplace transform reduces to a Fourier transform with

$$
\begin{aligned}
s &= j\omega \\
\Omega_o &= 0 \\
\tilde{f}(s) &= \tilde{f}(j\omega)
\end{aligned}
\tag{1.8}
$$

This note considers two typical types of waveforms for their characteristics in both frequency and time domains. These analytic waveforms (of simple mathematical form and thereby smooth in some respects) are considered for a variety of characteristics. The reader should note that a broad set of characteristics are important for EMP interaction implications. To only consider say the peak time domain value is generally an inadequate point of view.

II. Difference of Two Exponentials Times a Unit Step Function

For our first example let us choose the commonly used waveform formed by the difference of two exponentials times a unit step function as

$$E(t) = E_o f(t)$$

$$f(t) = [-e^{-\alpha t} + e^{-\beta t}]\; u(t) \tag{2.1}$$

$$\alpha \geq \beta \geq 0$$

where u(t) is the unit step function given by

$$u(t) = \begin{cases} 0 \text{ for } t < 0 \\ 1 \text{ for } t > 0 \end{cases} \tag{2.2}$$

and undefined at t=0 (although sometimes taken as 0.5 there).

This type of waveform is convenient from its simplicity. It starts at zero with a finite slope at t=0+, rises to a peak and smoothly decays. A capacitive pulse generator driving a frequency-independent resistive load (such as a well designed and constructed parallel-plate EMP simulator) gives this type of waveform assuming a rise characteristic dominated by a simple switch inductance. However, actual large pulse generators of this type have more complicated representations because of observed resonances (giving local peaks and notches in frequency spectrum (the $j\omega$ axis)) as well as more complicated rise characteristics (including prepulse, the details of switch closure, and local switch geometry). In addition, the actual EMP environment does not have a distinct discontinuity in slope as does this f(t) at t=0.

A. Time domain

Let us now consider some of the important features of the time domain waveform, particularly with regard to those features which directly impact portions of the frequency domain. For simplicity these are referred to as the rise, peak, and complete integral (over time) characteristics.

1. Rise characteristics

Consider first the initial (and maximum) time derivative as

$$\left.\frac{\partial}{\partial t}f(t)\right|_{t=0+} = \alpha - \beta$$
$$\left.\frac{\partial}{\partial t}E(t)\right|_{t=0+} = E_o(\alpha - \beta) \tag{2.3}$$

There is a slope discontinuity at t=0 which is related to the high frequency rolloff proportional to s^{-2}. Note for a rise fast compared to the decay we have

$$\alpha >> \beta \geq 0$$
$$\left.\frac{\partial}{\partial t}f(t)\right|_{t=0+} \simeq \alpha \tag{2.4}$$
$$\left.\frac{\partial}{\partial t}E(t)\right|_{t=0+} \simeq E_o\alpha$$

For a rise time one must be careful to choose a definition relevant to the problem at hand. One definition would be the maximum value or something related to that divided by the maximum time derivative. A choice consistent with this (called $t_{max\ rise}$ or t_{mr}) is just

$$t_{mr} \equiv \left[\left.\frac{d}{dt}f(t)\right|_{max}\right]^{-1} \simeq E_{max}\left[\left.\frac{d}{dt}E(t)\right|_{max}\right]^{-1} \tag{2.5}$$

which gives

$$t_{mr} = (\alpha - \beta)^{-1} \simeq \alpha^{-1} \quad (\text{for } \alpha >> \beta \geq 0) \tag{2.6}$$

Often a 10% to 90% rise time is defined. For the case of rise fast compared to decay

$$\begin{aligned} &\alpha >> \beta \geq 0 \\ &f(t) \simeq \left[-e^{-\alpha t} + 1\right]u(t) \quad (\text{during rise}) \\ &t_{10-90} \simeq \frac{2.2}{\alpha} \end{aligned} \tag{2.7}$$

