THEORETICAL NOTES

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THREE-DIMENSIONAL ELECTROMAGNETIC GROUND RESPONSE FOR MULTI-LAYERED EARTH: SURFACE INTEGRAL REPRESENTATION WITH FREQUENCY-DEPENDENT ELECTRICAL PARAMETERS*

Ira Kohlberg

Kohlberg Associates, Inc.
P.O. Box 23077
Alexandria, VA 22304

ABSTRACT

This investigation is concerned with the development of a mathematical proof that for an N-layer earth model--including three-dimensional variations in the electromagnetic fields and frequency dependence of electrical parameters--it is possible to express the three components of the electric field and the vertical component of the magnetic field on the surface of the earth as a space-time integration of the two horizontal components of the magnetic field. This result would appear to simplify considerably the numerical modeling of the high-altitude-burst electromagnetic pulse (HABEMP) when the ground response is coupled to finite-difference methods for solving the atmospheric part of the problem. Special-case solutions are developed which include the effect of coarse-graining from a finite-difference approximation.

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1. INTRODUCTION

This study concerns the formulation of a theoretical model of stratification for realistic ground structures. The model can be used with finite-difference methods to solve Maxwell's equations in predicting high-altitude-burst electromagnetic pulse (HABEMP). The objective of this effort is to improve the representation of electrical ground stratification and apply the results to the computation of HABEMP to support critical Army programs for HABEMP assessment and hardening.

For Army applications, the electromagnetic fields near the surface of the earth and in the ground are important. At very early times the fields near the ground may be determined by the use of reflection coefficients. However, as time progresses--into the millisecond range and eventually into the one second range--the coupling problem becomes more complex because of the larger wavelengths associated with the intermediate-to-late times. From skin depth considerations one easily sees that later times involve deeper penetration of the fields into the ground. Thus, for HABEMP assessment of systems it is necessary to characterize the relevant ground parameters on a regional basis, and to include realistic conductivity-versus-depth profiles in the calculation. Where appropriate, the effect of water content on the conductivity and dielectric constant should be included.

In this investigation we have been able to develop an analytical solution to the ground coupling problem which is applicable for all times. Also, this solution can be incorporated into the finite-difference numerical model being developed at the Electromagnetic Effects Survivability Laboratory of the Harry Diamond Laboratories (HDL). This can be done by expressing the three components of the electric field and the vertical component of the magnetic field on the surface of the earth as a space-time integration of the two horizontal components of the magnetic field. The aforementioned relationships are sufficient for satisfying the air-ground boundary condition; they essentially incorporate the feature of the conductivity-versus-depth profile for an N-layer structure, including frequency-dependent electrical parameters.

It would appear that this formulation could reduce the cost of running HABEMP problems, since it obviates the necessity of extending the computational grid into the ground. In addition, the special-case theoretical models (sect. 4), derived from the general solution (sect. 3), appear to provide a theoretical basis for simplifying the computations in selected cases.

2. PHYSICAL CONSIDERATIONS

Figure 1 describes the problem in geometric terms. We assume a layered structure, each layer having uniform electrical parameters σ_i, ϵ_i and magnetic permeability, μ_i. Horizontal variations of σ_i, ϵ_i and μ_i are neglected. However, although the fields will vary in x,y because the driving functions, that is, the fields above the ground, are assumed to vary in the horizontal directions.

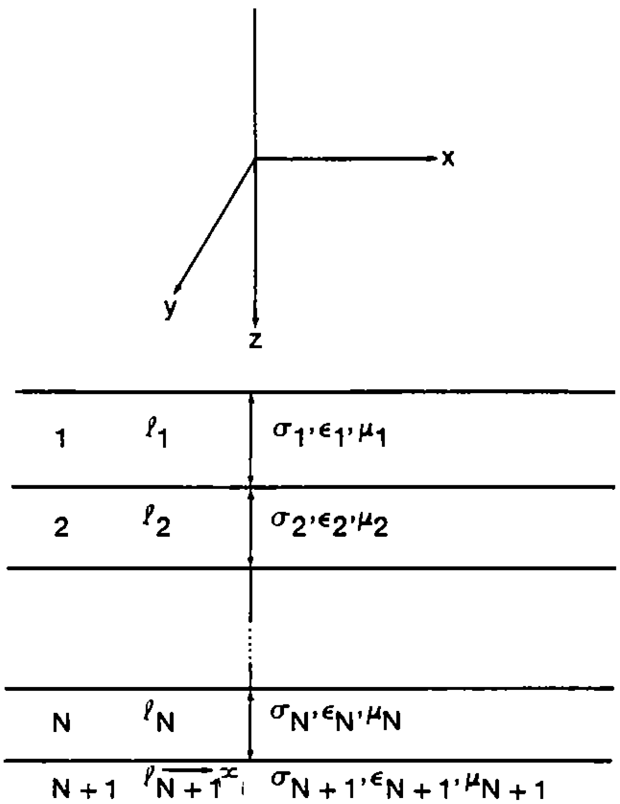

Figure 1. Geometric considerations.

If the ground were a perfect conductor, then the boundary conditions for Maxwell's equations on the earth's surface would be

$$
\begin{aligned}
&\vec{n} \times \vec{E}_s = 0 && (a)\\
&\vec{n} \cdot \vec{B}_s = 0 && (b)\\
&\vec{J}_s = \vec{n} \times \vec{H}_s && (c)\\
&\rho_s = \vec{n} \cdot \vec{D}_s && (d)\\
&\vec{B} = \mu_0 \vec{H}_s && (e)\\
&\vec{D} = \epsilon_0 \vec{E}_s && (f)
\end{aligned}
\qquad (1)
$$

where $\vec{n}$ denotes the normal to the surface, and the subscript "s" denotes the value on the surface. These boundary conditions become modified when the earth's conductivity is finite. In particular, it is shown in this study that if

$$Y_i(\vec{r}_s,t) = \{H_{zs}, E_{xs}, E_{ys}, E_{zs}\} \quad , \quad (2)$$

then every member of the set Y_i on the surface of a finite-conducting earth is related to the horizontal components of the magnetic field through the equation

$$
\begin{aligned}
Y_i(\vec{r}_s,t) = &\int_0^t \int_{\vec{r}_s'} T_{ix}(\vec{r}_s - \vec{r}_s', t - t')H_{xs}(\vec{r}_s', t')dt'dx'dy' \\
&+ \int_0^t \int_{\vec{r}_s'} T_{iy}(\vec{r}_s - \vec{r}_s', t - t')H_{ys}(\vec{r}_s', t')dt'dx'dy' \quad ,
\end{aligned}
\qquad (3)
$$

where $\vec{r}_s, \vec{r}_s'$ are points on the surface, t,t' are time, and T_{ix}, T_{iy} are functions which are calculated from the solution of the layered ground problem. Strictly speaking, the fields given by equations (2) and (3) are those in the ground itself. By matching these fields to those in the air at the earth's surface, we fulfill the boundary conditions leading to the solution of the problem. The thrust of this effort is the determination of T_{ix} and T_{iy} for all i.

The representation of the electromagnetic response in each layer is considered in detail. In particular, we assume that the current density, $\vec{J}$, in each layer is given by [1]

$$\vec{J}(\vec{r},t) = \sigma_0\vec{E}(\vec{r},t) + \varepsilon_\infty\frac{\partial\vec{E}(\vec{r},t)}{\partial t} + \int_0^t \frac{\partial\vec{E}(t')}{\partial t'} K(t - t')dt' \quad , \tag{4}$$

where σ_0 is the dc conductivity, ε_∞ is the dielectric constant at infinite frequency, and $K(t - t')$ is a function which accounts for the frequency dependence of the conductivity and dielectric constant attributed to a volume percentage of water. Longmire et al [1] and Scott [2] provide a good discussion of the dependence of K on water content. Notice that equation (4) is layer-dependent.

Using the notation of Longmire [1], $K(t - t')$ is given by

$$K(t - t') = \sum_n a_n e^{-\beta_n(t-t')} \quad . \tag{5}$$

Taking the Laplace transform of equation (4) gives

$$\vec{J}(\vec{r},s) = \sigma_0\vec{E}(\vec{r},s) + \varepsilon_\infty s\vec{E}(\vec{r},s) + \sigma^*(s)\vec{E}(\vec{r},s) \quad , \tag{6}$$

where

$$\sigma^*(s) = s \sum_n \frac{a_n}{s + \beta_n} \quad . \tag{7}$$

From an analytical point of view we can combine all three terms of equation (6) to write

$$\vec{J}(\vec{r},s) = \sigma(s)\vec{E}(\vec{r},s) \quad , \tag{8}$$

where $\sigma(s)$ can be regarded as a "generalized" conductivity, given by

$$\sigma(s) = \sigma_0 + \varepsilon_\infty s + \sigma^*(s) \quad . \tag{9}$$

The theoretical development of section 3 is carried out using the generalized conductivity of equation (9). The calculations for specific models presented in section 4 are executed for the class of generalized conductivity expressions given by

$$\sigma(s) = \sigma_0 + s\varepsilon \quad , \tag{10}$$

where σ_0 may be interpreted as a frequency-independent conductivity and ε as a dielectric constant. Within the context of equation (10) $s\varepsilon$ is recognized as the usual displacement term.

