Theoretical Notes

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EMP REFLECTED FROM A MOVING CONDUCTIVITY FRONT

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Abstract

This report calculates, by approximate analytical methods, the EMP from a nuclear burst reflected from the ground and then reflected from the conductivity front made in the atmosphere by the gamma rays from a second nuclear burst. It is shown that the re-reflected pulse is small in amplitude compared with the original EMP. However, the Fourier amplitude of the re-reflected pulse is less than that of the original EMP at all frequencies of interest, and probably at all frequencies.

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SECTION 1
INTRODUCTION

The gamma rays from a high-altitude nuclear burst interact with the atmosphere and create a downward-moving conductivity front traveling at the speed of light.[1] It is possible then that the upward-moving EMP from the ground reflection of one nuclear burst can be reflected from the downward-moving conductivity due to a second nuclear burst. A question has arisen concerning frequency enhancement of EMP reflected in this way. It is the purpose of this report to investigate the magnitude and time variation of the reflected wave.

Since the moving conductivity travels at the speed of light, it is not difficult to show that if the conductivity were an infinite step (analogous to a perfectly-conducting, moving surface) the reflected EMP, regardless of the incident waveshape, would be an impulse. In effect, the moving surface integrates the incident pulse to create the reflection and considerable frequency enhancement can occur.

In the actual situation, the conductivity is finite and varies with time, space, and the net electric field, so that analysis of the time-variation of the reflected wave is considerably more complex. It will be seen in the following material that the magnitude of the reflection is still related to the time integral of the incident pulse; however, the time variation of the reflection is strongly related to the time variation of the conductivity. We shall see that the amplitude of the wave reflected from the conductivity front is small compared with that of the upward-moving incident wave.

SECTION 2
ANALYSIS

Figure 1 illustrates the geometry of the problem. It is assumed that EMP (W1) from a high-altitude nuclear burst is reflected from the ground (W2) and coincidentally another high-altitude burst occurs creating a downward-moving conductivity front. Interest centers on the reflection (W3) of the upward-moving wave from the moving conductivity.

For convenience in analysis, a planar geometry is assumed; that is, it is assumed that the ground is flat and all wavefronts are homogeneous, planar, and moving along the z axis. Furthermore, the residual conductivity remaining from the first burst is neglected. Inclusion of this conductivity would only further reduce the amplitude of the conductivity reflection. It will be seen that the assumption of a planar geometry, while preserving all of the pertinent features of the problem allows a clear picture of the reflection physics to emerge from the analysis.

In the following material, all units are MKS. Maxwell's equations for the geometry assumed; i.e., for the transverse waves, are

$$\mu_0 \frac{\partial H_y}{\partial t} = - \frac{\partial E_x}{\partial z} , \qquad (1)$$

$$\epsilon_0 \frac{\partial E_x}{\partial t} + \sigma E_x = - \frac{\partial H_y}{\partial z} , \qquad (2)$$

