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Modeling a Lens with Dielectric Constant Inversely Proportional to Radius Squared

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Abstract

A lens design procedure, with constant wavelength to cross section ratio as $\varepsilon_r$ increases, is used to obtain better focusing at the second focal point of a prolate-spheroidal IRA. It is shown that the lens behavior can be analyzed by a correspondence to a lens behaving like an exponential transmission line.
1 Introduction

In [3,4] the analytical, numerical and experimental focal waveforms of a prolate spheroidal IRA are presented. Early papers [1,2] discussed addition of a lens at the second focal point of a prolate spheroidal IRA for better concentrating at the second focal point. Figure 1 shows this lens geometry and \( z \) is the transmission-line coordinate.

\[ \varepsilon_r = 1 \]

\[ \varepsilon_r(r_{\text{max}}) = 1 \]

plane-wave approximation and transmission-line-approximation region

\[ z = 0 \]

\[ z = r_{\text{max}} - r_{\text{min}} \]

\[ z = r_{\text{max}} \]

Figure 1. Lens Geometry

\( \varepsilon_r(r) \) is the dielectric constant of the tapered lens [5,2]. For \( r_{\text{min}} < r < r_{\text{max}} \) region we use the transmission line results to estimate peak and droop of wave reaching \( r_{\text{min}} \). Inside \( r_{\text{min}} \) the dielectric constant can be defined as

\[ \varepsilon_r = \varepsilon_r(r_{\text{min}}) \quad 0 < r < r_{\text{min}} \], \quad (1) \]

And this is a focusing region for which the transmission-line results do not apply however the spot size analysis in [3] can be applied.
2 Analytical Calculations for $r_{min} < r < r_{max}$ Region

Let’s assume $\lambda \propto r$ in the $\epsilon_r(r)$ medium in which we have a converging wave toward $r = 0$. One can define $\lambda$ as the wavelength for some characteristic frequency $f$ in the pulse, giving

$$f\lambda = v, \quad v = c \epsilon_r(r),$$

$$f \propto 1/\Delta t, \text{ a constant}$$

where $v$ is the speed of propagation of the lens. $c$ is the speed of light in the free space and $\Delta t$ is the pulse width (or time of interest after the initial pulse “step” rise). One uses these approximations

$$\lambda \propto r,$$

$$v \propto \lambda / \Delta t \propto r / \Delta t.$$  (3)

Then $v$ can be defined as

$$v = c \epsilon_r(r) = Cr.$$  (4)

From (4) $\epsilon_r(r)$ can be found as

$$\epsilon_r(r) = \frac{c}{Cr},$$

$$\epsilon_r(r) = \frac{D}{r^2} = \left[ \frac{r_{max}}{r} \right]^2.$$  (5)

The speed of propagation is

$$v(r) = c/\sqrt{\epsilon_r(r)} = cr / r_{max},$$  (6)

the ratio can be found as

$$c / v = r_{max} / r.$$  (7)

3 Changing the Radial Coordinate

We want to create a new coordinate $\zeta$ in which we want to make the wave propagation speed equal to the speed of light $c$ locally as

$$\frac{d\zeta}{dt} = c \quad \text{for } r_{min} < r < r_{max}.$$  (8)

Integrating (8) gives us
\[ \zeta = ct, \quad (9) \]

where \( \zeta \) is the equivalent spatial coordinate or equivalent transmission-line coordinate. \( t \) is the time for the wave to reach \( \zeta \) we have in our \( \varepsilon_r \) varying medium

\[ \frac{dr}{dt} = v. \quad (10) \]

By substituting (10) in (8) and integrating it one can obtain

\[ \zeta = \int_{r}^{r_{\text{max}}} \frac{c}{v} dr > [r_{\text{max}} - r], \quad (11) \]

slower velocity implies \( \zeta > [r_{\text{max}} - r] \) except as \( \zeta \to 0, v \to c \).

One can substitute (7) in (10) as

\[ \zeta = \int_{r}^{r_{\text{max}}} \frac{r_{\text{max}}}{r'} dr' = r_{\text{max}} \ln(r') \bigg|_{r_{\text{max}}}^{r_{\text{max}}} = r_{\text{max}} \ln \left( \frac{r_{\text{max}}}{r} \right) \quad (12) \]

(12) shows that as \( r \to 0, \zeta \to \infty \) therefore our lens is spatially limited as in [2].

The ratio of \( r / r_{\text{max}} \) can be found as

\[ \frac{r}{r_{\text{max}}} = e^{-\zeta / r_{\text{max}}}. \quad (13) \]

Note that \( r / r_{\text{max}} \) has an exponential behavior as \( \zeta \to \infty, r \to 0 \).