The degree to which it is necessary to incorporate several earth layers in the calculation depends on the frequency content and hence on the time of interest; it also depends on the electrical parameters and thickness of each region. For intermediate-to-late times which are of interest it is shown that the dominant ground effect is the conduction term. The skin depth for the top layer is given by

$$\delta_1 = \frac{1}{\sqrt{\pi f \mu_1 \sigma_{01}}} \quad , \tag{11}$$

where f is the frequency, μ_1 is the magnetic permeability, and σ_{01} is the conductivity in mhos/meter. Assuming $\mu_1 = \mu_0 = 4\pi \times 10^{-7}$ H/m gives

$$\delta_1 = 0.51 \frac{1}{\sqrt{f\sigma_{01}}} = 0.51 \sqrt{T\rho_1} \ (\text{km}) \quad , \tag{12}$$

where $T = 1/f$ = period of signal and $\rho_1 = 1/\sigma_{01}$ = resistivity in ohm-m. Thus, if ℓ_1 is the thickness of the top region, lower regions may be neglected when

$$\delta_1 < \ell_1 \quad . \tag{13}$$

To be sure, there may be a wide variation in the values of $T\rho_1$ which are of interest for systems assessment. For example, using a combination of a relatively long period (corresponding to late times) of 0.1 s and a relatively large value of resistivity of $\rho = 10^4$ gives

$$\delta_1 = 0.51\sqrt{0.1(10^4)} = 16 \text{ km} \quad . \tag{14}$$

On the other hand, for sea water ($\sigma = 0.3$) and intermediate periods ($T = 10^{-3}$ s) the skin depth would be as small as

$$\delta_1 = 0.51\sqrt{10^{-3}(3.3)} = 0.029 \text{ km} = 29 \text{ m} \quad . \tag{15}$$

Because of the wide range of electrical properties of the earth, it is desirable to model the earth as accurately as possible for results to be meaningful. Information on ground conductivity structures is available from magnetotelluric experiments. These experiments are usually concerned with deep-lying layers which are related to relatively long periods. Nevertheless, it appears that sufficient data have accrued to provide adequate models for the crust (top layers). Much of these data have been assembled by the author for subsequent use in the numerical computation. Samples of the information are shown in figures 2 through 5, which are taken from references 3 through 6, respectively. In particular, figure 5 shows an attempt to deduce lateral variations in conductivity from the magnetotelluric sounding. Gregori and Lanzerotti [7] provide a comprehensive summary of crustal conductivity. On a national scale, figure 6 [8] shows a model of crustal conductivity deduced from radio station measurements at 10 kHz.

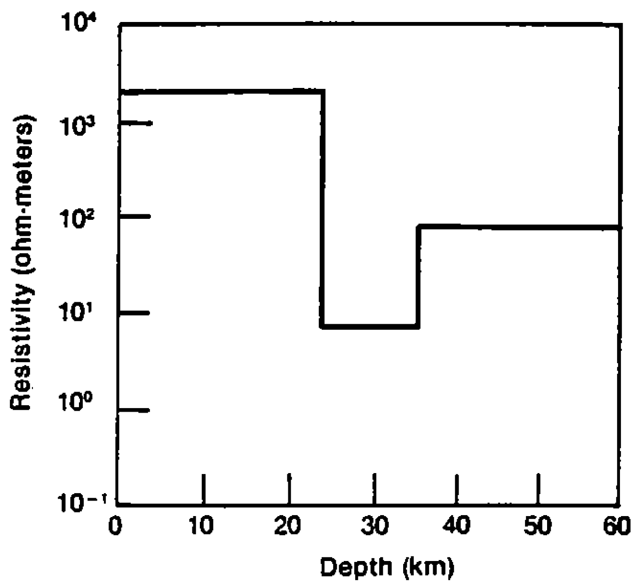

Figure 2. Resistivity for southern Rio Grande rift zone [ref 3].

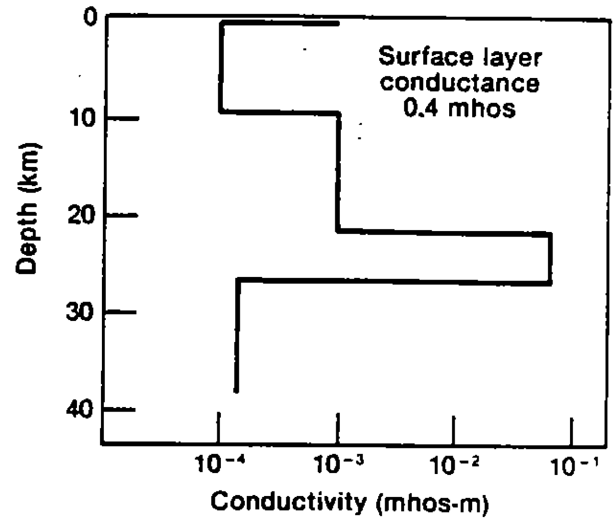

Figure 3. Adirondack conductivity-depth profile [ref 4].

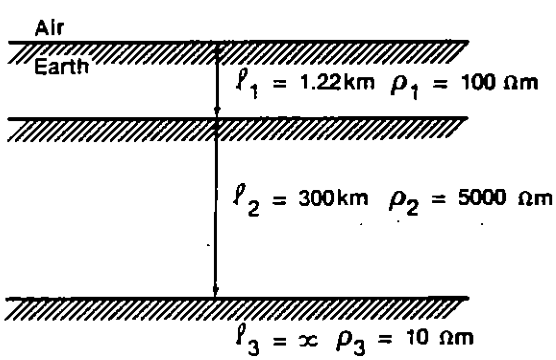

Figure 4. Three-layer resistivity model for Plano, IL, vicinity [ref 5].

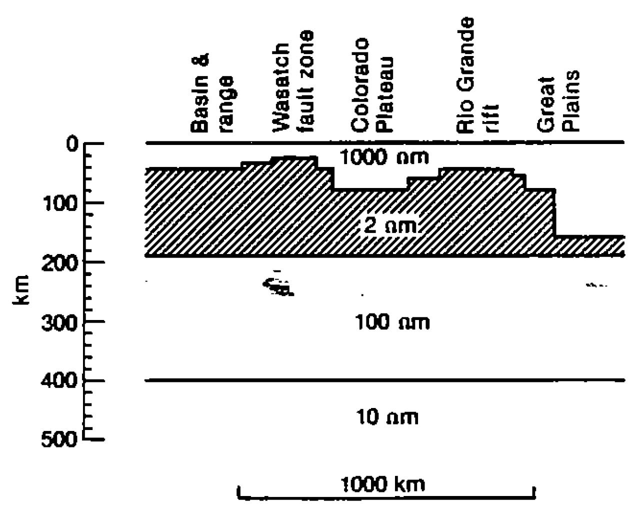

Figure 5. Conductivity structure for Colorado plateau [ref 6].

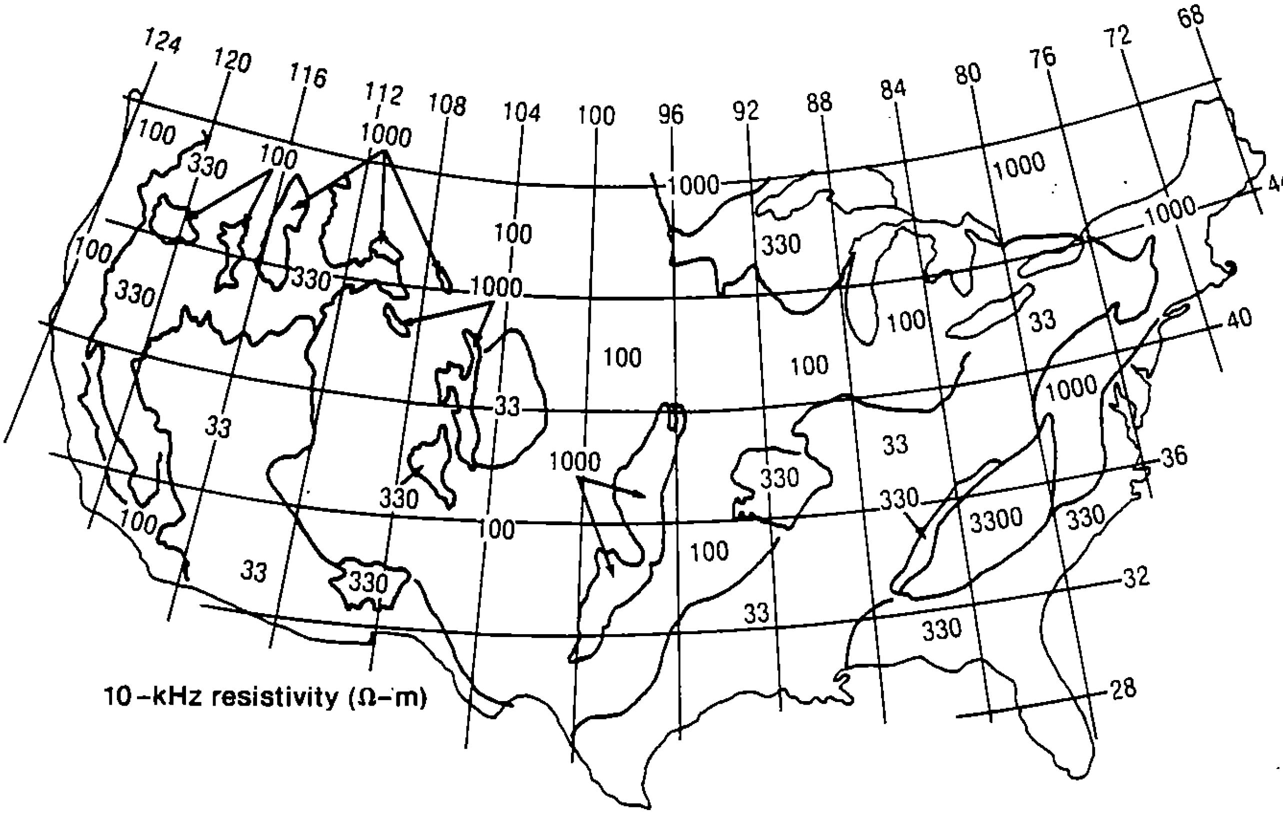

Figure 6. Surface conductivity determined from radio station measurements.