Other types of rise times can be defined as well, such as the e-fold time of the rise t_r appearing in the waveform function as

$$t_r = \frac{1}{\alpha} \tag{2.8}$$

2. Peak characteristics

The peak occurs at a time when the time derivative is zero. Defining

$$\begin{aligned} &T \equiv \alpha t \\ &\zeta \equiv \frac{\beta}{\alpha} \quad , \quad 0 \leq \zeta \leq 1 \end{aligned} \tag{2.9}$$

then

$$f(t) = \left[-e^{-T} + e^{-\zeta T}\right]u(T) \tag{2.10}$$

with a defining equation for the time t_{max} that the maximum occurs as

$$\begin{aligned} &1 = \zeta e^{(1-\zeta)T_{max}} \\ &T_{max} \equiv \alpha t_{max} = \frac{1}{1-\zeta} \ln\left(\frac{1}{\zeta}\right) = \frac{1}{\zeta - 1} \ln(\zeta) \\ &t_{max} = \frac{1}{\alpha - \beta} \ln\left(\frac{\alpha}{\beta}\right) \quad . \end{aligned} \tag{2.11}$$

Note for small β (and constant α) we have

$$t_{max} = \frac{1}{\alpha} \ln\left(\frac{\alpha}{\beta}\right)\left[1 + O\left(\frac{\beta}{\alpha}\right)\right]$$

$$\to\infty \text{ as } \frac{\beta}{\alpha} \to 0 \tag{2.12}$$

while for β approaching α we have

$$t_{max} = \frac{1}{\alpha} \frac{1}{1 - \frac{\beta}{\alpha}} \ln\left(\frac{1}{1 - \left(1 - \frac{\beta}{\alpha}\right)}\right)$$

$$= \frac{1}{\alpha}\left[1 + O\left(1 - \frac{\beta}{\alpha}\right)\right]$$

$$\to \frac{1}{\alpha} \quad \text{as} \quad \frac{\beta}{\alpha} \to 1 \tag{2.13}$$

The peak of the waveform is given by

$$f_{max} \equiv f(t_{max})$$

$$= -e^{-T_{max}} + e^{-\zeta T_{max}}$$

$$= e^{-T_{max}}\left[-1 + \frac{1}{\zeta}\right]$$

$$= \zeta^{\frac{1}{1-\zeta}}\left[\frac{1}{\zeta} - 1\right] \tag{2.14}$$

for small ζ this is

$$f_{max} = \zeta e^{\frac{\zeta}{1-\zeta}\ln(\zeta)}\left[\frac{1}{\zeta} - 1\right]$$

$$= \left[1 - \zeta\right]\left[1 + \frac{\zeta\ln(\zeta)}{1 - \zeta} + O((\zeta\ln(\zeta))^2)\right]$$

$$= 1 + \zeta[\ln(\zeta) - 1] + O((\zeta\ln(\zeta))^2)$$

$$\to 1 \quad \text{as } \zeta \to 0 \tag{2.15}$$

For ζ near 1 this is

$$
\begin{aligned}
f_{max} &= e^{\frac{1}{1-\zeta}\ln(\zeta)} \frac{1-\zeta}{\zeta} \\
&= e^{-1+\frac{1}{2}(1-\zeta)+O((1-\zeta)^2)} \frac{1-\zeta}{\zeta} \\
&= e^{-1}\left[1 + \frac{1}{2}(1-\zeta) + O((1-\zeta)^2)\right] \frac{1-\zeta}{\zeta} \\
&= e^{-1}\left[\frac{1-\zeta}{\zeta} + \frac{1}{2}\frac{(1-\zeta)^2}{\zeta} + O((1-\zeta)^3)\right] \\
&= (1-\zeta)e^{-1} + O(1-\zeta) \\
&\to 0 \text{ as } \zeta \to 1
\end{aligned}
\tag{2.16}
$$

Note, of course, that

$$
E_{max} \equiv E(t_{max}) = E_o f_{max} \tag{2.17}
$$

Generally we are concerned with $\alpha >> \beta \geq 0$, for which case the approximation of (2.15) applies. In this case, the peak field is approximately E_o, although actually a little less.