Let us change the transmission-line coordinate as

\[ z = r_{\text{max}} - r \]

\[ \frac{z}{r_{\text{max}}} = 1 - \frac{r}{r_{\text{max}}} \quad (14) \]

as \( \varepsilon_r \to \infty \) \( r_{\text{max}} \) is like the length of the transmission line. One can find \( z \) as a function of \( \zeta \) and \( r_{\text{max}} \) as

\[ z = r_{\text{max}} \left( 1 - e^{-\zeta / r_{\text{max}}} \right) \quad (15) \]
4. Relation to the Telegrapher Equations

The wave propagation can be described by the source-free telegrapher equations [5(2.3)] as

\[ \frac{d}{dz} \tilde{V}(z,s) = -\tilde{Z}'(z,s) \tilde{I}(z,s) \]
\[ \frac{d}{dz} \tilde{I}(z,s) = -\tilde{Y}'(z,s) \tilde{V}(z,s) \]

where \( \tilde{\cdot} \) shows the two-sided Laplace transform over time \( t \) and \( s = \Omega + j \omega \) is the complex frequency. For transmission line assumed lossless with \( \tilde{Z}'(z,s) = sL'(z) \) where \( L'(z) \) is the inductance per unit and \( \tilde{Y}'(z,s) = sC'(z) \) where \( C'(z) \) is the capacitance per unit length. Under the assumption that the tapered transmission line consists of perfect conductors with lossless dielectric one can define \( L'(z) \) and \( C'(z) \) as

\[ L'(z) = \mu f_g(z), \quad L'(z) = \varepsilon / f_g(z) \]

(17)

where \( f_g(z) \) is geometric impedance factor and it is

\[ f_g(z) = e^{-\zeta/r_{max}}. \]

(18)

We have an analogy between

\[ V \rightarrow E, I \rightarrow H. \]

(19)

Substituting (17) and (18) in (16) one can transform the 1D wave equation to an equivalent \( \zeta \) space coordinate using (15) as

\[ \frac{dV(\zeta,s)}{d\zeta} = -s\mu_0 e^{-\zeta/r_{max}} I, \]
\[ \frac{dI(\zeta,s)}{d\zeta} = -s\varepsilon_0 e^{\zeta/r_{max}} V. \]

(20)

This result is similar as the previous spatially limited case [2(4.10)] and both of the results have the same exponential behavior. Therefore we can use these results for a spatially limited lens.

We can use the exact solution of the transfer function neglecting the reflection from the beginning of the lens in [5(A.8)] as
\[
\tilde{T} = e^{S+G} \left[ \cosh \left( S^2 + G^2 \right)^{1/2} + \frac{S}{(S^2 + G^2)^{1/2}} \sinh \left( S^2 + G^2 \right)^{1/2} \right]^{-1}
\]

(21)

where S is the normalized complex frequency

\[
S = s t_{\xi_{\text{max}}} = (\Omega + j \omega) t_{\xi_{\text{max}}},
\]

(22)

where \( t_{\xi_{\text{max}}} \) is the transit time through lens. The high-frequency gain is defined in [5(3.4)] as

\[
\frac{V(r_{\text{min}})}{V_0} = g = e^{G} = \sqrt{\frac{Z_2}{Z_1}} = \varepsilon_r(r_{\text{min}})^{-1/4} = e^{-\xi_{\text{max}}/(2r_{\text{max}})}
\]

(23)

This is actually a decrease which will be overcome by the wave convergence in the lens. One can also define \( I(r_{\text{min}})/I_0 \) and the wave impedance at \( \xi = \xi_{\text{max}} \) or \( r = r_{\text{min}} \) as

\[
\frac{I(r_{\text{min}})}{I_0} = 1/g = e^{-G} = \sqrt{\frac{Z_1}{Z_2}} = \varepsilon_r(r_{\text{min}})^{1/4} = e^{\xi_{\text{max}}/(2r_{\text{max}})}
\]

\[
\frac{V(r_{\text{min}})}{I(r_{\text{min}})} = Z_w(r_{\text{min}}) = \frac{Z_0}{\varepsilon_t(r_{\text{min}})^{1/2}}
\]

(24)

Note that as one goes down in frequency this also neglects any reflection (small) at \( r = r_{\text{min}} \) where the lens goes into a constant \( \varepsilon_r(r_{\text{min}}) \) region.

5. Relation the Telegrapher Equations to the Fields

Figure 2a and 2b show the transmission-line representation of the fields in the lens. In this case as the wave is focused toward \( r = 0 \), the height \( h \) and the width \( w \) of a transmission line representing an incremental portion in a spherical cross section of the lens are decreasing toward \( r = 0 \) as

\[
h \propto r / r_{\text{max}},
\]

\[
w \propto r / r_{\text{max}},
\]

(25)

where \( h/w \) is independent of \( r \).
The fields in the transmission line are thus proportional to $V$ and $I$ as

$$E \propto V \frac{r_{\text{max}}}{r}, \quad H \propto I \frac{r_{\text{max}}}{r}.$$  \hfill (26)

So $E$ and $H$ are increasing relative to $V$ and $I$ as the wave propagates through the lens.