In summary, it would appear that for most of the range of interest, only one, or possibly two, layers would be necessary to account for earth effects. For late times (≥ 1 s), this could conceivably involve three layers in special cases.

3. GENERAL SOLUTION OF SURFACE INTEGRAL REPRESENTATION

In this section we shall present the theoretical formalism for relating the three components of the electric field and the vertical component of the magnetic field to the two horizontal components of the magnetic field at the earth's surface. The functional form describing the relationships is given by equations (2) and (3). Specifically, we are concerned with developing a method for calculating T_{ix} and T_{iy}.

As we shall see, the process of computing $T_{ix}(\vec{r}_s - \vec{r}_s', t - t')$ and $T_{iy}(\vec{r}_s - \vec{r}_s', t - t')$, which are essentially Green's functions, involves the process of performing an inverse Laplace transform (time) and inverse Fourier transform (space) in succession. It is not clear that this can always be done analytically. However, for a number of cases which are relevant to the HABEMP effort this is possible; these special-case results are presented in section 4.

The geometry of the problem is defined by figure 1. If we replace $\partial/\partial t$ by the Laplace transform variable "s," Maxwell's equations in the n^{th} layer become

$$
\begin{aligned}
\nabla \times \vec{H}_n &= \vec{J}_n = \sigma_n \vec{E}_n \quad , && (a)\\
\nabla \times \vec{E}_n &= -\mu_n s \vec{H}_n \quad , && (b)\\
\nabla(\sigma_n \vec{E}_n) &= 0 \quad , \text{ and} && (c)\\
\nabla(\mu_n \vec{H}_n) &= 0 \quad , && (d)
\end{aligned}
\qquad (16)
$$

where σ_n is the generalized conductivity within the layer. As previously mentioned, $\sigma_n(s)$ is not a function of x or y, and is uniform throughout the thickness of the layer.

The boundary conditions between the n^{th} and n + 1 regions are given by

$$
\begin{aligned}
\mu_n H_{z,n} &= \mu_{n+1} H_{z,n+1} \quad , && (a)\\
\sigma_n E_{z,n} &= \sigma_{n+1} E_{z,n+1} \quad , && (b)\\
H_{x,n} &= H_{x,n+1} \quad , && (c)\\
H_{y,n} &= H_{y,n+1} \quad , && (d)\\
E_{x,n} &= E_{x,n+1} \quad , \text{ and} && (e)\\
E_{y,n} &= E_{y,n+1} \quad . && (f)
\end{aligned}
\qquad (17)
$$

It is essential in solving the problem to notice that equations (17 a-f) are not entirely independent [9], as can be seen from Maxwell's equations. Thus, for example, if we consider the expression for $E_{z,n}$ and $E_{z,n+1}$ as given by the z-component of equation (16) we have

$$\frac{\partial H_{y,n}}{\partial x} - \frac{\partial H_{x,n}}{\partial y} = \sigma_n E_{z,n} \quad , \qquad (a)$$

and

$$\frac{\partial H_{y,n+1}}{\partial x} - \frac{\partial H_{x,n+1}}{\partial y} = \sigma_{n+1} E_{z,n+1} \quad . \qquad (b) \qquad (18)$$

If H_y and H_x are continuous (compare eq(17c,d)) across the boundary, so are their derivatives; hence, we immediately deduce from equation (18) that $\sigma_n E_{z,n} = \sigma_{n+1} E_{z,n+1}$, which is the result given by equation (17b). Similarly, we have

$$\frac{\partial E_{y,n}}{\partial x} - \frac{\partial E_{x,n}}{\partial y} = -\mu_n s H_{z,n} \quad , \qquad \text{(a)}$$

(19)

and

$$\frac{\partial E_{y,n+1}}{\partial x} - \frac{\partial E_{x,n+1}}{\partial y} = -\mu_{n+1} s H_{z,n+1} \quad . \qquad \text{(b)}$$

Using equations (17e,f) at the boundary and assuming the continuity of E_y and E_x reduces equation (19) to equation (17a).

The principal conclusion drawn from the aforementioned discussion is that only four of the six continuity equations between the boundaries are necessary. In all cases we shall use equations (17c-f).

Taking the curl of equation (16a) and using equations (16b-d) gives the result

$$\nabla^2 \vec{H}_n = s\mu_n \sigma_n \vec{H}_n \quad , \qquad (20)$$

with an identical equation for $\vec{E}_n$ being derived from the curl operation on equation (16b); that is,

$$\nabla^2 \vec{E}_n = s\mu_n \sigma_n \vec{E}_n \quad . \qquad (21)$$

If we now let

$$\vec{H}_n(s,z,x,y) = \int_{\kappa_x} \int_{\kappa_y} \hat{\vec{H}}_n(s,z,\kappa_x,\kappa_y) e^{i\kappa_x x} e^{i\kappa_y y} d\kappa_x d\kappa_y \quad , \qquad \text{(a)}$$

$$\vec{E}_n(s,z,x,y) = \int_{\kappa_x} \int_{\kappa_y} \hat{\vec{E}}_n(s,z,\kappa_x,\kappa_y) e^{i\kappa_x x} e^{i\kappa_y y} d\kappa_x d\kappa_y \quad , \qquad \text{(b)}$$

(22)

$$\hat{\vec{H}}_n(s,z,\kappa_x,\kappa_y) = \left(\frac{1}{2\pi}\right)^2 \iint_{xy} \vec{H}_n(s,z,x,y) e^{-i\kappa_x x} e^{-i\kappa_y y} dxdy \quad , \qquad \text{(c)}$$

and

$$\hat{\vec{E}}_n(s,z,\kappa_x,\kappa_y) = \left(\frac{1}{2\pi}\right)^2 \iint_{xy} \vec{E}_n(s,z,x,y) e^{-i\kappa_x x} e^{-i\kappa_y y} dxdy \quad , \qquad \text{(d)}$$

and substitute equations (22a,b) into equations (20) and (21), we obtain

$$\frac{\partial^2 \hat{\vec{H}}_n}{\partial z^2} = \left(s\mu_n \sigma_n + \kappa_x^2 + \kappa_y^2\right) \hat{\vec{H}}_n \qquad \text{(a)}$$

and (23)

$$\frac{\partial^2 \hat{\vec{E}}_n}{\partial z^2} = \left(s\mu_n \sigma_n + \kappa_x^2 + \kappa_y^2\right) \hat{\vec{E}}_n \quad . \qquad \text{(b)}$$

The z-dependence in regions $1 \rightarrow N$ is of the form $\exp(-\lambda_n z)$ and $\exp(+\lambda_n z)$; for the $N + 1$ region there can only be a downward travelling wave (recall that z is positive in downward direction) of the form $\exp(-\lambda_{N+1} z)$. We have

$$\frac{\partial}{\partial z} = \gamma_n = \pm\sqrt{\lambda_n} \qquad \text{(a)}$$

and (24)

$$\lambda_n = \sqrt{s\mu_n\sigma_n + \kappa_x^2 + k_y^2} \quad . \qquad \text{(b)}$$

The interrelation between the components of $\hat{\vec{H}}_n$ and $\hat{\vec{E}}_n$ is determined from equations (16a,b). Replacing

$$\frac{\partial}{\partial x} = i\kappa_x \quad , \qquad \text{(a)}$$

$$\frac{\partial}{\partial y} = i\kappa_y \quad , \qquad \text{(b)} \qquad (25)$$

and

$$\frac{\partial}{\partial z} = \gamma_n \qquad \text{(c)}$$

in the aforementioned equations gives

$$i\kappa_y \hat{E}_{z,n} - \gamma_n \hat{E}_{y,n} = -s\mu_n \hat{H}_{x,n} \quad , \qquad \text{(a)}$$

$$\gamma_n \hat{E}_{x,n} - i\kappa_x \hat{E}_{z,n} = -s\mu_n \hat{H}_{y,n} \quad , \qquad \text{(b)}$$

$$i\kappa_x \hat{E}_{y,n} - i\kappa_y \hat{E}_{x,n} = -s\mu_n \hat{H}_{z,n} \quad , \qquad \text{(c)} \qquad (26)$$

$$i\kappa_y \hat{H}_{z,n} - \gamma_n \hat{H}_{y,n} = \sigma_n \hat{E}_{y,n} \quad , \qquad \text{(d)}$$

$$\gamma_n \hat{H}_{x,n} - i\kappa_x \hat{H}_{z,n} = \sigma_n \hat{E}_{y,n} \quad , \qquad \text{(e)}$$

and

$$i\kappa_x \hat{H}_{y,n} - i\kappa_y \hat{H}_{x,n} = \sigma_n \hat{E}_{z,n} \quad . \qquad \text{(f)}$$

Alternatively, we can recast equation (26) in the form

$$M_n \hat{\vec{E}}_n = -s\mu_n \hat{\vec{H}}_n \quad , \qquad \text{(a)}$$

$$M_n \hat{\vec{H}}_n = \sigma_n \hat{\vec{E}}_n \quad , \qquad \text{(b)} \qquad (27)$$

where M_n is the following matrix:

$$M_n = \begin{pmatrix} 0 & -\gamma_n & i\kappa_y \\ \gamma_n & 0 & -i\kappa_x \\ -i\kappa_y & i\kappa_x & 0 \end{pmatrix} \tag{28}$$

and $\hat{\vec{E}}_n$, $\hat{\vec{H}}_n$ are expressed as column vectors. Equation (27) can be cast as an eigenvalue problem for the propagation constant γ_n (assuming we did not already know it) by operating on equation (27b) with M_n and then substituting equation (27a). We have

$$M_n^2\hat{\vec{H}}_n = \sigma_n M_n \hat{\vec{E}}_n = -s\mu_n\sigma_n\hat{\vec{H}}_n \quad . \tag{29}$$

Working out the matrix multiplication gives the following result for equation (29):

$$\begin{pmatrix} -\gamma_n^2 + \kappa_y^2 & -\kappa_y\kappa_x & i\gamma_n\kappa_x \\ -\kappa_y\kappa_x & -\gamma_n^2 + \kappa_x^2 & i\gamma_n\kappa_y \\ i\gamma_n\kappa_x & i\gamma_n\kappa_y & \kappa_y^2 + \kappa_x^2 \end{pmatrix} \begin{pmatrix} \hat{H}_{x,n} \\ \hat{H}_{y,n} \\ \hat{H}_{z,n} \end{pmatrix} = 0 \quad . \tag{30}$$

It can be shown from the determinant of the foregoing matrix that a nontrivial ($\hat{\vec{H}}_n \neq 0$) solution exists when γ_n satisfies the equation

$$\gamma_n = \pm\sqrt{s\mu_n\sigma_n + \kappa_x^2 + \kappa_y^2} \quad , \tag{31}$$

which as promised is the same as equation (24).