3. Complete-integral characteristics

The complete time integral is just

$$
\int_{-\infty}^{\infty} f(t)\, dt = \frac{1}{\beta} - \frac{1}{\alpha} = \frac{\alpha - \beta}{\alpha\beta} \tag{2.18}
$$

which can also be written as $\tilde{f}(0)$. For fast rise and slow decay we have

$$
\begin{aligned}
\tilde{f}(0) &= \frac{1}{\beta}\left[1 - \frac{\beta}{\alpha}\right] = \frac{1}{\beta}\left[1 - \zeta\right] \\
&\to \frac{1}{\beta} \text{ as } \zeta \to 0
\end{aligned}
\tag{2.19}
$$

For the case of β near α the complete integral is proportional to $1-\zeta$ and we have

$$\tilde{f}(0) \to 0 \quad \text{as} \quad \zeta \to 1 \tag{2.20}$$

The time constant of the decay is approximately β^{-1} for $\alpha >> \beta > 0$. However, as noted previously it is the complete time integral which is important because of its low-frequency implications. In terms of the electric field

$$\int_{-\infty}^{\infty} E(t)\, dt = E_o \frac{\alpha - \beta}{\alpha\beta} \tag{2.21}$$

B. Frequency domain

In complex frequency domain (2.1) becomes

$$\begin{aligned} \tilde{E}(s) &= E_o \tilde{f}(s) \\ \tilde{f}(s) &= -\frac{1}{s + \alpha} + \frac{1}{s + \beta} = \frac{\alpha - \beta}{(s + \alpha)(s + \beta)} \end{aligned} \tag{2.22}$$

In Fourier transform sense with $s=j\omega$ the frequency domain form is

$$\tilde{f}(j\omega) = -\frac{1}{j\omega + \alpha} + \frac{1}{j\omega + \beta} = \frac{\alpha - \beta}{(j\omega + \alpha)(j\omega + \beta)} \tag{2.23}$$

with magnitude

$$|\tilde{f}(j\omega)| = \frac{\alpha - \beta}{\left[(\omega^2 + \alpha^2)(\omega^2 + \beta^2)\right]^{1/2}} \tag{2.24}$$

and phase

$$\arg(\tilde{f}(j\omega)) = -\arctan\left(\frac{\omega}{\alpha}\right) - \arctan\left(\frac{\omega}{\beta}\right) \tag{2.25}$$

1. Low-frequency characteristics

In the low frequency limit we have

$$\tilde{f}(0) = \frac{1}{\beta} - \frac{1}{\alpha} = \frac{\alpha - \beta}{\alpha\beta} \tag{2.26}$$

corresponding to the complete integral discussed in section II.A.3. For small s we have

$$\tilde{f}(s) = \frac{\alpha - \beta}{\alpha\beta} \left[1 + \frac{s}{\alpha}\right]^{-1}\left[1 + \frac{s}{\beta}\right]^{-1}$$

$$= \tilde{f}(0)\left[1 - \left[\frac{1}{\alpha} + \frac{1}{\beta}\right]s + O(s^2)\right] \tag{2.27}$$

Of course in terms of the electric field we have

$$\tilde{E}(0) = E_o\tilde{f}(0) = E_o \frac{\alpha - \beta}{\alpha\beta} \tag{2.28}$$

2. Intermediate-frequency characteristics

Assuming $\alpha >> \beta \geq 0$ we have for $\alpha >> |s| >> \beta$ the approximate form

$$\tilde{f}(s) \simeq \frac{\alpha - \beta}{\alpha\beta} \simeq \frac{1}{s}$$

$$|\tilde{f}(j\omega)| \simeq \frac{\alpha - \beta}{\alpha\omega} \simeq \frac{1}{\omega} \tag{2.29}$$

$$\arg(\tilde{f}(j\omega)) \simeq -\frac{\pi}{2}$$

with ω assumed positive, and for the electric field

$$\tilde{E}(s) \simeq E_o \frac{\alpha - \beta}{\alpha s} \simeq \frac{E_o}{s}$$

$$|\tilde{E}(j\omega)| \simeq E_o \frac{\alpha - \beta}{\alpha\omega} \simeq \frac{E_o}{\omega} \tag{2.30}$$

Since E_o represents approximately the peak (a little over the peak) of the electric field waveform as discussed in section II.A.2, then we see that this intermediate frequency regime has a magnitude approximately proportional to the waveform peak.