where

$$\mu_0 = 4\pi \times 10^{-7} \text{ henries/meter} ,$$

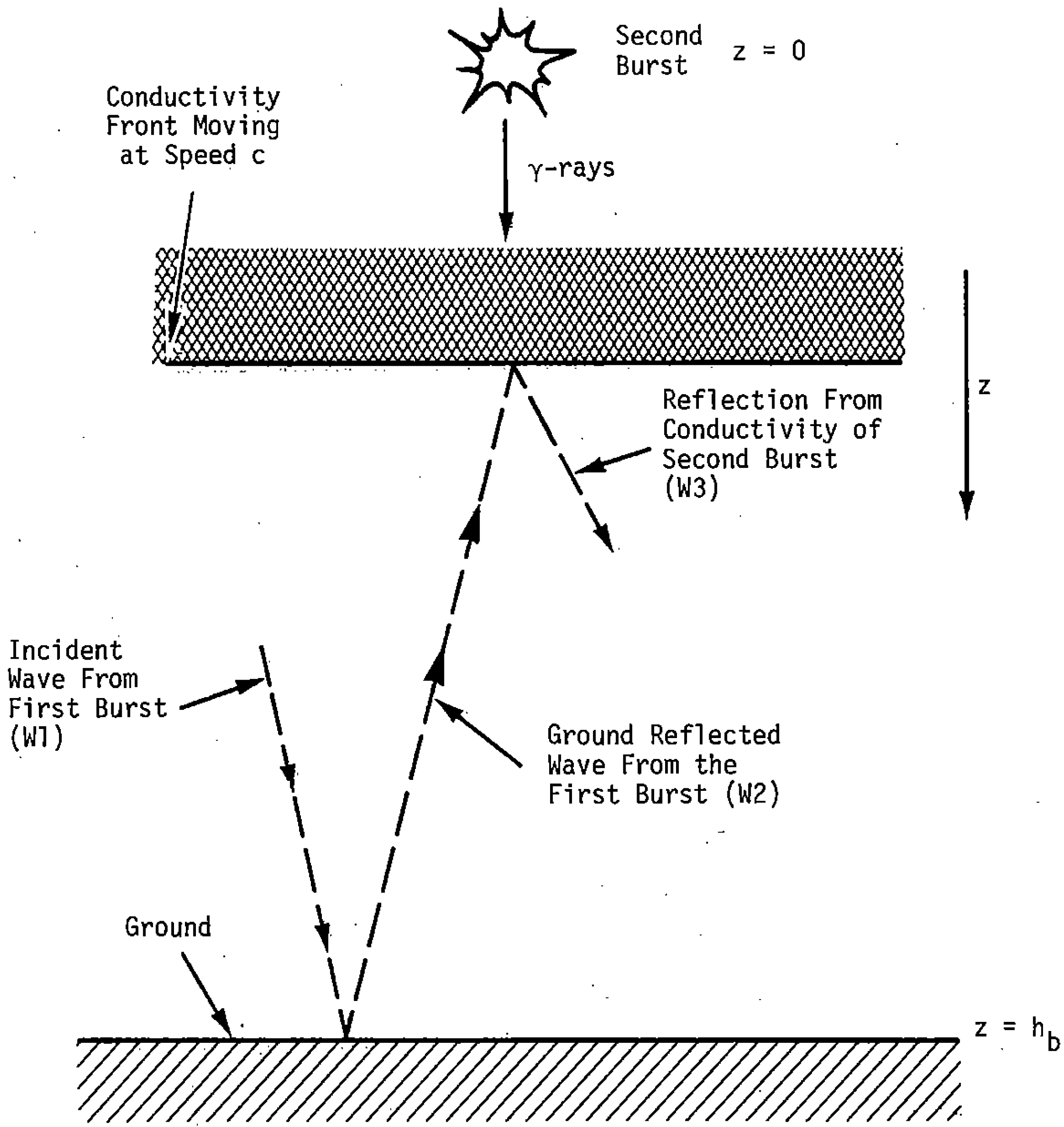

Figure 1. Geometry of the burst.

$$\varepsilon_0 = \frac{10^{-9}}{36\pi} \text{ farads/meter} ,$$

$$\sigma \equiv \sigma(z, t - \frac{z}{c}) = \text{conductivity (mhos/meter)} ,$$

c is the speed of light.

Since the conductivity, σ, is a function of $t - z/c$, it is convenient to define time in this form; that is, define τ as

$$\tau = t - \frac{z}{c} .$$

It follows that

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \tau} ,$$

$$\frac{\partial}{\partial z} \rightarrow \frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial \tau} ,$$

and Equations 1 and 2, in terms of τ, become

$$\left.\begin{aligned} \mu_0 \frac{\partial H_y}{\partial \tau} &= -\frac{\partial E_x}{\partial z} + \frac{1}{c}\frac{\partial E_x}{\partial \tau} , \\ \varepsilon_0 \frac{\partial E_x}{\partial \tau} + \sigma E_x &= -\frac{\partial H_y}{\partial z} + \frac{1}{c}\frac{\partial H_y}{\partial \tau} . \end{aligned}\right\} \qquad (3)$$

It is convenient to define functions F and G as*

$$\left.\begin{aligned} F &= E_x + \eta_0 H_y \\ G &= E_x - \eta_0 H_y \end{aligned}\right\} \Rightarrow \left.\begin{aligned} E_x &= \frac{F + G}{2} \\ H_y &= \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{(F - G)}{2} \end{aligned}\right\} \qquad (4)$$