When equation (31) is substituted back into equation (30), we obtain

$$\begin{pmatrix} -\kappa_x^2 & -\kappa_y\kappa_x & i\gamma_n\kappa_x \\ -\kappa_y\kappa_x & -\kappa_y^2 & i\gamma_n\kappa_y \\ i\gamma_n\kappa_x & i\gamma_n\kappa_y & \gamma_n^2 \end{pmatrix} \begin{pmatrix} \hat{H}_{x,n} \\ \hat{H}_{y,n} \\ \hat{H}_{z,n} \end{pmatrix} = 0 \quad . \tag{32}$$

Notice that each component of equation (32) reduces to the same equation, namely,

$$i\kappa_x\hat{H}_{x,n} + i\kappa_y\hat{H}_{y,n} + \gamma_n\hat{H}_{z,n} = 0 \quad . \tag{33}$$

Equation (33) could of course have been written down immediately from equation (16d). However, the formalism of the present method shows that there are two linearly independent solutions driven by $\hat{H}_{x,n}$ and $\hat{H}_{y,n}$. We thus rewrite equation (33) in the form

$$\hat{H}_{z,n} = \frac{1}{\gamma_n}\left[(-i\kappa_x)\hat{H}_{x,n} + (-i\kappa_y)\hat{H}_{y,n}\right] \quad . \tag{34}$$

The corresponding functional relationships involving the three components of the electric field are determined from equation (27a). We have

$$\hat{\vec{E}}_n = -s\mu_n M_n^{-1}\hat{\vec{H}}_n \quad , \tag{35}$$

where M_n^{-1} is the inverse matrix of M_n. The result of the matrix inversion, combined with the use of equation (34) gives the following result for $\hat{E}_{x,n}$, $\hat{E}_{y,n}$, $\hat{E}_{z,n}$:

$$\hat{E}_{x,n} = \frac{\gamma_n}{\sigma_n}\left[-\hat{H}_{y,n} + \frac{1}{\gamma_n^2}\left(\kappa_y\kappa_x\hat{H}_{x,n} + \kappa_y^2\hat{H}_{y,n}\right)\right] \quad , \quad \text{(a)}$$

$$\hat{E}_{y,n} = \frac{\gamma_n}{\sigma_n}\left[\hat{H}_{x,n} - \frac{1}{\gamma_n^2}\left(\kappa_x^2\hat{H}_{x,n} + \kappa_x\kappa_y\hat{H}_{y,n}\right)\right] \quad , \quad \text{(b)} \tag{36}$$

and

$$\hat{E}_{z,n} = \frac{1}{\sigma_n}\left[(-i\kappa_y)\hat{H}_{x,n} + (i\kappa_x)\hat{H}_{y,n}\right] \quad . \quad \text{(c)}$$

The solution in each region is written as follows. First, we introduce the parameters

$$L_n = \sum_{i=1}^{n} \ell_i \quad . \tag{37}$$

Using equation (37) the range of z in the n^{th} region is given by

$$L_{n-1} \leq z \leq L_n \quad . \tag{38}$$

For the N + 1 region the range of z is

$$L_N \leq z \leq \infty \quad . \tag{39}$$

We now define

$$g_n(z) = e^{-\lambda_n(z-L_{n-1})} \quad \text{(a)}$$

and (40)

$$h_n(z) = e^{\lambda_n(z-L_{n-1})} \quad . \quad \text{(b)}$$

Using equation (36) for $\gamma_n = \pm\sqrt{\lambda_n}$ and recalling that the solutions in each layer except the N+1 consist of both $\exp(-\lambda_n z)$ and $\exp(+\lambda_n z)$ terms give the following expressions for the fields:

$1 \leq n \leq N$

$$\hat{H}_{x,n} = A_n g_n(z) + B_n h_n(z) \quad , \tag{41}$$

$$\hat{H}_{y,n} = C_n g_n(z) + D_n h_n(z) \quad , \tag{42}$$

$$\hat{H}_{z,n} = \frac{1}{\lambda_n}[i\kappa_x A_n + i\kappa_y C_n]g_n(z) - \frac{1}{\lambda_n}[i\kappa_x B_n + i\kappa_y D_n]h_n(z) \quad , \tag{43}$$

$$\hat{E}_{x,n} = \frac{-1}{\sigma_n\lambda_n}[-(s\mu_n\sigma_n + \kappa_x^2)C_n + \kappa_x\kappa_y A_n]g_n(z) + \frac{1}{\sigma_n\lambda_n}[-(s\mu_n\sigma_n + \kappa_x^2)D_n + \kappa_x\kappa_y B_n]h_n(z) \quad , \tag{44}$$

$$\hat{E}_{y,n} = -\frac{1}{\sigma_n\lambda_n}[(s\mu_n\sigma_n + \kappa_y^2)A_n - \kappa_x\kappa_y C_n]g_n(z) + \frac{1}{\sigma_n\lambda_n}[(s\mu_n\sigma_n + \kappa_y^2)B_n - \kappa_x\kappa_y D_n]h_n(z) \quad , \tag{45}$$

and

$$\hat{E}_{z,n} = \frac{1}{\sigma_n}[-i\kappa_y A_n + i\kappa_x C_n]g_n(z) + \frac{1}{\sigma_n}[-i\kappa_y B_n + i\kappa_x D_n]h_n(z) \quad . \tag{46}$$

$n = N + 1 =$ last layer

$$\hat{H}_{x,N+1} = A_{N+1}g_{N+1}(z) \quad , \tag{47}$$

$$\hat{H}_{y,N+1} = C_{N+1}g_{N+1}(z) \quad , \tag{48}$$

$$\hat{H}_{z,N+1} = \frac{1}{\lambda_{N+1}}[i\kappa_x A_{N+1} + i\kappa_y C_{N+1}]g_{N+1}(z) \quad , \tag{49}$$

$$\hat{E}_{x,N+1} = -\frac{1}{\sigma_{N+1}\lambda_{N+1}}[-(s\mu_{N+1}\sigma_{N+1} + \kappa_x^2)C_{N+1} + \kappa_x\kappa_y A_{N+1}]g_{N+1}(z) \quad , \tag{50}$$

$$\hat{E}_{y,N+1} = -\frac{1}{\sigma_{N+1}\lambda_{N+1}}[(s\mu_{N+1}\sigma_{N+1} + \kappa_y^2)A_{N+1} - \kappa_x\kappa_y C_{N+1}]g_{N+1}(z) \quad , \tag{51}$$

and

$$\hat{E}_{z,N+1} = \frac{1}{\sigma_{N+1}}[-i\kappa_y A_{N+1} + i\kappa_x C_{N+1}]g_{N+1}(z) \quad . \tag{52}$$

Before proceeding with the details of solving the boundary layer equations between the layers, it is desirable to restate the ultimate objective of these procedures. As previously mentioned, the goal is to express the three components of the electric field and the vertical component of the magnetic field on the earth's surface in terms of the two horizontal components of the magnetic field. In "s,κ_x,κ_y" space, the desired relationship will be an algebraic relationship in the transform variables s,κ_x,κ_y.

For brevity, we now define the entities:

$$\left.\begin{aligned} \alpha_n &= s\mu_n\sigma_n + \kappa_x^2 \\ \beta_n &= s\mu_n\sigma_n + \kappa_y^2 \end{aligned}\right\} \text{ all } n \qquad \begin{aligned} &(a) \\ &(b) \end{aligned}$$

$$\rho = \kappa_x\kappa_y \qquad (c) \tag{53}$$

In anticipation of their use we also introduce the variables:

$1 \leq n \leq N$

$$\hat{A}_n = A_n\bar{g}_n(\ell_n) \quad , \quad (a)$$

$$\hat{B}_n = B_n\bar{h}_n(\ell_n) \quad , \quad (b)$$

$$\hat{C}_n = C_n\bar{g}_n(\ell_n) \quad , \quad (c) \tag{54}$$

and

$$\hat{D}_n = D_n\bar{h}_n(\ell_n) \quad , \quad (d)$$

where

$$\bar{g}_n(\ell_n) = \exp(-\lambda_n \ell_n) \quad , \qquad (a)$$

and (55)

$$\bar{h}_n(\ell_n) = \exp(\lambda_n \ell_n) \quad . \qquad (b)$$

It should also be noted from equations (37) to (40) that at the top of every layer

$$g_n(z) = h_n(z) = 1 \quad , \qquad (56)$$

since at these points the corresponding values of z are L_{n-1}, respectively.