The transition between low and intermediate frequencies occurs at a frequency ω_1 found by equating the results of (2.26) and (2.29), giving

$$\omega_1 \simeq \beta \tag{2.31}$$

which is the reciprocal of the approximate decay time constant.

3. High-frequency characteristics

Again assuming $\alpha >> \beta \geq 0$ we have for $|s| >> \alpha$ the approximate form

$$\begin{aligned} \tilde{f}(s) &\simeq \frac{\alpha - \beta}{s^2} \simeq \frac{\alpha}{s^2} \\ |\tilde{f}(j\omega)| &\simeq \frac{\alpha - \beta}{\omega^2} \simeq \frac{\alpha}{\omega^2} \\ \arg(\tilde{f}(j\omega)) &\simeq -\pi \end{aligned} \tag{2.32}$$

with ω assumed positive, and for the electric field

$$\begin{aligned} \tilde{E}(s) &\simeq E_o \frac{\alpha - \beta}{s^2} \simeq E_o \frac{\alpha}{s^2} \\ |\tilde{E}(j\omega)| &\simeq E_o \frac{\alpha - \beta}{\omega^2} \simeq E_o \frac{\alpha}{\omega^2} \end{aligned} \tag{2.33}$$

Since $E_o(\alpha-\beta)$ represents the maximum time derivative as discussed in section II.A.1, then we see that this high frequency regime has a magnitude approximately proportional to the waveform peak time derivative.

The transition between intermediate and high frequencies occurs at a frequency ω_2 found by equating the results of (2.28) and (2.31), giving

$$\omega_2 \simeq \alpha \tag{2.34}$$

which is the reciprocal of the rise time defined according to maximum time derivative in (2.6) and the e-fold time of the rise in (2.7).

III. Reciprocal of the Sum of Two Exponentials

For our second example let us choose a waveform formed by the reciprocal of the sum of two exponentials as

$$E(t) = E_o f(t)$$
$$f(t) = \left[e^{-\alpha(t-t_o)} + e^{\beta(t-t_o)}\right]^{-1} \qquad (3.1)$$
$$\alpha \geq 0 \ , \ \beta \geq 0$$

Here the waveform is in general nonzero for the complete time axis provided $\alpha \neq 0$ and $\beta \neq 0$. As will become clearer later a good choice for the reference time t_o for translation of the waveform will be $t_o = 0$. The case of typical interest will have $\alpha >> \beta > 0$.

This type of waveform is similar to that in section II in its decay characteristics. However the rise is of the form of a rising exponential with no discontinuity in the function or any of its derivatives with respect to time. This form of rise is also more consistent with an exponential type of rise of the nuclear radiation sources of the fields. This type of waveform is coming into more popular usage; the form above is written so as to simplify the form and make it more symmetric for better analytic understanding.

A. Time domain

In a manner similar to the previous section let us now consider some of the detailed time-domain characteristics of this waveform.

1. Rise characteristics

At early time for $\alpha >> \beta$ one has the case of $\beta \to 0$ giving

$$f(t) \simeq \left[e^{-\alpha(t-t_o)} + 1\right]^{-1} \quad . \qquad (3.2)$$

For $t \ll t_o$ this is

$$f(t) \approx e^{\alpha(t-t_o)} \tag{3.3}$$

giving an exponential characteristic to the rise which is thereby rather smooth.

One might define a characteristic time for the rise according to the e-fold as

$$t_r \equiv \frac{1}{\alpha} \tag{3.4}$$

For a 10% to 90% rise time one might use (3.2) and obtain

$$t_{10-90} \approx \frac{4.4}{\alpha} \tag{3.5}$$

Note also that the 50% point in (3.2) is given at $t = t_o$, where the time derivative is also maximum with a value of approximately $\alpha/4$. This gives a rise time according to the maximum time derivative as

$$t_{mr} \equiv \left[\frac{d}{dt}f(t)\Big|_{max}\right]^{-1} \approx \frac{4}{\alpha} \tag{3.6}$$