* These functions have been used by us before in EMP analysis. We derive them here in MKS units.

where

$$\eta_0 = \sqrt{\mu_0/\varepsilon_0} \quad \text{and} \quad c = 1/\sqrt{\mu_0\varepsilon_0} \; .$$

It turns out that F and G are outgoing and ingoing waves respectively; that is, F is a wave traveling in the positive z direction, while G is a wave traveling in the negative z direction. In terms of F and G, Equations 3 and 4 become

$$\frac{2}{c}\frac{\partial G}{\partial \tau} = \frac{\partial (F+G)}{\partial z} \quad , \tag{5}$$

$$\frac{2}{c}\frac{\partial G}{\partial \tau} + \eta_0\sigma(F+G) = -\frac{\partial (F-G)}{\partial z} \; . \tag{6}$$

Finally, adding and subtracting Equations 5 and 6 and rearranging terms, the following equations emerge

$$\frac{\partial F}{\partial z} + \eta_0 \frac{\sigma}{2} F = -\eta_0 \frac{\sigma}{2} G \; , \tag{7}$$

$$\frac{\partial G}{\partial z} - \frac{2}{c}\frac{\partial G}{\partial \tau} = \eta_0 \frac{\sigma}{2} (F+G) \; . \tag{8}$$

In terms of the geometry of Figure 1, the outgoing wave F can be associated with the conductivity reflected wave (W3), while the ingoing wave G can be associated with the incident or ground reflected wave (W2). As can be seen from Equation 7, σG is the source of the conductivity reflected wave F. It follows that if σ is small*, then F will be correspondingly small, and Equation 8 is approximately

$$\frac{\partial G}{\partial z} - \frac{2}{c}\frac{\partial G}{\partial \tau} = \eta_0 \frac{\sigma}{2} G \; . \tag{9}$$

The characteristics of this equation are determined by

$$dz = -\frac{c}{2}\, d\tau = 2\eta_0 \frac{dG}{\sigma G} \; ,$$

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* We shall verify later that this is the case of most interest.

which results in the independent equations

$$dz = -\frac{c}{2}\,d\tau \ , \tag{10}$$

$$-\frac{c}{2}\,d\tau = 2\eta_0 \frac{dG}{\sigma G} \ . \tag{11}$$

The solutions to Equations 10 and 11 are

$$\tau + \frac{2z}{c} = c_1 \ , \tag{12}$$

and

$$G \exp\left(\frac{c}{4}\,\eta_0 \int_0^{\tau} \sigma(z',\tau')d\tau'\right) = c_2 \ , \tag{13}$$

where

c_1 and c_2 are constants.

Note that the integration of σ in Equation 13 must be done along a characteristic determined by Equation 12 (in z' and τ'), since σ is, in general, a function of both z and τ. (It turns out, however, that σ depends much more rapidly on τ' than on z'/c, so that the variation of z' along the characteristic is not important.) Also, conductivity is a function of electric field; thus, in obtaining Equation 13, it has been assumed that for the entire reflection process the fields are either very low, or that they do not significantly alter the fields already present in the conducting medium, associated with the EMP from the second burst.

From the theory of partial differential equations[2] the general solution to Equation 9 is

$$G(z,\tau) = \exp\left(-\frac{c}{4}\,\eta_0 \int_0^{\tau} \sigma(z',\tau')d\tau'\right)\, g\left(\tau + \frac{2z}{c}\right) \ , \tag{14}$$

where $\tau' + 2z'/c = \tau + 2z/c$ and g is a general function. The function G can be interpreted as a wave traveling in the negative z direction, since

$$\tau + \frac{2z}{c} = t + \frac{z}{c} .$$

Furthermore, since for all practical purposes, no gamma radiation penetrates to the ground, it is apparent that $\sigma \equiv 0$ when $z = h_b$ (the burst height). It follows that the function g is equal to the incident EMP function (W2). Equation 14, then, indicates how the incident EMP is attenuated as it penetrates into the downward-moving conductivity.