Now let us consider the boundary condition equations between the N and N+1 regions. According to the previous discussion, the relationship between the fields in the two regions is obtained by matching $\hat{E}_x$, $\hat{E}_y$, $\hat{H}_x$, $\hat{H}_y$ at the interface. Using the results of equations (53) through (56) we get

$$\hat{A}_N + \hat{B}_N = A_{N+1} \quad , \qquad (57)$$

$$\hat{C}_N + \hat{D}_N = C_{N+1} \quad , \qquad (58)$$

and

$$\frac{1}{\sigma_N \lambda_N} [\alpha_N \hat{C}_N - \rho \hat{A}_N] + \frac{1}{\sigma_N \lambda_N} [-\alpha_N \hat{D}_N + \rho \hat{B}_N] = \frac{1}{\sigma_{N+1} \lambda_{N+1}} [\alpha_{N+1} C_{N+1} - \rho A_{N+1}] \quad , \text{ and} \qquad (59)$$

$$\frac{1}{\sigma_N \lambda_N} [-\beta_N \hat{A}_N + \rho \hat{C}_N] + \frac{1}{\sigma_N \lambda_N} [\beta_N \hat{B}_N - \rho \hat{D}_N] = \frac{1}{\sigma_{N+1} \lambda_{N+1}} [-\beta_{N+1} A_{N+1} + \rho C_{N+1}] \quad . \qquad (60)$$

For computational purposes we recast equations (59) and (60) in the matrix form

$$R_N \begin{pmatrix} \hat{A}_N \\ \hat{C}_n \end{pmatrix} - R_N \begin{pmatrix} \hat{B}_N \\ \hat{D}_N \end{pmatrix} = R_{N+1} \begin{pmatrix} A_{N+1} \\ C_{N+1} \end{pmatrix} \quad , \qquad (61a)$$

where

$$R_N = \frac{1}{\sigma_n \lambda_n} \begin{pmatrix} -\rho & \alpha_N \\ -\beta_N & \rho \end{pmatrix} \quad , \qquad (61b)$$

and R_{N+1} is the corresponding matrix with N replaced by N+1. From equation (61a) we have

$$\begin{pmatrix} A_{N+1} \\ C_{N+1} \end{pmatrix} = (R_{N+1}^{-1} R_N) \begin{pmatrix} \hat{A}_N \\ \hat{C}_N \end{pmatrix} - (R_{n+1}^{-1} R_N) \begin{pmatrix} \hat{B}_N \\ \hat{D}_N \end{pmatrix} , \tag{62}$$

where R_{n+1}^{-1} is the inverse matrix. The matrix equivalent of equations (57) and (58) is

$$\begin{pmatrix} \hat{B}_N \\ \hat{D}_N \end{pmatrix} = - \begin{pmatrix} \hat{A}_N \\ \hat{C}_N \end{pmatrix} + \begin{pmatrix} A_{N+1} \\ C_{N+1} \end{pmatrix} . \tag{63}$$

Substituting equation (62) into equation (63) yields

$$\begin{pmatrix} \hat{B}_N \\ \hat{D}_N \end{pmatrix} = Q_N \begin{pmatrix} \hat{A}_N \\ \hat{C}_N \end{pmatrix} , \tag{64}$$

where Q_N is a 2 × 2 matrix given by

$$Q_N = -(I + R_{N+1}^{-1} R_N)^{-1} (I - R_{N+1}^{-1} R_N) , \tag{65}$$

and I is the identity matrix. The important feature of equation (64) is that we have been able to express the upward components of the waves $\hat{B}_N$, $\hat{D}_N$ in terms of the downward components $\hat{A}_N$, $\hat{B}_N$. This technique will be used repeatedly to work "up the ladder" to the first region.

Now let us consider the boundary conditions between the N-1 and N regions. In matrix form, these equations are given by

$$\begin{pmatrix} \hat{A}_{N-1} \\ \hat{C}_{N-1} \end{pmatrix} + \begin{pmatrix} \hat{B}_{N-1} \\ \hat{D}_{N-1} \end{pmatrix} = \begin{pmatrix} A_N \\ C_N \end{pmatrix} + \begin{pmatrix} B_N \\ D_N \end{pmatrix} , \tag{66}$$

and

$$R_{N-1} \begin{pmatrix} \hat{A}_{N-1} \\ \hat{C}_{N-1} \end{pmatrix} - R_{N-1} \begin{pmatrix} \hat{B}_{N-1} \\ \hat{D}_{N-1} \end{pmatrix} = R_N \begin{pmatrix} A_N \\ C_N \end{pmatrix} - R_N \begin{pmatrix} B_N \\ D_N \end{pmatrix} . \tag{67}$$

Using equations (54) and (64) the foregoing equations become

$$\begin{pmatrix} \hat{A}_{N-1} \\ \hat{C}_{N-1} \end{pmatrix} + \begin{pmatrix} \hat{B}_{N-1} \\ \hat{D}_{N-1} \end{pmatrix} = \bar{g}_N^{-1} \begin{pmatrix} \hat{A}_N \\ \hat{C}_N \end{pmatrix} + \bar{h}_N^{-1} Q_N \begin{pmatrix} \hat{A}_N \\ \hat{C}_N \end{pmatrix} . \tag{68}$$

$$R_{N-1} \begin{pmatrix} \hat{A}_{N-1} \\ \hat{C}_{N-1} \end{pmatrix} - R_{N-1} \begin{pmatrix} \hat{B}_{N-1} \\ \hat{D}_{N-1} \end{pmatrix} = \bar{g}_N^{-1} R_N \begin{pmatrix} \hat{A}_N \\ \hat{C}_N \end{pmatrix} - \bar{h}_N^{-1} R_N Q_N \begin{pmatrix} \hat{A}_N \\ \hat{C}_N \end{pmatrix} . \tag{69}$$

From equation (69) we have

$$\begin{pmatrix}\hat{A}_N\\ \hat{C}_N\end{pmatrix} = (J_N^{-1}R_{N-1})\begin{pmatrix}\hat{A}_{N-1}\\ \hat{C}_{N-1}\end{pmatrix} - (J_N^{-1}R_{N-1})\begin{pmatrix}\hat{B}_{N-1}\\ \hat{D}_{N-1}\end{pmatrix} \quad , \tag{70}$$

where J_N^{-1} is the inverse matrix of

$$J_N = \bar{g}_N^{-1}R_N - \bar{h}_N^{-1}R_NQ_N \quad . \tag{71}$$

Inserting equation (70) into equation (68) then yields

$$\begin{pmatrix}\hat{B}_{N-1}\\ \hat{D}_{N-1}\end{pmatrix} = Q_{N-1}\begin{pmatrix}\hat{A}_{N-1}\\ \hat{C}_{N-1}\end{pmatrix} \quad , \tag{72}$$

where

$$Q_{N-1} = -(I + U_{N-1})^{-1}(I - U_{N-1}) \quad \text{(a)}$$

(73)

and

$$U_{N-1} = (\bar{g}_N^{-1}I + \bar{h}_N^{-1}Q_N)J_N^{-1}R_{N-1} \quad . \quad \text{(b)}$$

It is not necessary to carry out equation (73) any further to see that the ultimate result will be the generation of the relationship

$$\begin{pmatrix}\hat{B}_1\\ \hat{D}_1\end{pmatrix} = Q_1\begin{pmatrix}\hat{A}_1\\ \hat{C}_1\end{pmatrix} \quad , \tag{74}$$

where Q_1 is a 2 × 2 matrix obtained by repeated application of the procedure just developed, beginning at the N+1 layer and ending at the first layer.

The method for calculating the four components of Q_1 is straightforward, although lengthy for a large number of layers. It is not important, however, to carry out the computation in this section to see how the result given by equation (74) yields the result expressed by equation (3), which defines the surface relationships between field quantities.

Let us now examine equations (41) through (46) at the earth's surface, defined by $z = 0$, $h_1(0) = 1$, $g_1(0) = 1$. We denote the fields at the surface by the subscript s (e.g., $\hat{H}_{xs}$). Evaluating equations (41) and (42) at the earth's surface gives

$$\hat{H}_{xs} = A_1 + B_1 \tag{75}$$

and

$$\hat{H}_{ys} = C_1 + D_1 \quad . \tag{76}$$

Using equations (53) and (54) in equation (74) we obtain

$$\bar{h}_1\begin{pmatrix} B_1 \\ D_1 \end{pmatrix} = \bar{g}_1 Q_1 \begin{pmatrix} A_1 \\ C_1 \end{pmatrix} \quad , \tag{77}$$

or equivalently

$$\begin{pmatrix} B_1 \\ D_1 \end{pmatrix} = e^{-2\lambda_1 \ell_1} Q_1 \begin{pmatrix} A_1 \\ C_1 \end{pmatrix} \quad \text{(a)}$$

$$= \bar{Q}_1 \begin{pmatrix} A_1 \\ C_1 \end{pmatrix} \quad , \quad \text{(b)} \tag{78}$$

where

$$\bar{Q} = e^{-2\lambda_1 \ell_1} Q \quad . \tag{79}$$

As observed, $\bar{Q}$ is also a 2 × 2 matrix whose components are simply $\exp(-2\lambda_1 \ell_1)$ times the components of Q. Writing equations (75) and (76) in matrix form and using equation (78b) gives

$$\begin{pmatrix} \hat{H}_{xs} \\ \hat{H}_{ys} \end{pmatrix} = (I + \bar{Q}) \begin{pmatrix} A_1 \\ C_1 \end{pmatrix} \quad . \tag{80}$$

We can alternately write

$$\begin{pmatrix} A_1 \\ C_1 \end{pmatrix} = W \begin{pmatrix} \hat{H}_{xs} \\ \hat{}_{ys} \end{pmatrix} , \tag{81}$$

where W is the inverse matrix defined by

$$W = (I + \bar{Q})^{-1} \quad . \tag{82}$$

Substituting equation (81) into equation (78b) gives

$$\begin{pmatrix} B_1 \\ D_1 \end{pmatrix} = \bar{Q}_1 W \begin{pmatrix} \hat{H}_{xs} \\ \hat{H}_{ys} \end{pmatrix} = V \begin{pmatrix} \hat{H}_{xs} \\ \hat{H}_{ys} \end{pmatrix} \quad , \tag{83}$$

where V is the matrix

$$V = \bar{Q}_1 W \quad . \tag{84}$$

The essential point of equations (81) and (83) is that each of the coefficients, A_1, B_1, C_1, D_1, is a linear combination of $\hat{H}_{xs}$ and $\hat{H}_{ys}$.