2. Peak characteristics

The peak occurs at a time determined by setting the derivative of f(t), or more simply of 1/f(t), to zero. Defining

$$T \equiv \alpha(t - t_o)$$
$$\zeta \equiv \frac{\beta}{\alpha} \tag{3.7}$$

then

$$f(t) = \left[e^{-T} + e^{\zeta T}\right]^{-1} \tag{3.8}$$

The time t_{max} of the maximum occurs as

$$1 = \zeta e^{(1+\zeta)T_{max}}$$

$$T_{max} \equiv \alpha(t_{max} - t_o) = \frac{1}{1+\zeta}\ln\left(\frac{1}{\zeta}\right) = -\frac{1}{1+\zeta}\ln(\zeta)$$

$$t_{max} - t_o = \frac{1}{\alpha+\beta}\ln\left(\frac{\alpha}{\beta}\right) \tag{3.9}$$

Some limiting cases are

$$t_{max} - t_o \begin{cases} \to 0 \text{ for } \alpha \to \infty \quad (\beta \text{ constant}) \\ \to 0 \text{ for } \beta \to \infty \quad (\alpha \text{ constant}) \\ \to -\infty \text{ for } \alpha \to 0 \ (\beta \neq 0) \\ \to +\infty \text{ for } \beta \to 0 \ (\alpha \neq 0) \\ = 0 \text{ for } \alpha = \beta \neq 0 \end{cases} \tag{3.10}$$

The peak of the waveform is given by

$$\begin{aligned} f_{max} &\equiv f(t_{max}) \\ &= \left[e^{-T_{max}} + e^{\zeta T_{max}}\right]^{-1} \\ &= e^{T_{max}}\left[1 + \frac{1}{\zeta}\right]^{-1} \\ &= \zeta^{-\frac{1}{1+\zeta}}\left[\frac{1}{\zeta} + 1\right]^{-1} \end{aligned} \tag{3.11}$$

For small ζ this is

$$\begin{aligned} f_{max} &= \frac{\zeta^{\frac{\zeta}{1+\zeta}}}{1+\zeta} = \frac{e^{\frac{\zeta}{1+\zeta}\ln(\zeta)}}{1+\zeta} \\ &= [1+\zeta]^{-1}\left[1 - \frac{\zeta\ln(\zeta)}{1+\zeta} + O((\zeta\ln(\zeta))^2)\right]^{-1} \\ &= \left[1 - \zeta[\ln(\zeta) - 1] + O((\zeta\ln(\zeta))^2)\right]^{-1} \\ &= 1 + \zeta[\ln(\zeta) - 1] + O((\zeta\ln(\zeta))^2) \\ &\to 1 \text{ as } \zeta \to 0 \end{aligned} \tag{3.12}$$

For the special case of $\zeta = 1$ we have

$$f_{max} = \frac{1}{2} \quad \text{for} \quad \alpha = \beta \neq 0 \tag{3.13}$$

Again note that

$$E_{max} \equiv E(t_{max}) = E_o f_{max} \tag{3.14}$$

The case of usual concern is $\alpha >> \beta > 0$, for which case the approximation of (3.11) applies. Again the peak field is a little less than E_o.

3. Complete-integral characteristics

As will be derived in section III.B.1, the complete time integral for $\alpha \geq \beta > 0$ is given by

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)dt &= \tilde{f}(0) \\ &= \frac{\pi}{\alpha + \beta}\left[\sin\left(\frac{\pi\beta}{\alpha + \beta}\right)\right]^{-1} \end{aligned} \tag{3.15}$$

For fast rise and slow decay we have

$$\begin{aligned} \tilde{f}(0) &= \frac{1}{\beta}\left[1 - \frac{1}{3!}\left(\frac{\pi\beta}{\alpha + \beta}\right)^2 + O(\zeta^4)\right]^{-1} \\ &= \frac{1}{\beta}\left[1 - \frac{\pi^2\zeta^2}{6} + O(\zeta^3)\right]^{-1} \\ &= \frac{1}{\beta}\left[1 + \frac{\pi^2\zeta^2}{6} + O(\zeta^3)\right] \quad \text{as } \zeta \to 0 \end{aligned} \tag{3.16}$$

