Part I. From the text: Chapter 1: Problems 7, 12, 15, 36 and 37

Part II. Additional exercises:

Exercise 1.
Let $\mathcal{B}$ be the collection of all Borel sets in $\mathbb{R}$. For any such Borel set $U$, define $\mathcal{B} \cap U$ as the collection of all intersections between $U$ and the members of $\mathcal{B}$, that is $\mathcal{B} \cap U = \{B \cap U : B \in \mathcal{B}\}$. Show that $\mathcal{B} \cap U$ is a $\sigma$-algebra in $U$.

Exercise 2.
Recall the “dart” experiment discussed in the notes and class. Let $(\Omega, \mathcal{F})$ be the corresponding measurable space. Consider the random variable $X_2$ defined as by

$$X_2(\omega) = \begin{cases} 2 & \text{if } \omega \in D_s \\ 1 & \text{if } \omega \in D \setminus D_s \\ 0 & \text{if } \omega = \text{miss} \end{cases}$$

where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $D_s = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1/4\}$.

Describe the $\sigma$-algebra generated by $X$, i.e., what is $\sigma(X)$?

Exercise 3.
Let $P$ be a probability measure defined on $(\Omega, \mathcal{F})$. We proved in the notes that if $A_1 \subset A_2 \subset A_3 \ldots \in \mathcal{F}$, then $\lim_{n \to \infty} P(A_n) = P(\bigcup_{n=1}^{\infty} A_n)$. Use this fact to show that if $A_1 \supset A_2 \supset A_3 \ldots$, then $\lim_{n \to \infty} P(A_n) = P(\bigcap_{n=1}^{\infty} A_n)$. (Hint: Use set complementation.)

Exercise 4.
Consider a probability space $(\Omega, \mathcal{F}, P)$ and a random variable $X$ defined on it. We define the distribution of $X$, denoted by $\mu_X$, as a function on the set of Borel sets $\mathcal{B}$ as follows: $\mu_X(B) = P(X^{-1}(B))$, $B \in \mathcal{B}$. Show that $\mu_X$ defines a probability measure on $\mathcal{B}$.

Exercise 5.
Show that $X$ is a random variable if and only if $X^{-1}([r, \infty)) \in \mathcal{F}$ for every $r \in \mathbb{R}$.

Exercise 6.
Show that $X$ is a random variable if and only if $X^{-1}((r, \infty)) \in \mathcal{F}$ for every $r \in \mathbb{R}$.

Exercise 7.
Consider a probability space $(\Omega, \mathcal{F}, P)$ and a random variable $X$ defined on it. Show that the collection of subsets $\{X^{-1}(B) : B \in \mathcal{B}\}$ is a sub $\sigma$-algebra of $\mathcal{F}$. 