Lecture 7: Variance & conditional pmf

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Example to review

Problem 3.18
A computer reserves a path in a network. To extend the reservation the computer must successfully refresh message before the expiry time. The messages are lost with probability $\frac{1}{2}$. Suppose that it take 10 sec. to send a refresh message and receive acknowledgement. When should the computer start sending a refresh messages in order to have a 99% chance of successfully extending the reservation time.
• Let $X$ be the times that a refresh request needs to be sent till successfully received.
  • $S_X$ is \{1, 2, 3, \ldots\}
• Let $Y$ be the seconds that a refresh request needs to be sent till successfully received.
  • $S_Y$ is \{10, 20, 30, \ldots\}
• Success rate of a message is $1/2$

What is the set $B$, which begins with 1 and increases accordingly, has 99% chance of successfully extending reservation?
pmf of X?

\[ p_x(x_k) = p(x = K) = \left(\frac{1}{2}\right)^{k-1} \cdot \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^k \]

Set B probability?

\[ P[B] = \sum_{k \in B} p(x_k) = \sum_{k=1}^N \left(\frac{1}{2}\right)^k \]

P[B]>99%

\[ p[B] \geq \begin{pmatrix} 9 & 9 \\ 1 & 0 & 0 \end{pmatrix} \]

\[ \sum_{k=1}^N \frac{1}{2}^k \geq \begin{pmatrix} 9 & 9 \\ 1 & 0 & 0 \end{pmatrix} \]

\[ \begin{pmatrix} 0 \cdot 0.5^{n} \\ 1 - 0 \cdot 0.5^{n} \end{pmatrix} \]

\[ \begin{pmatrix} 1 \\ 1 & 0 & 0 \end{pmatrix} \geq \begin{pmatrix} 0 \cdot 0.5^{n} \\ 1 - 0 \cdot 0.5^{n} \end{pmatrix} \]

\[ n \geq 7 \]
• N=7
• 70 seconds before expiry time
• What is the expected time (seconds) that it takes to renew the reservation?

\[
E_7[Y] = \sum_{k=10}^{70} y \cdot p_X(X_k)
\]

\[
= 10 \cdot \left(\frac{1}{2}\right)^1 + 20 \cdot \left(\frac{1}{2}\right)^2 + 30 \cdot \left(\frac{1}{2}\right)^3 + \ldots + 70 \cdot \left(\frac{1}{2}\right)^7
\]

\[
= 19.30
\]
Quiz 3
Reading

- This class: Section 3.3, 3.4
- Next class: Section 3.5
Outline

• Variance of a random variable
• Moment of a random variable
• Conditional Probability mass function

• Samples
Random variable X and Y
Histogram of variable $X$ and $Y$
• Variance

\[ \sigma^2 = \text{VAR}[X] = E \left[ (X - m_X)^2 \right] \]
where, \( m_X \) is the expected value of \( X \).

\[ \sigma^2 = E[(X - m_X)^2] = \sum_{x \in S_X} (x-m_X)^2 \cdot p_x(x) \]

• Standard deviation

\[ \sigma = \text{STD}[X] = \text{VAR}[X]^{1/2} \]
\[ \text{VAR}[X] = \mathbb{E} \left[ (x-m_x)^2 \right] \]
\[ = \mathbb{E} \left[ x^2 - 2xm_x + m_x^2 \right] \]
\[ = \mathbb{E}[x^2] - 2m_x\mathbb{E}[x] + m_x^2 \]
\[ = \mathbb{E}[x^2] - m_x^2 \]

\( \mathbb{E}[x^2] \) is the second moment of \( X \)

\( \mathbb{E}[x^n] \) is the \( n \)th moment of \( X \)
Example 1

- Uniform random variable
- \( \text{VAR}[X] = ? \)
- \( \text{STD}[X] \)

\[ E[X] = 3.5 \]

\[ \text{VAR}[X] = \sum_{k=1}^{6} (k - 3.5)^2 \cdot \frac{1}{6} = 2.9167 \]

\[ \text{VAR}[X] = \left( \sum_{k=1}^{6} k^2 \cdot \frac{1}{6} \right) - 3.5^2 = \frac{91}{6} - 12.25 = 2.9167 \]

\[ \text{STD}[X] = 1.707 \]
Example2

• Let X be the number of heads in three tosses of a fair coin

• $\text{VAR}[X] = \text{E}[(X - m_x)^2] = \frac{3}{4}$

• $\text{VAR}[X] = \text{E}[X^2] - m_x^2 = \frac{3}{4}$
1) \(X = c\)

\[\text{VAR}[X] = ? \quad \text{E}[(x-c)^2] = 0\]

2) \(Y = X + c;\)

\[\text{VAR}[Y] = \text{VAR} [X + c] = \text{E} \left[ (x + c - (\text{E}[X] + c))^2 \right] \]

\[= \text{E} \left[ (x - \text{E}[X])^2 \right] \]

\[= \text{E} \left[ (x - m_x)^2 \right] \]

\[= \text{VAR}[X] \]

3) \(Y = cX;\)

\[\text{VAR}[Y] = \text{VAR} [cX] = \text{E} \left[ (cx - cE[X])^2 \right] \]

\[= \text{E} \left[ c^2 (x - E[X])^2 \right] \]

\[= c^2 \text{E} \left[ (x - mx)^2 \right] \]

\[= c^2 \text{VAR}[X] \]
Conditional probability mass function

• Let $X$ be a discrete random variable with a pmf $p_X(x)$. Let $C$ be an event with $P[C] > 0$. The conditional probability mass function of $X$: 

$$p_X(x|C) = \frac{P[ X=x \mid C]}{P[C]}$$
\[ S \]

\[ A_k \]

\[ C \]

\[ p_X(x_k) \]

\[ p_X(x_k|C) \]
Example 3

Residual waiting time

Let $X$ be the time required to send a message, where $X$ is the a uniform random variable with $S_x \{1,2,3,\ldots,L\}$. Suppose a message has been transmitting for $m$ time units, find the probability that remaining transmission time is $j$ time units.
- Given $X > m$; Let $C = \{X > m\}$
- $P[X = m+j]; \ L \geq m+j > m$

$p_{x}(X = m+j \mid C)$

$$p_{x}(x = m+j \mid C) = \frac{p(x = m+j \cap x > m)}{p[C]} = \frac{1/L}{(L - m)/L} = \frac{1}{L - m}$$
Example 4 - Device lifetime

- Type 1 (a): $(1-r)^{k-1}r$

- Type 2 (1-a): $(1-s)^{k-1}s$
\[ p_X(x) = \sum_{i=1}^{n} p_X(x \mid B_i) p[B_i] \]

\[ \sum_{i=1}^{n} B_i = S_X; B_i \text{ are disjoint events} \]

\[ p_X(x) = p_X(x_k \mid B_1) p[B_1] + p_X(x_k \mid B_2) p[B_2] \]
\[ = (1-r)^{k-1} r\alpha + (1-s)^{k-1} s(1-\alpha) \]
Conditional expected value

- $E[X|B]$

$$E[X|B] = \sum_{x \in S_x} x \cdot p_X(x|B)$$
Conditional variance

- $\text{VAR}[X|B]$

\[
\text{VAR}[X \mid B] = E[(X - m_{x|B})^2 \mid B] \\
= \sum_{k=1}^{\infty} (x_k - m_{x|B})^2 \cdot p_X(x_k \mid B) \\
= E[X^2 \mid B] - m_{x|B}^2
\]
HW 3

• problems 3.11, 3.20, 3.21, 3.31

• Due date: Feb. 20 at the beginning of class