For the special case of $\beta = \alpha$ we have from (3.14)

$$\tilde{f}(0) = \frac{\pi}{2}\frac{1}{\beta} \tag{3.17}$$

The time constant of the decay is approximately β^{-1} for $\alpha >> \beta > 0$. However, the complete time integral is more important because of its low-frequency implications. In terms of the electric field this is

$$\int_{-\infty}^{\infty} E(t)dt = \tilde{E}(0) = E_o \frac{\pi}{\alpha + \beta} \left[\sin\left(\frac{\pi\beta}{\alpha + \beta}\right)\right]^{-1} \tag{3.18}$$

B. Frequency domain

In complex frequency domain (3.1) can be evaluated from

$$\tilde{E}(s) = E_o \tilde{f}(s)$$

$$\tilde{f}(s) = \int_{-\infty}^{\infty} \left[e^{-\alpha(t-t_o)} + e^{\beta(t-t_o)}\right]^{-1} e^{-st} dt$$

$$= e^{-st_o} \tilde{F}(s) \tag{3.19}$$

$$\tilde{F}(s) = \int_{-\infty}^{\infty} \left[e^{-\alpha t'} + e^{\beta t'}\right]^{-1} e^{-st'} dt', \text{ with } t' \equiv t - t_o$$

This integral can be evaluated through some transformations as

$$\tilde{F}(s) = \int_{-\infty}^{\infty} \left[e^{-(\alpha+\beta)t'} + 1\right]^{-1} e^{-(s+\beta)t'} dt'$$

$$= \frac{1}{\alpha + \beta} \int_{-\infty}^{\infty} \left[e^{-\tau} + 1\right]^{-1} e^{-\frac{s+\beta}{\alpha+\beta}\tau} d\tau \text{ with } \tau \equiv (\alpha + \beta)t' \tag{3.20}$$

The integral is now in a form to be found in standard tables [2, vol. 1, sect. 3.2, (15)] generalized to the Laplace transform (two-sided) as

$$\tilde{F}(s) = \frac{\pi}{\alpha + \beta} \csc\left(\pi \frac{s + \beta}{\alpha + \beta}\right) = \frac{\pi}{\alpha + \beta} \csc\left(\pi \frac{\alpha - s}{\alpha + \beta}\right) \tag{3.21}$$

From this we have

$$\tilde{f}(s) = e^{-st_o} \frac{\pi}{\alpha + \beta} \left[\sin\left(\pi \frac{s + \beta}{\alpha + \beta}\right)\right]^{-1} \tag{3.22}$$

which can be expanded out as

$$\tilde{f}(s) = e^{-st_o} \frac{\pi}{\alpha + \beta}\left[\sin\left(\frac{\pi s}{\alpha + \beta}\right)\cos\left(\frac{\pi\beta}{\alpha + \beta}\right) + \cos\left(\frac{\pi s}{\alpha + \beta}\right)\sin\left(\frac{\pi\beta}{\alpha + \beta}\right)\right]^{-1} \tag{3.23}$$

For $s = j\omega$ this becomes

$$\tilde{f}(j\omega) = e^{-j\omega t_o} \frac{\pi}{\alpha + \beta}\left[j\sinh\left(\frac{\pi\omega}{\alpha + \beta}\right)\cos\left(\frac{\pi\beta}{\alpha + \beta}\right) + \cosh\left(\frac{\pi\omega}{\alpha + \beta}\right)\sin\left(\frac{\pi\beta}{\alpha + \beta}\right)\right]^{-1}$$

$$|f(j\omega)| = \frac{\pi}{\alpha + \beta}\left[\sinh^2\left(\frac{\pi\omega}{\alpha + \beta}\right)\cos^2\left(\frac{\pi\beta}{\alpha + \beta}\right) + \cosh^2\left(\frac{\pi\omega}{\alpha + \beta}\right)\sin^2\left(\frac{\pi\beta}{\alpha + \beta}\right)\right]^{-1/2}$$

$$= \frac{\pi}{\alpha + \beta}\left[\sinh^2\left(\frac{\pi\omega}{\alpha + \beta}\right) + \sin^2\left(\frac{\pi\beta}{\alpha + \beta}\right)\right]^{-1/2} \tag{3.24}$$

$$\arg(\tilde{f}(j\omega)) = -\omega t_o - \arctan\left[\tanh\left(\frac{\pi\omega}{\alpha + \beta}\right) \cot\left(\frac{\pi\beta}{\alpha + \beta}\right)\right]$$

which would suggest $t_0 = 0$ for minimum phase variation at high frequencies.