In order to demonstrate how the foregoing result translates into the form of equation (3) let us work out a detailed demonstration for $\hat{H}_{zs}$. From equation (43) we have

$$\hat{H}_{zs} = \frac{1}{\lambda_1}[(i\kappa_x)(A_1 - B_1) + i\kappa_y(C_1 - D_1)] \quad . \tag{85}$$

Using equations (81) and (83) we have

$$\begin{aligned} A_1 &= W_{11}\hat{H}_{xs} + W_{12}\hat{H}_{ys} \quad &(a)\\ C_1 &= W_{21}\hat{H}_{xs} + W_{22}\hat{H}_{ys} \quad &(b)\\ B_1 &= V_{11}\hat{H}_{xs} + V_{12}\hat{H}_{ys} \quad &(c)\\ D_1 &= V_{21}\hat{H}_{xs} + V_{22}\hat{H}_{ys} \quad &(d) \end{aligned} \tag{86}$$

where W_{ij} and V_{ij} are the components of the respective matrices. Substituting equation (86) into equation (85) gives

$$\begin{aligned} \hat{H}_{zs}(s,\kappa_x,\kappa_y) &= \hat{T}_{zx}(s,\kappa_x,\kappa_y)\hat{H}_{xs}(s,\kappa_x,\kappa_y) \\ &\quad + \hat{T}_{zy}(s,\kappa_x,\kappa_y)\hat{H}_y(s,\kappa_x,\kappa_y) \quad , \end{aligned} \tag{87}$$

with $\hat{T}_{zs}$ and $\hat{T}_{zy}$ being given by

$$\hat{T}_{zx} = \frac{1}{\lambda_1}[i\kappa_x(W_{11} - V_{11}) + i\kappa_y(W_{21} - V_{21})] \qquad (a)$$

and

$$\hat{T}_{zy} = \frac{1}{\lambda_1}[i\kappa_x(W_{12} - V_{12}) + i\kappa_y(W_{22} - V_{22})] \quad . \qquad (b) \tag{88}$$

We interpret the $\hat{T}$'s as transfer coefficients. Using equation (86) in equations (44) and (46) at $z = 0$ gives analogous functional forms for $\hat{E}_{xs}$, $\hat{E}_{ys}$, and $\hat{E}_{zs}$.

The space-time relationships at the earth's surface are obtained by examining a typical term generated in the s, κ_x, κ_y space. If we let

$$\hat{Y} = \{\hat{H}_{zs}, \hat{E}_{xs}, \hat{E}_{ys}, \hat{E}_{zs}\} \quad , \qquad (a)$$

$$(89)$$

$$\hat{X} = \{\hat{H}_{xs}, \hat{H}_{ys}\} \quad , \qquad (b)$$

and

$\hat{T}$ = the representative transfer coefficient, (c)

then

$$\hat{Y}(\vec{\kappa},s) = \hat{T}(\vec{\kappa},s)\hat{X}(\vec{\kappa},s) \quad , \qquad (90)$$

where $\vec{\kappa} = (\kappa_x, \kappa_y)$. Taking the inverse Laplace transform of equation (90) gives

$$\hat{Y}^{(t)}(\vec{\kappa},t) = \int_0^t \hat{T}^{(t)}(\vec{\kappa},t - t')\hat{X}^{(t)}(\vec{\kappa},t')dt' \qquad (91)$$

(the superscript (t) indicates time space). We now write

$$Y^{(t)}(\vec{r}_s,t) = \int_{\vec{\kappa}} e^{i\vec{\kappa}\cdot\vec{r}_s}\hat{Y}^{(t)}(\vec{\kappa},t)d\vec{\kappa} \quad , \qquad (92)$$

and

$$\hat{X}^{(t)}(\vec{\kappa},t') = \left(\frac{1}{2\pi}\right)^2 \int_{\vec{r}_s'} e^{-i\vec{\kappa}\cdot\vec{r}_s}X^{(t)}(\vec{r}_s',t')d\vec{r}_s' \quad . \qquad (93)$$

Substituting equations (91) and (93) into equation (92) yields

$$Y^{(t)}(\vec{r}_s,t) = \int_0^t \int_{\vec{r}_s'} T(\vec{r}_s - \vec{r}_s',t - t')X^{(t)}(\vec{r}_s',t')d\vec{r}_s'dt' \quad , \qquad (94)$$

where

$$T(\vec{r}_s - \vec{r}_s' ,t - t') = \left(\frac{1}{2\pi}\right)^2 \int_{\vec{\kappa}} e^{i\vec{\kappa}\cdot(\vec{r}_s-\vec{r}_s')}T(\vec{\kappa},t - t')d\vec{\kappa} \quad . \qquad (95)$$

We thus complete the proof establishing the relationship between the field components at the earth's surface.

The utility of the formalism developed depends on the ease with which one can perform the inverse Laplace and Fourier transforms. In the next section we develop special-case solutions.

4. SPECIAL SOLUTIONS

In this section we shall apply the general theory outlined in section 3 to specific cases which appear to be of interest in HABEMP modelling.

4.1 One Layer/One Dimension

This problem has been studied before by numerous authors, but is repeated here because of its connection with the more complicated models considered in sections 4.2 and 4.3. Since we are dealing with only one layer, we have taken the liberty of dropping the subscript "1." Working from equations (41) through (46), we note that here we set

$$k_x = k_y = 0 \qquad \text{(a)}$$

$$B = D = 0 \qquad \text{(b)} \qquad (96)$$

$$\lambda = \sqrt{s\mu\sigma} \qquad \text{(c)}$$

and

$$\hat{H}_{xs} = A \qquad \text{(a)}$$

$$\hat{H}_{ys} = C \qquad \text{(b)}$$

$$\hat{H}_{zs} = 0 \qquad \text{(c)}$$

$$\hat{E}_{xs} = \frac{s\mu}{\lambda} C = \frac{s\mu}{\lambda} \hat{H}_{ys} \qquad \text{(d)} \qquad (97)$$

$$\hat{E}_{ys} = - \frac{s\mu}{\lambda} A = - \frac{s\mu}{\lambda} \hat{H}_{xs} \qquad \text{(e)}$$

$$\hat{E}_{zs} = 0 \qquad \text{(f)}$$

Since equations (97d) and (97e) are mathematically equivalent, we consider the solution for $\hat{E}_{xs}$ with $\sigma = \sigma_0 + \epsilon s$. We have

$$\hat{E}_{xs} = \frac{s\mu}{\sqrt{s\sigma\mu}} \hat{H}_{ys} = Z \left(\frac{1}{\sqrt{s}\ \sqrt{s + \omega_0}} \right) \left(s\hat{H}_{ys} \right) \quad , \qquad (98)$$

where

$$Z = \sqrt{\frac{\mu}{\varepsilon}} \quad , \qquad \text{(a)}$$

$$\omega_0 = \left(1/T_R\right) \quad , \qquad \text{(b)} \qquad (99)$$

and

$$T_R = \left(\varepsilon/\sigma_0\right) = \text{relaxation time} \quad . \qquad \text{(c)}$$

Taking the inverse Laplace transform of equation (98) yields

$$\hat{E}_{xs}^{(t)} = Z \int_0^t \left\{ e^{-\frac{\omega_0}{2}(t-t')} I_0\left(\frac{\omega_0}{2}(t-t')\right)\right\} \hat{\dot{H}}_{xs}^{(t)}(t')dt' \quad , \qquad (100)$$

where the superscript (t) denotes time-space (compared with "s" space), and the {} term is the inverse Laplace transform of $1/\sqrt{s}\left(\sqrt{s + \omega_0}\right)$; that is,

$$L^{-1}\left(\frac{1}{\sqrt{s}\sqrt{s + \omega_0}}\right) = e^{-\frac{\omega_0 t}{2}} I_0\left(\frac{\omega_0 t}{2}\right) \quad , \qquad (101)$$

where I_0 is the modified Bessel function.

Equation (100) is exact. In the limit where ω_0 is large compared to the frequencies in $\hat{\dot{H}}_{xy}$, we can replace I_0 by its asymptotic value:

$$I_0(\xi) \to \frac{e^{-\xi}}{\sqrt{2\pi\xi}} \quad , \quad \text{Lim } \xi \to \infty \qquad (102)$$

and we obtain

$$\hat{E}_{xs}^{(t)} = \sqrt{\frac{\mu}{\pi\sigma_0}} \int_0^t \frac{1}{\sqrt{t - t'}} \hat{\dot{H}}_{ys}^{(t)}(t')dt' \quad . \qquad (103)$$

Equation (103) (or eq (100)) can be used when the skin depth is less than the thickness for the first layer, and horizontal variations in the fields are assumed to be negligible.