From Equations 7 and 14, the reflected wave W3 is the solution to

$$\frac{\partial F}{\partial z} \approx - \eta_0 \frac{\sigma(z,\tau)}{2} \exp\left(- \frac{c}{4} \eta_0 \int_0^{\tau} \sigma(z',\tau')d\tau'\right) g\left(\tau + \frac{2z}{c}\right) ,$$

which is

$$F(z,\tau) = -\int_{\xi=-\infty}^{z} \eta_0 \frac{\sigma(\xi,\tau)}{2} \exp\left(- \frac{c}{4} \eta_0 \int_0^{\tau} \sigma d\tau'\right) g\left(\tau + \frac{2\xi}{c}\right) d\xi. \quad (15)$$

Here we have temporarily neglected the attenuation term $\eta_0 \sigma F/2$ in Equation 7. The ξ integration of Equation 15 is done at constant τ. Figure 2 is a portrayal of the z-τ plane which aids in the interpretation of Equation 15. For illustrative purposes, it is assumed that the incident pulse is of length δ.

The integral of Equation 15 can be simplified, with certain realistic assumptions. Suppose that the pulse length, δ, of (W2) is on the order of a microsecond (or less) and consider the reflected pulse (W3) over this same time span. The range of σG along the z axis is 150 meters. At fixed τ, the distance in which σ changes appreciably is the gamma ray scattering length λ, which is of the order of one or a few kilometers. The variation of σ over 150 meters (at fixed τ) can therefore be neglected, and we may approximate Equation 15 as