1. Low-frequency characteristics

In the low frequency limit we now have

$$\tilde{f}(0) = \frac{\pi}{\alpha + \beta}\left[\sin\left(\frac{\pi\beta}{\alpha + \beta}\right)\right]^{-1} \tag{3.25}$$

corresponding to the complete integral discussed in section III.A.3. For small s we have

$$\tilde{f}(s) = e^{-st_o} \frac{\pi}{\alpha + \beta}\left[\frac{\pi s}{\alpha + \beta}\cos\left(\frac{\pi\beta}{\alpha + \beta}\right) + \sin\left(\frac{\pi\beta}{\alpha + \beta}\right) + O(s^2)\right]^{-1} \tag{3.26}$$

For $\alpha >> \beta \geq 0$ this is

$$\tilde{f}(s) \simeq e^{-st_o}\left[s + \beta + O(s^2) + O(\zeta^2)\right]^{-1} \tag{3.27}$$

In terms of the electric field we have

$$\tilde{E}(0) = E_o\tilde{f}(0) = E_o \frac{\pi}{\alpha + \beta}\left[\sin\left(\frac{\pi\beta}{\alpha + \beta}\right)\right]^{-1} \tag{3.28}$$

2. Intermediate-frequency characteristics

Assuming $\alpha >> \beta \geq 0$ we have for $\alpha >> |s| >> \beta$ the approximate form

$$\begin{aligned} \tilde{f}(s) &\simeq \frac{e^{-st_o}}{s} \\ |\tilde{f}(j\omega)| &\simeq \frac{1}{\omega} \\ \arg(\tilde{f}(j\omega)) &\simeq -\omega t_o - \frac{\pi}{2} \end{aligned} \tag{3.29}$$

with ω assumed positive, again suggestive of the choice $t_o = 0$. For the electric field

$$\begin{aligned} \tilde{E}(s) &\simeq E_o \frac{e^{-st_o}}{s} \\ |\tilde{E}(j\omega)| &\simeq \frac{E_o}{\omega} \end{aligned} \tag{3.30}$$

Since E_o is a little larger than the peak of the electric field waveform for $\alpha >> \beta > 0$ as discussed in section III.A.2, then the intermediate portion of the frequency spectrum is approximately proportional to the time domain peak.

The transition between the low and intermediate frequency regimes occurs at a frequency ω_1 found from (3.22) or (3.27) from the combination $s+\beta$ giving

$$\omega_1 \simeq \beta \tag{3.31}$$

Neglecting t_o note that the function $\tilde{f}(s)$ is a function of $s+\beta$ alone (or $\alpha-s$ alone) for the frequency dependence. Note that ω_1 is then the reciprocal of the approximate decay time constant.

3. High-frequency characteristics

For high frequencies rewrite (3.21) as

$$\tilde{f}(s) = e^{-st_o} \frac{2\pi j}{\alpha + \beta} \left[e^{j\pi\frac{s+\beta}{\alpha+\beta}} - e^{-j\pi\frac{s+\beta}{\alpha+\beta}}\right]^{-1} \tag{3.32}$$

The high frequency asymptotic form depends on what direction $s \to \infty$ in the complex plane. Note that on the Re[s] axis there are poles spaced at integer values of $(s+\beta)/(\alpha+\beta)$. If we set $s=j\omega$ with ω positive we have

$$\begin{aligned}\tilde{f}(j\omega) &= -\frac{2\pi j}{\alpha + \beta} e^{-j\omega t_o} e^{-\pi\frac{\omega-j\beta}{\alpha+\beta}} \left[1 - e^{-2\pi\frac{\omega-j\beta}{\alpha+\beta}}\right]^{-1} \\ &= -\frac{2\pi j}{\alpha + \beta} e^{-j(\omega t_o - \frac{\pi\beta}{\alpha+\beta})} e^{-\pi\frac{\omega}{\alpha+\beta}} \left[1 + O(e^{-2\pi\frac{\omega}{\alpha+\beta}})\right] \\ &\quad \text{as } \omega \to +\infty\end{aligned} \tag{3.33}$$