At the beginning of the lens, at early time, we have
\[ \frac{E_0}{H_0} = Z_0, \]

In the \( r_{\text{min}} < r < r_{\text{max}} \) region we have, at early time,

\[ \frac{V}{V_0} = \frac{E}{E_0} \frac{r}{r_{\text{max}}}, \quad \frac{I}{I_0} = \frac{H}{H_0} \frac{r}{r_{\text{max}}}. \] \tag{27}

On the wave front the wave impedance is

\[ Z_w(r) = \frac{V(r)}{I(r)} = \frac{Z_0}{\varepsilon_r(r)^{1/2}}. \] \tag{28}

Power in the transmission-line, \( V_0I_0 \), and on the wavefront, \( VI \), are, therefore, equal because of the conservation of energy with no high-frequency reflection, giving

\[ VI = V_0I_0, \] \tag{29}

The power in an equivalent transmission line is converging toward the focus. The power density, at early times, in the lens is

\[ VI \left( \frac{r_{\text{max}}}{r} \right)^2, \] \tag{30}

consistent with (27).

In our case as indicated in Figures 2a and 2b the distance is decreasing. Therefore, we have higher a electric field in a smaller region. This statement also can be made by conservation of energy in the wavefront. The wave impedance in the \( r_{\text{min}} < r < r_{\text{max}} \) region is

\[ \frac{E}{H} = \frac{Z_0}{\varepsilon_r^{1/2}} = Z_0 e^{-\xi/r_{\text{max}}}. \] \tag{31}

This lens behavior is the same as that for a spatially limited lens [2(Section 4)]. Therefore, one can use the same equations for this lens.

One can define the transit, normalized and droop time [5(A.11)] parameters as follow
\[ t_{\zeta_{\text{max}}} \equiv \frac{\zeta_{\text{max}}}{c} \quad \text{transit time through lens} \]
\[ \tau \equiv \frac{t}{t_{\zeta}} \quad \text{normalized time} \]
\[ \tau_d = 2 \ln^{-2}(g) \quad \text{normalized droop time} \]
\[ = \frac{t_d}{t_{\zeta_{\text{max}}}} \quad (32) \]

\( t_d \) is the droop time. While this is quite accurate for an exponential transmission-line [5], it is only approximate here. This is due to the approximate number of wavelengths across the spherical surface of radius \( r \) through which the wave is propagating. However, we are dealing with a pulse for which going to later times involves lower frequencies, and therefore, larger wavelengths, making the approximation less valid.

The electric-field gain, step response quality factor for lens improvement, can be defined as

\[ g_E = \frac{E_{\text{out}}}{E_0} = \epsilon_r^{1/2} g_V e^{-\Delta t/t_d} u(t) \approx \epsilon_r^{1/2} g_V \left(1 - \Delta t / t_d\right) u(t) \]
\[ \approx \epsilon_r^{1/4} \left(1 - \Delta t / t_d\right) u(t). \quad (33) \]

where \( \epsilon_r^{1/2} \) is the wave convergence factor (defined as enhancement factor \( F_0 \) in [1]), \( \Delta t \) is pulse width, \( e^{-\Delta t/t_d} \) is the droop, \( g_V e^{-\Delta t/t_d} \) is the high-frequency transmission-line gain for early times, \( g_V = \epsilon_r^{-1/4} \) and \( E_0 \) is the electric field at the focal point when the lens is not there.

One can define \( t_d \) from (32) and (13) as

\[ t_d = -\frac{r_{\text{max}}}{c} \ln \left(\frac{r_{\text{min}}}{r_{\text{max}}} \right) 2 \ln^{-2}(\epsilon_r^{-1/4}) \quad (34) \]

In [6], the numerical simulations show us when the outer radii of the lenses are \( r_{\text{max}} = 15 \text{cm and 24 cm} \), we obtain acceptable focusing at the focal point. The outer and inner radii and droop times calculated from (34) for \( \epsilon_r = 9 \) and 81 are presented in Table 1.

<table>
<thead>
<tr>
<th>( r_{\text{max}} ) (meters)</th>
<th>0.15</th>
<th>0.24</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon_r )</td>
<td>9</td>
<td>81</td>
</tr>
<tr>
<td>( r_{\text{min}} ) (meters)</td>
<td>0.05</td>
<td>0.017</td>
</tr>
<tr>
<td>( t_d ) (ns)</td>
<td>3.6</td>
<td>1.8</td>
</tr>
</tbody>
</table>

One can see from (33) and Table 1 that higher dielectric constant and larger radius give smaller droop times, while obtaining higher electric-field gain.
6. Conclusion

In this paper we start with a lens design with constant wavelength to cross section ratio. $\varepsilon_r$ varies [8] and we come up with a spatially limited lens [2(Section 4)]. A lens is designed in which we want to make the wave propagation speed in $r_{min} < r < r_{max}$ region equal to the speed of light $c$ in an equivalent $\zeta$ coordinate system. The 1D wave equation is transformed to an equivalent $\zeta$ coordinate and this equation has the same exponential behavior as a spatially limited lens [2(Section 4)]. The region $0 < r < r_{min}$ is a focusing region for which the transmission-line results do not apply however the spot size analysis in [3] can be applied.

Higher dielectric constant and larger radius give smaller droop times, while obtaining higher electric-field gain. Even though $t_d$ is one parameter for lens design one should also consider other parameters such as loss and dispersion. More realistic results can be obtained from experiments.

References