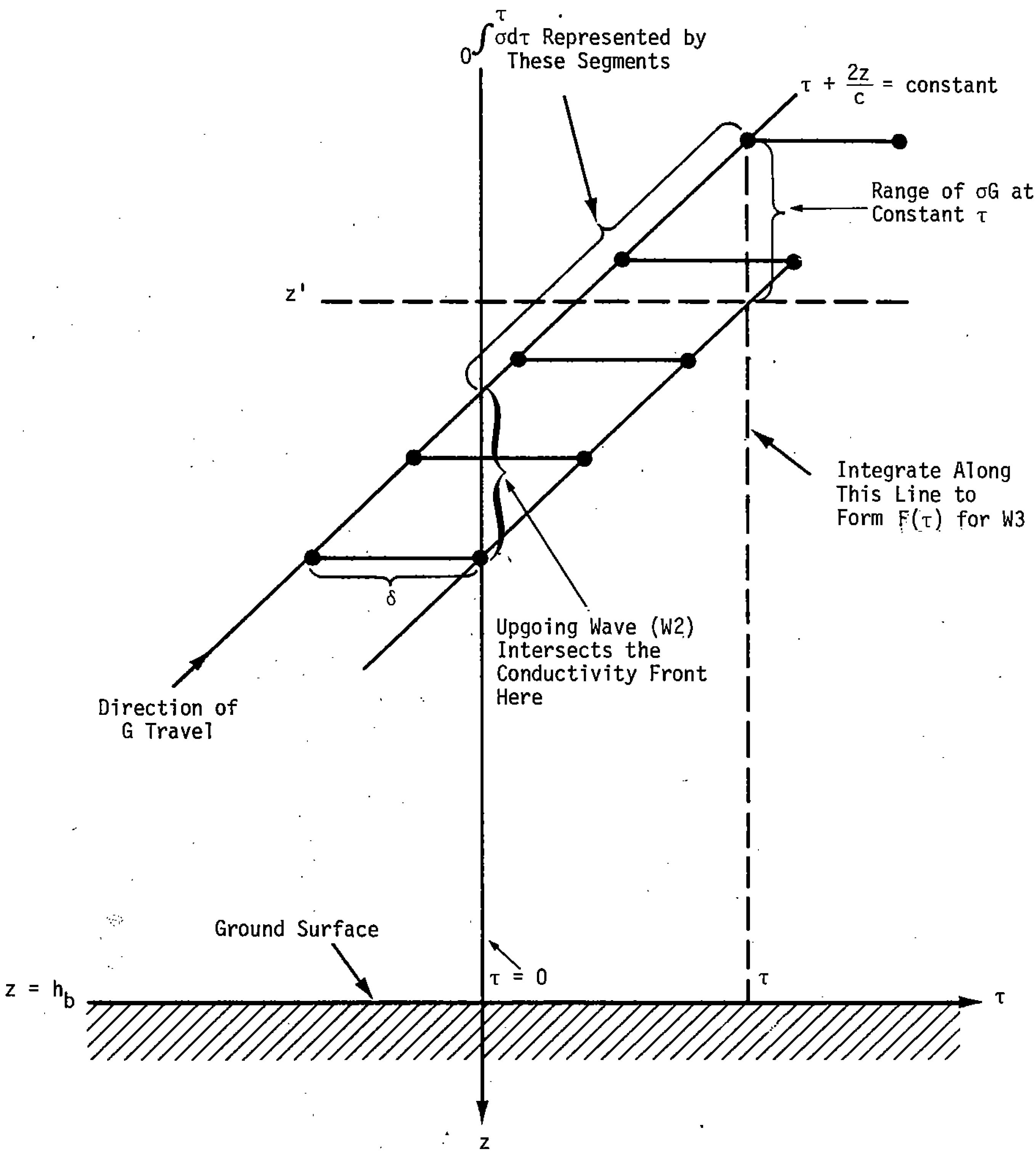

Figure 2. Geometry of the z-τ plane.

$$F(z,\tau) \approx -\eta_0 \frac{\sigma(z,\tau)}{2} \exp(-\frac{c}{4}\eta_0 \int_0^{\tau} \sigma(z,\tau')d\tau') \int_{-\infty}^{z} g(\tau + \frac{2\xi}{c})d\xi$$

$$\approx -\eta_0 \frac{\sigma(z,\tau)}{2} A\, c \exp(-\frac{c}{4}\eta_0 \int_0^{\tau} \sigma(z,\tau')d\tau') \;,$$

where A is the time integral of the incident EMP pulse. The F wave is further attenuated (by the term $\eta_0\sigma F/2$ in Equation 7) in traveling from z to the surface of the earth; therefore, the reflected wave at the ground is

$$F(0,\tau) \approx -\eta_0 \frac{\sigma(z_i,\tau)}{2} A\, c \exp(-\frac{c}{4}\eta_0 \int_0^{\tau} \sigma(z_i,\tau')d\tau') \exp(-\frac{1}{2}\eta_0 \int_{z_i}^{h_b} \sigma(z,\tau)dz) \;. \tag{16}$$

In this equation z_i indicates the approximate altitude at which the wave W2 interacts with the conductivity front.

It is seen from Equation 16 that the reflected wave magnitude is dependent upon the area of the incident pulse, while the time variation of the reflected wave is strongly related to the time variation of the conductivity.

The exponentials in Equation 16 can be approximated as follows. Assume that the gamma pulse from the second burst, and therefore the conductivity, rises exponentially with time constant T, i.e., as $\exp(\tau/T)$. Then the argument of the first exponential in Equation 16 is $-(cT/4)\eta_0\sigma(z_i,T)$. Further, assume that the gamma attenuation (scattering) length at the interaction altitude, and therefore that of the conductivity, is λ. Then the argument of the second exponential in Equation 16 is $-(\lambda/2)\eta_0\sigma(z_i,\tau)$. Since cT is of the order of a few meters while λ is of the order of kilometers, the first argument may be neglected compared with the second. Finally, since, for small σ, little G wave is generated in W3 ($G_3 \ll F_3$), we have from Equations 4

$$E_3 \approx F/2 \;.$$

Thus Equation 16 becomes, for the electric field at the ground surface,

$$E_3(\tau) \approx -\eta_0 \frac{\sigma(z_i,\tau)}{2} A\, c \exp(-\eta_0 \frac{\lambda}{2} \sigma(z_i,\tau)) . \qquad (17)$$