4.2 One Layer/Two Dimensions

For this case we neglect variations in the y-direction (equivalent to neglecting variations in x) and thus set $\kappa_y = 0$. As in section 4.1, we again drop the subscript "1" notation. In this two-dimensional case we have

$$\lambda = \sqrt{s\mu\sigma + \kappa_x^2} \quad . \tag{104}$$

From equations (41) through (46) we deduce

$$\hat{H}_{zs} = \left(\frac{i\kappa_x}{\lambda}\right)\hat{H}_{xs} \quad , \qquad \text{(a)}$$

$$\hat{E}_{ys} = -\left(\frac{s\mu}{\lambda}\right)\hat{H}_{xs} \quad , \qquad \text{(b)} \tag{105}$$

and

$$\hat{E}_{zs} = \left(\frac{i\kappa_x}{\sigma}\right)\hat{H}_{ys} \quad , \qquad \text{(a)}$$

$$\hat{E}_{xs} = \frac{\left(s\mu\sigma + \kappa_x^2\right)}{\sigma\lambda}\hat{H}_{ys} = \left(\frac{s\mu}{\lambda}\right)\hat{H}_{ys} + \left(\frac{\kappa_x^2}{\sigma\lambda}\right)\hat{H}_{ys} \quad . \qquad \text{(b)} \tag{106}$$

Notice from equations (105) and (106) the decoupling of the general solution into two independent sets of solutions--those driven by $\hat{H}_{xs}$ (eq (105)), and those driven by $\hat{H}_{ys}$ (eq (106)).

The first step in converting equations (105) and (106) into time and space dependence is to take the inverse Laplace transform. Using

$$\sigma = \sigma_0 + s\varepsilon \tag{107}$$

and

$$\lambda = \sqrt{\mu\varepsilon s^2 + s\mu\sigma_0 + \kappa_x^2} \quad , \tag{108}$$

we can readily show [10]

$$\mathcal{L}^{-1}\left(\frac{1}{\sigma}\right) = \left(\frac{1}{\varepsilon}\right)e^{\omega_0 t'} = G_1(t') \quad , \tag{109}$$

$$\mathcal{L}^{-1}\left(\frac{1}{\lambda}\right) = \frac{1}{\sqrt{\mu\varepsilon}}\, e^{-\omega_0 t'/2}\, J_0\left[\left(\ell^2\kappa_x^2 - 1\right)^{1/2} \frac{\omega_0 t'}{2}\right] = G_2(\kappa_x, t') \quad , \tag{110}$$

and

$$L^{-1}\left(\frac{1}{\sigma\lambda}\right) = \int_0^{t'} G_1(t'-t'')G_2(t'')dt'' = G_3(\kappa_x,t') \quad , \tag{111}$$

where

$$G_3(\kappa_x,t') = \frac{e^{-\omega_0 t'}}{\varepsilon^{3/2}\mu^{1/2}} \int_0^{t'} e^{\frac{\omega_0 t''}{2}} J_0\left[(\ell^2\kappa_x^2 - 1)^{1/2} \frac{\omega_0 t''}{2}\right]dt'' \quad . \tag{112}$$

The parameters, ω_0 and ℓ, are given by

$$\omega_0 = (1/T_R) = \frac{\sigma_0}{\varepsilon} \quad , \quad T_R = \text{relaxation time} \quad , \tag{113}$$

and

$$\ell = 2\sqrt{\frac{\varepsilon}{\sigma_0^2\mu}} \quad . \tag{114}$$

The length, "ℓ," which is derived from the theory, has an interesting physical significance. It is proportional to the skin depth,

$$\delta = \sqrt{\frac{T}{\pi\mu\sigma_0}} \quad , \tag{115}$$

calculated for a period T equal to the relaxation time, T_R. Substituting T_R into equation (115) gives

$$\delta^* = \sqrt{\frac{\varepsilon}{\pi\mu\sigma_0^2}} = \frac{1}{2\sqrt{\pi}}\,\ell \quad . \tag{116}$$

Using the convolution theorem for the inverse Laplace transform, we now take the inverse Fourier transform of equations (105) and (106). We shall carry out the computation explicitly for equation (105a) and then state the resulting equations for equations (105b) and (106).

Once again using the superscript (t) to denote time space, there results

$$H_{zs}^{(t)} = \int_0^t \int_{\kappa_x} (i\kappa_x)G_2(\kappa_x,t-t')\hat{H}_{xs}^{(t)}(\kappa_x,t')e^{i\kappa_x x}d\kappa_x dt' \qquad (a)$$

(117)

$$= \frac{\partial}{\partial x}\int_0^t \int_{\kappa_x} G_2(\kappa_x,t-t')\hat{H}_{xs}^{(t)}(\kappa_x,t')e^{i\kappa_x x}d\kappa_x dt' \quad . \qquad (b)$$

Writing

$$\hat{H}_{xs}^{(t)}(\kappa_x,t') = \frac{1}{2\pi}\int_{x'} e^{-i\kappa_x x'} H^{(t)}(x',t')dx' \quad , \tag{118}$$

and substituting equation (118) in equation (117b) yields

$$H_{zs}^{(t)} = \frac{\partial}{\partial x}\int_0^t \int_{x'} T(x-x',t-t')H_{xs}^{(t)}(x',t')dx'dt' \quad , \tag{119}$$

where

$$T(x-x',t-t') = \left(\frac{1}{2\pi}\right)\int_{\kappa_x} e^{i\kappa_x(x-x')} G_2(\kappa_x,t-t')d\kappa_x \quad . \tag{120}$$

Using the foregoing techniques, we can write

$$E_{ys}^{(t)} = -\mu\int_0^t \int_{x'} T(x-x',t-t')\dot{H}_{xs}^{(t)}(x',t')dx'dt' \quad , \tag{121}$$

$$E_{zs}^{(t)} = \int_0^t G_1(t-t')\left(\frac{\partial H_{ys}^{(t)}(t')}{\partial x}\right)dt' \quad , \tag{122}$$

and

$$\begin{aligned} E_{xs}^{(t)} = {} & \mu\int_0^t \int_{x'} T(x-x',t-t')\dot{H}_{ys}^{(t)}(x',t')dx'dt' \\ & - \frac{\partial^2}{\partial x^2}\int_0^t \int_{x'} T^*(x-x',t-t')H_{ys}^{(t)}(x',t')dx'dt' \quad , \end{aligned} \tag{123}$$

where

$$\begin{aligned} T^* &= \left(\frac{1}{2\pi}\right)\int_{\kappa_x} e^{i\kappa_x(x-x')} G_3(\kappa_x,t-t')d\kappa_x \\ &= \int_0^{t-t'} G_1(t-t'-t'')T(x-x',t'')dt'' \quad . \end{aligned} \tag{124}$$

It is evident from the foregoing analysis that the fundamental building block in the calculations is

$$T(\Delta x,\tau) = \frac{1}{\sqrt{\mu\varepsilon}} e^{-\omega_0\tau/2} \left(\frac{1}{2\pi}\right) \int_{\kappa_x} e^{i\kappa_x\Delta x} J_0\left[\left(\ell^2\kappa_x^2 - 1\right)^{1/2} \frac{\omega_0\tau}{2}\right] d\kappa_x \quad ; \tag{125}$$

Because $J_0[\ ]$ is an even function of κ_x, the integral in equation (125) becomes

$$F = \int_{\kappa_x} e^{i\kappa_x\Delta x} J_0[\cdot]\, d\kappa_x = 2\int_0^\infty \cos(\kappa_x\Delta x) J_0[\cdot] d\kappa_x \quad . \tag{126}$$

$$\begin{aligned} F = {} & 2\int_0^{\ell^{-1}} \cos(\kappa_x\Delta x) I_0\left[\left(1 - \kappa_x^2\ell^2\right)^{1/2} \frac{\omega_0\tau}{2}\right] d\kappa_x \\ & + 2\int_{\ell^{-1}}^\infty \cos(\kappa_x\Delta x) J_0\left[\left(\ell^2\kappa_x^2 - 1\right)^{1/2} \frac{\omega_0\tau}{2}\right] d\kappa_x \quad . \end{aligned} \tag{127}$$

Using equation (127) we find a number of relatively simple (e.g., power series, asymptotic expansions) methods one could use to calculate F exactly. However, it is easy to show that for all cases which appear to be of interest, the upper limit of ℓ (computed from eq (114)) is less than 1 km. Since ℓ is then smaller than the grid spacing for numerical computation, the contribution from the $J_0[\cdot]$ integral of equation (127) is negligible.