This indicates that for minimum phase variation as $\omega \to +\infty$ we should choose

$$t_o \equiv 0 \tag{3.34}$$

In high frequency limit we then have

$$\begin{aligned}|\tilde{f}(j\omega)| &\simeq \frac{2\pi}{\alpha + \beta} e^{-\pi\frac{\omega}{\alpha+\beta}} \\ \arg(\tilde{f}(j\omega)) &\simeq -\frac{\pi}{2} - \omega t_o + \frac{\pi\beta}{\alpha + \beta} \\ &= -\frac{\pi}{2} + \frac{\pi\beta}{\alpha + \beta} \qquad \text{for } t_o = 0\end{aligned} \tag{3.35}$$

For the electric field we have (for $t_o=0$, $\omega \to +\infty$)

$$\tilde{E}(j\omega) \simeq -E_o \frac{2\pi j}{\alpha + \beta} e^{j\frac{\pi\beta}{\alpha+\beta}} e^{-\pi\frac{\omega}{\alpha+\beta}}$$

$$|\tilde{E}(j\omega)| \simeq E_o \frac{2\pi}{\alpha + \beta} e^{-\pi\frac{\omega}{\alpha+\beta}} \tag{3.36}$$

Note that the high frequency content of this type of waveform falls off exponentially with frequency. This characteristic is associated with the smooth nature of the early time rise of the waveform. This behavior can be contrasted with that for the waveform with a slope discontinuity with a high frequency content proportional to s^{-2} in section II.B.3.

The transition between intermediate and high frequencies occurs at a corner frequency ω_2 corresponding to the deviation of the sine function from its argument by some specified amount. One might define this by the power series expansion of the sine function as

$$\sin\left(\pi\frac{j\omega + \beta}{\alpha + \beta}\right) \simeq \pi\frac{j\omega + \beta}{\alpha + \beta} - \frac{1}{3!}\left(\pi\frac{j\omega + \beta}{\alpha + \beta}\right)^3 + 0(\omega^5) \tag{3.37}$$

setting the second term equal in magnitude to the first for $\omega=\omega_2$ we obtain the approximation

$$1 \simeq \left|\frac{1}{6}\left(\pi\frac{j\omega_2 + \beta}{\alpha + \beta}\right)^2\right| \tag{3.38}$$

For $\alpha >> \beta \geq 0$ this gives

$$\frac{1}{6}\left(\frac{\pi\omega_2}{\alpha}\right)^2 \simeq 1$$

$$\omega_2 \simeq \frac{\sqrt{6}}{\pi}\alpha \simeq .78\alpha \tag{3.39}$$

As expected ω_2 is proportional to α. However one can obtain various constants of proportionality depending on how the corner frequency is quantitatively defined.

IV. Summary

We have now considered in some detail two waveform functions for approximately describing EMP environments. Each of these is described by three parameters:

α: time constant for the rise

β: time constant for the decay

E_o: approximate peak value

In frequency domain we have three frequency regions of interest for $\alpha >> \beta > 0$ governed by:

$\frac{E_o}{\beta}$: approximate low frequency content

$\omega_1 \simeq \beta$: approximate transition from low to intermediate frequencies

$\frac{E_o}{s}$: approximate intermediate frequency content

$\omega_2 \simeq$ (constant times α): approximate transition from intermediate frequencies to high frequencies

The high frequencies are proportional to:

$E_o \frac{\alpha}{s^2}$: for difference of two exponentials times a unit step function

$E_o \frac{2\pi}{\alpha} e^{-\frac{\pi\omega}{\alpha+\beta}}$: for reciprocal of the sum of two exponentials

One can use these simple features of the waveform in conjunction with sets of data, calculations, etc. for the frequency content (magnitude) of EMP waveforms in order to approximate these with simple analytic forms. One might construct "typical" or "worst case" waveforms by this procedure.

References

1. C. E. Baum, Electromagnetic Pulse Interaction Close to Nuclear Bursts and Associated EMP Environment Specification, Interaction Note 76, July 1971.

2. A. Erdelyi et al., Tables of Integral Transforms, McGraw Hill, 1954.