The area A in this expression is that of the pulse W2. However, because the reflection factor of low frequency waves from the ground approaches unity, the area of W2 is nearly the same as that of the original EMP, W1. W2 is generally smaller in amplitude but longer in duration than W1. Since we will want to compare the reflected pulse W3 with W1, we shall relate A to the amplitude and duration of W1. Let us assume that the electric field in the wave W1 has peak amplitude E_1 and effective duration ΔT. Then the time integral of the G wave in W2 is

$$A \approx 2E_1 \Delta T , \qquad (18)$$

and we can write Equation 17 as

$$E_3(\tau) \approx - E_1 \frac{2c\Delta T}{\lambda} u(z_i,\tau) e^{-u(z_i,\tau)} , \qquad (19)$$

where

$$u(z_i,\tau) \equiv \eta_0 \frac{\lambda}{2} \sigma(z_i,\tau) . \qquad (20)$$

Now the maximum value of ue^{-u} is e^{-1} and occurs at $u = 1$. Thus the peak value of E_3 is

$$\text{peak } E_3 = - \frac{2c\Delta T}{2.71\lambda} E_1 , \qquad (21)$$

provided the function $u(z_i,\tau)$ reaches the value unity during the rising part of the gamma pulse.

Since ΔT is of the order of 0.2 μsec, $c\Delta T$ is of the order of 60 meters. Since λ is of the order of kilometers, it is clear that the peak E_3 is relatively small compared to E_1. Further, since λ is shorter at lower altitudes, it is clear that the largest peak E_3 will occur for the lowest interaction altitude z_i for which $u(z_i,\tau)$ reaches the value unity. According to Equation 20, this is the altitude at which σ just reaches the value

$$\sigma(z_i,\tau) \approx 2/\eta_0\lambda \ , \qquad (22)$$

during the rising part of the gamma pulse. We shall examine this condition numerically in the next section. We note here that this value of σ_i satisfies the smallness criterion assumed above in connection with Equation 9 and the following equations.

SECTION 3
NUMERICAL EVALUATION

We wish to examine the conditions under which the conductivity can reach the value given by Equation 22 before the peak of the gamma pulse.

Suppose we have a given gamma fluence F_γ (MeV/m^2), integrated up to the peak of the gamma flux. The energy deposited in Compton recoil electrons is then $\frac{1}{2} F_\gamma/\lambda$ MeV/m^3, since about half the gamma energy is transferred in the first Compton collision. Compton recoil electrons lose their energy to ionization in a retarded time interval

$$\Delta\tau \approx R/5c \quad , \tag{23}$$

where R is the mean range of the Compton recoil electrons. The factor 5 occurs here because the Compton recoil electrons move forward with speed not much less than c. If the gamma flux rises as $\exp(\tau/T)$, then the energy deposited in ionization at the peak of the gamma pulse is approximately $\frac{1}{2} (F_\gamma/\lambda)(T/\Delta\tau)$ MeV/m^3, provided $T < \Delta\tau$. One MeV deposited in ionization makes about 3×10^4 secondary electrons; at the low altitudes of interest here we can assume that this ionization is completed instantaneously after the energy is lost by the Compton electrons. Thus, we will have an electron density N_e at the peak of the gamma pulse

$$N_e = 3 \times 10^4 \frac{F_\gamma}{2\lambda} \frac{5cT}{R} \text{ elect/m}^3, \text{if } \frac{5cT}{R} < 1 \; . \tag{24}$$

Since $R \approx 10$ meters at 10 km altitude and increases with altitude, 5cT/R will usually be less than unity at altitudes greater than 10 km.

The conductivity is

$$\sigma = e\mu N_e \ , \qquad (25)$$

where -e is the electron charge and μ is the electron mobility. Thus the condition (22) becomes

$$3 \times 10^4 \ e\mu \frac{F_\gamma}{2\lambda} \frac{5cT}{R} = \frac{2}{\eta_0 \lambda} \ ,$$

or

$$F_\gamma = \frac{4}{15 \times 10^4} \frac{R}{e\mu cT\eta_0} \ . \qquad (26)$$

Now R and μ are both proportional to the reciprocal of the air density, so R/μ is independent of altitude and can be evaluated at sea level. With

$$R \approx 2 \text{ m} \ ,$$

$$\mu \approx 0.3 \ (\frac{\text{m}}{\text{sec}})/(\frac{\text{V}}{\text{m}}) \ ,$$

$$e = 1.6 \times 10^{-19} \text{ Coulomb} \ ,$$

$$\eta_0 = 377 \text{ ohms} \ ,$$

$$T = 5 \times 10^{-9} \text{ sec} \ ,$$

we find that

$$F_\gamma \approx 2 \times 10^{12} \frac{\text{MeV}}{\text{m}^2} \qquad (27)$$

is needed to reach the condition (22).