One may conjecture that for a large number of situations the spatial variations of the fields would be such that

$$\kappa_x^2\ell^2 < 1 \quad . \tag{128}$$

In this case we can write

$$\left(1 - \kappa^2\ell^2\right)^{1/2} \approx 1 - \frac{1}{2}\kappa_x^2\ell^2 \quad . \tag{129}$$

A further simplification of equation (127) occurs when we assume

$$\frac{\omega_0\tau}{2} = \frac{\tau}{2T_R} \gg 1 \quad . \tag{130}$$

Most of the values of T_R which will arise are less than 1 μs; in certain cases, T_R will only be a few microseconds. Thus, for the times of interest in this study, equation (130) is valid. Under these conditions we can use the asymptotic expression for I_0 (see eq (102)) and obtain

$$I_0 \approx \frac{1}{\sqrt{2\pi(\omega_0\tau/2)}} e^{(\omega_0\tau/2)} e^{-(\omega_0\tau/2)[\kappa_x^2 \ell^2/2]} \quad . \tag{131}$$

Using equation (128) in equation (127) and approximating ℓ^{-1} by ∞ (an excellent approximation when σ_0 is large) gives

$$F \approx \frac{e^{\omega_0\tau/2}}{\sqrt{\pi\omega_0\tau}} \sqrt{\frac{\pi\mu\sigma_0}{\tau}} \, e^{-\left(\frac{\Delta x^2 \mu\sigma_0}{4\tau}\right)} \quad . \tag{132}$$

Substituting equation (129) into equation (125) yields

$$T(\Delta x,\tau) = \left(\frac{1}{2\pi\tau}\right) e^{-\left(\frac{\Delta x^2 \mu\sigma_0}{4\tau}\right)} \quad . \tag{133}$$

When we substitute $\Delta x = x-x'$ and $\tau = t-t'$ into equation (130) and insert the resulting expression into equations (119), (121), and (123), we deduce the required space-time relationships between the field variables. For example, equation (121) becomes

$$E_{ys}^{(t)} = \frac{-\mu}{2\pi} \int_0^t \int_{x'} \frac{1}{t-t'} e^{-\left[\frac{(x-x')^2\mu\sigma_0}{4(t-t')}\right]} \dot{H}_{xs}^{(t)}(x',t')dx'dt' \quad . \tag{134}$$

When $\dot{H}_{xs}$ is space-independent, the resultant relationship reduces to the case presented in section 4.1.

4.3 One Layer/Three Dimensions

Virtually all the mathematical steps performed in the two-dimensional case are repeated here; the major difference is the use of the two-dimensional space transform (κ_x,κ_y) instead of only κ_x. As in the previous section, closed-form expressions for the $T(x - x',y - y',t - t')$ functions are obtained when "ℓ" is smaller than the mesh spacing, and for times of interest which are greater than the relaxation time, T_R. For brevity, we give the results in two representative cases. If we let

$$\Gamma(x - x',y - y',t - t') = e^{-\frac{\mu\sigma}{4(t-t')}[(x - x')^2 + (y - y')^2]} \quad , \tag{135}$$

we obtain

$$H_{zs}^{(t)}(x,y,t) = \sqrt{\pi\mu\sigma_0} \int_0^t \frac{dt'}{(t-t')^{3/2}} \left[\frac{\partial}{\partial x} \int_{x'} \int_{y'} \Gamma H_{xs}^{(t)}(x',y',t')dx'dy' + \frac{\partial}{\partial y} \int \int \Gamma H_{xs}^{(t)}(x',y',t')dx'dy'\right] \quad (136)$$

and

$$E_{xs}^{(t)}(x,y,t) = \mu^{3/2} \sqrt{\pi\sigma} \int_0^t \frac{dt'}{(t-t')^{3/2}} \left[\int_{x'} \int_{y'} \Gamma \dot{H}_{ys}^{(t)}(x',y',t')dx'dy'\right] + \sqrt{\frac{\pi\mu}{\sigma}} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \int_0^t \frac{dt'}{(t-t')^{3/2}} \left[\int_{x'} \int_{y'} \Gamma H_{ys}^{(t)}(x',y',t')dx'dy'\right] . \quad (137)$$

Similar expressions may be written for $E_{zs}^{(t)}$ and $E_{ys}^{(t)}$. However, in all cases the analytical forms can be incorporated into numerical methods.

4.4 Two-Layers/One Dimension

For many situations, particularly involving one or more highly conducting layers in the upper regions of the crust, the relaxation times will be small, the relative depth of penetration will be shallow, and the importance of horizontal variations will be negligible. In this situation, the one-dimensional set of equations applies, and we have

$$\begin{aligned}
\hat{H}_{x1} &= A_1 e^{-\lambda_1 z} + B_1 e^{+\lambda_1 z} && \text{(a)} \\
\hat{E}_{y1} &= -\frac{\lambda_1}{\sigma_1} A_1 e^{-\lambda_1 z} + \frac{\lambda_1}{\sigma_1} B_1 e^{\lambda_1 z} && \text{(b)} \\
\hat{H}_{xs} &= A_1 + B_1 && \text{(c)} \\
\hat{E}_{ys} &= -\frac{\lambda_1}{\sigma_1}(A_1 - B_1) && \text{(d)} \\
\hat{H}_{xz} &= A_2 e^{-\lambda_2 z} && \text{(e)} \\
\hat{E}_{yz} &= -\frac{\lambda_2}{\sigma_2} A_2 e^{-\lambda_2 z} && \text{(f)} \\
\sigma_1 &= \text{constant} && \text{(g)} \\
\sigma_2 &= \text{constant} && \text{(h)} \\
\lambda_1 &= \sqrt{s\mu_1\sigma_1} && \text{(i)} \\
\lambda_2 &= \sqrt{s\mu_2\sigma_2} && \text{(j)}
\end{aligned} \quad (138)$$

Matching boundary conditions at $z = \ell_1$ gives

$$A_1 e^{-\lambda_1 \ell_1} + B_1 e^{\lambda_1 \ell_1} = A_2 \qquad \text{(a)} \tag{139}$$

and

$$-\frac{\lambda_1}{\sigma_1}\left(A_1 e^{-\lambda_1 \ell_1} + B_1 e^{\lambda_1 \ell_1}\right) = -\frac{\lambda_2}{\sigma_2} A_2 \quad . \qquad \text{(b)}$$

There results

$$B_1 = e^{-2\lambda_1 \ell_1} \xi A_1 \quad , \tag{140}$$

where

$$\xi = \left(\frac{1 - f}{1 + f}\right) \tag{141}$$

and

$$f = \sqrt{\frac{\mu_2 \sigma_1}{\mu_1 \sigma_2}} \quad . \tag{142}$$

Using equation (139) in equations (138c,d) gives

$$\hat{E}_{ys} = \hat{E}_\infty \left(\frac{1 - \xi e^{-2\lambda_1 \ell_1}}{1 + \xi e^{-2\lambda_1 \ell_1}}\right) \quad , \tag{143}$$

where $\hat{E}_\infty$ is the solution for an infinite layer:

$$\hat{E}_\infty = -\frac{\lambda_2}{\sigma_2} \hat{H}_{xs} \quad . \tag{144}$$

Using the expansion

$$\frac{1}{1 + y} = \sum_{n=0}^{\infty} (-1)^n y^n \quad , \tag{145}$$

combined with algebraic manipulation gives the result

$$\hat{E}_{ys} = \hat{E}_\infty + \sum_{m=1}^{\infty} \hat{E}_m \quad , \tag{146}$$

where

$$\hat{E}_m = Q_m \hat{E}_\infty \tag{147}$$

and

$$Q_m = 2(-1)^m \xi^m e^{-2m\lambda_1 \ell_1} \quad . \tag{148}$$

The time behavior is given by

$$\hat{E}_{ys}(t) = \hat{E}(t) + \sum_{m=1}^{\infty} \hat{E}_m(t) \quad , \tag{149}$$

where $\hat{E}_m(t)$ is determined from the inverse Laplace transform of equation (146). It is convenient to organize the inverse transform of equation (147) by rearranging it as

$$\hat{E}_m = \left(\frac{Q_m}{s}\right)(s\hat{E}_\infty) \quad . \tag{150}$$

We have

$$L^{-1}\left(\frac{Q_m}{s}\right) = 2(-1)^m \xi^m \mathrm{Erfc}\left(m\sqrt{\frac{\tau_d}{\tau}}\right) \quad , \tag{151}$$

where τ_d is the diffusion time to traverse the top layer; we have

$$\tau_d = \mu_1 \sigma_1 \ell_1^2 = \text{diffusion time} \quad . \tag{152}$$

In most cases of interest $\tau_d/\tau >> 1$, and we can use the asymptotic expansion for Erfc. For large θ we have

$$\mathrm{Erfc}(\theta) \to 1 - \frac{e^{-\theta^2}}{\theta\sqrt{2\pi}}\left(1 - \frac{1}{2\theta^2} + \ldots\right) \quad , \tag{153}$$

which when applied for $\theta = m\sqrt{\tau_d/t}$ yields

$$\mathrm{Erfc}\left(m\sqrt{\tau_d/t}\right) = \frac{e^{-m^2\left(\frac{\tau_d}{t}\right)}}{m\sqrt{\frac{\tau_d}{t}}\sqrt{2\pi}} \quad . \tag{154}$$

Using equation (150) and applying the convolution theorem gives

$$\hat{E}_m(t) = \frac{2(-1)^m \xi^m}{\sqrt{2\pi}\, m} \int_0^t \left(\frac{\tau_d}{t}\right)^{-1/2} e^{-m^2 \tau_d/\tau'} \, \hat{\dot{E}}_\infty(t)(t-t')dt' \quad . \qquad (155)$$

Equation (151) appears to be a rapidly diminishing function of "m" so that perhaps only a few terms would be required to calculate $\hat{E}_{ys}(t)$ as given by equation (149).

5. CONCLUSION

In this investigation we have developed a mathematical proof that for a layered earth model--including three-dimensional variations in the electromagnetic fields and frequency dependence of electrical parameters--it is possible to express the three components of the electric field and the vertical component of the magnetic field on the surface of the earth as a space-time integration of the two horizontal components of the magnetic field. This result would appear to considerably simplify the numerical modelling of the HABEMP when the ground response is coupled to finite-difference methods for solving the atmospheric part of the problem. Special-case solutions are developed which include the effect of coarse-graining due to finite-difference approximations.

In addition, we have provided examples of realistic models of crustal conductivity-vs-depth profiles which could be used in various geometric locations.

Further exploitation of approximations based on the developed formalism would appear to offer a greater range of application of the technique.

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