As an example, a 1-megaton burst at 100 km altitude will put about 1×10^{14} MeV/m^2 on the top of the atmosphere (at about 30 km altitude) by the peak of the gamma pulse. Thus the fluence (27) will occur at $\ell n(50) = 4$ scattering mean free paths into the atmosphere. Since this mean free path is about 20 gm/cm^2, we seek the altitude above which the mass of air is 80 gm/cm^2. This altitude is about 18 km.

At 18 km altitude the scattering mean free path of gammas is about

$$\lambda \approx 1.6 \text{ km} . \tag{28}$$

Returning to Equation 21 and using the pulse length $\Delta T = 0.2$ μsec, we then find

$$\text{peak } E_3 \equiv E_{3p} \approx \frac{2 \times 60\text{m}}{2.71 \times 1600\text{m}} E_1$$

$$\approx 0.028 \, E_1 . \tag{29}$$

Thus the reflected pulse is indeed small compared with the original EMP.

According to Equation 19, the wave form of the reflected pulse, for an exponentially rising gamma pulse is

$$E_3(\tau) \approx E_3(\text{peak}) \exp(1 + \frac{\tau}{T}) \exp(-e^{\tau/T}) . \tag{30}$$

The shape of the time-dependent factor hence is shown in Figure 3. It can be seen from this figure that the reflected pulse falls very rapidly in time after its peak, due to the second exponential factor in Equation 30, which itself contains an exponential. As a result the pulse contains somewhat more high-frequency content than the original EMP would for the same amplitude. The Fourier transform of $E_3(\tau)$ is

$$E_3(\omega) = E_{3p} e \int_{-\infty}^{\infty} e^{i\omega\tau} e^{\tau/T} \exp(-e^{\tau/T}) d\tau$$

$$= 2.71 T E_{3p} \int_{-\infty}^{\infty} e^{i\omega T x} e^{x} \exp(-e^{x}) dx .$$

On changing the variable of integration to $u = e^x$, this equation becomes

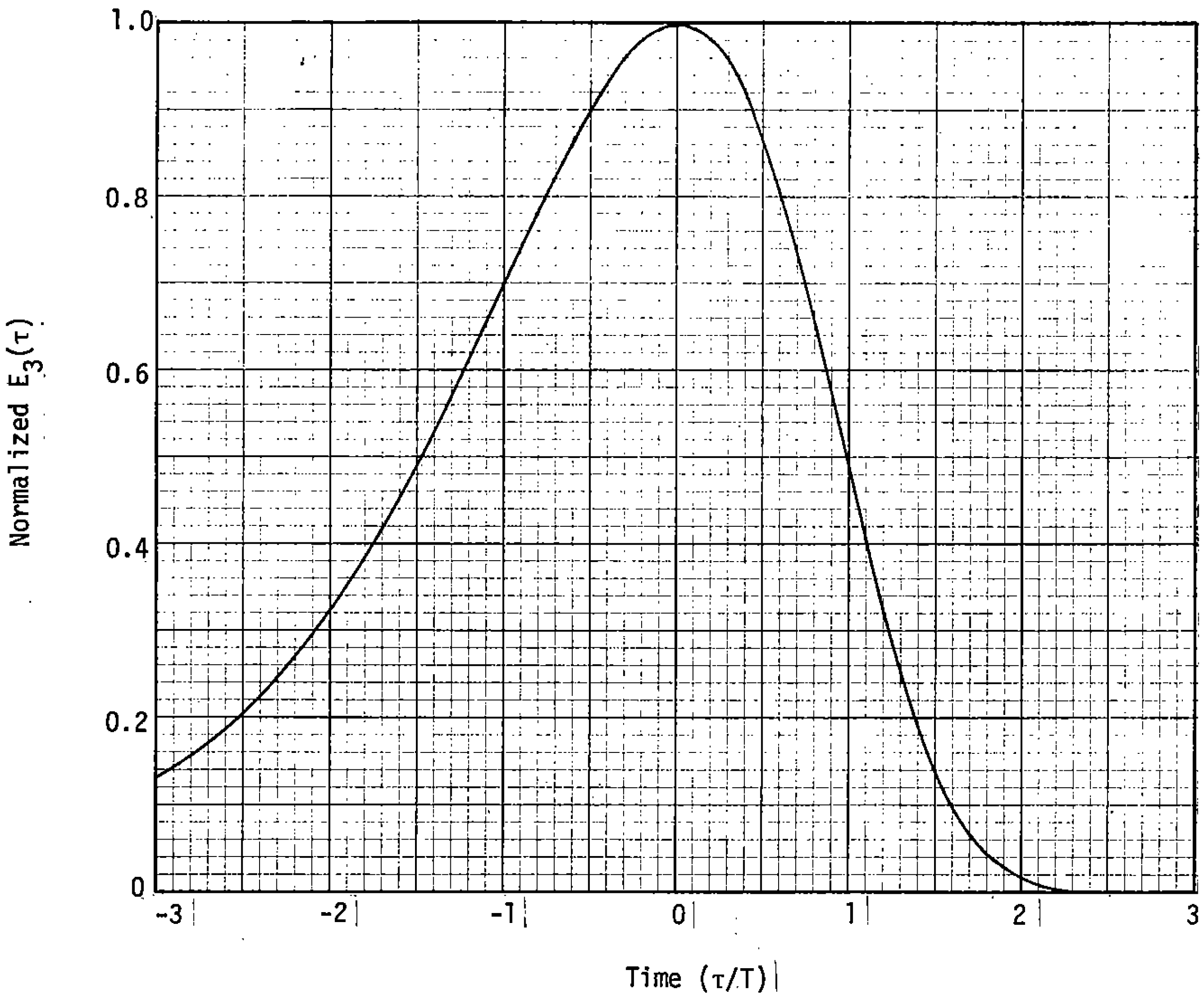

Figure 3. Shape of the reflected pulse for gamma pulse rising as $\exp(\tau/T)$.

$$E_3(\omega) = 2.71TE_{3p}\int_0^\infty u^{i\omega T}e^{-u}du$$

$$= 2.71TE_{3p}\Gamma(1+i\omega T)$$

$$= 2.71i\omega T^2E_{3p}\Gamma(i\omega T) \ . \tag{31}$$

From the asymptotic behavior of the Γ function, we find, for large ωT

$$|E_3(\omega)| \to 2.71E_{3p}T\sqrt{2\pi\omega T}\ \exp(-\tfrac{\pi}{2}\,\omega T) \ . \tag{32}$$

Thus the Fourier transform falls approximately exponentially with ω. This result may be compared with that for the normal EMP. For example, for a pulse of the form

$$E(t) = E_1\ \frac{2e^{\tau/T}}{1 + e^{2\tau/T}} \ , \tag{33}$$

which has peak value E_1, rises with time constant T, and falls with the same time constant, we have

$$|E(\omega)| \to 2\pi E_1T\ \exp(-\tfrac{\pi}{2}\,\omega T) \ . \tag{34}$$

The Fourier amplitude of Equation 32 becomes higher than that of Equation 34 for

$$\omega T > \left(\frac{\sqrt{2\pi}\ E_1}{2.71\ E_{3p}}\right)^2$$

$$\gtrsim \frac{2\pi}{(2.71 \times 0.028)^2} = 1.1 \times 10^3 \ . \tag{35}$$

The numerical result here is for the relation (29) between E_{3p} and E_1.

Thus the original EMP has more high frequency content than the reflected pulse up to quite high frequencies. Furthermore, at those extremely high frequencies at which the form Equation 30 has more Fourier content, it is doubtful if this form is correctly representative. These frequencies come from the final part of $E_3(\tau)$, when E_3 is vanishing rapidly because it is assumed that the conductivity continues to rise exponentially. The truth is probably that the original EMP has more high-frequency content at all frequencies.

SECTION 4
CONCLUSIONS

We have calculated, by approximate analytical methods, the EMP from a nuclear burst reflected from the ground and then reflected from the conductivity front made in the atmosphere by the gamma rays from a second nuclear burst. We have seen that the re-reflected pulse is small in amplitude compared with the original EMP, but has relatively more high-frequency content for its amplitude than the original EMP. However, the Fourier amplitude of the re-reflected pulse is less than that of the original EMP at all frequencies of interest, and probably at all frequencies.

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