Problem 1. Consider the noisy observed process

\[ X(t) = Z(t) + W(t), \]

where \( W(t) \) is zero-mean white Gaussian noise and \( Z(t) \) is a WSS process. Assume that the processes \( Z(t) \) and \( W(t) \) are orthogonal. Assume further that the power spectral densities of \( W \) and \( Z \) are

\[ S_{WW}(\omega) = \frac{N_0}{2}, \]

\[ S_{ZZ}(\omega) = \frac{1}{9 + \omega^2}. \]

a) Derive the noncausal Wiener filter, \( h_{nc}(t) \), to estimate \( Z(t) \) from the data \( X(s), -\infty < s < \infty \).

b) State the orthogonality principle associated with the non-causal filter.

c) Derive the causal Wiener filter \( h_c(t) \) to estimate \( Z(t) \) from \( X(s), s \leq t \).

d) Compute the mean-square error \( \epsilon^2 \triangleq E[(\hat{Z}(t) - Z(t))^2] \), where \( \hat{Z}(t) \) is the output of the causal Wiener filter.

Problem 2. A standard Poisson process \( N(t) \) is used as an input to an LTI system with impulse response \( h(t) = u(t) - u(t - 1) \), where \( u \) is the unit-step function. Calculate the mean and the variance of the output. Carry out the required analysis to simplify your answer as much as possible.

Problem 3. White noise \( W(t) \) is used as an input to an LTI system with impulse response \( h(t) = 3e^{-3t}u(t) \). Calculate the power spectral density of the output and compute the output’s variance.

Problem 4. Use a rigorous probabilistic argument to calculate

\[ \lim_{k \to \infty} \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}^k. \]

Interpret this limit using Markov chains. For full credit you must justify your answers by citing the appropriate theorems.

Problem 5. Consider a sequence \( N_n = X_1 + \ldots + X_n \), where the \( X_i \)’s are iid, Poisson distributed, and \( E[X_1] = \lambda \).

a) What can you say about the quantity \( n^{-1}N_n \) when \( n \) is large? In other words, does \( \lim_{n \to \infty} n^{-1}N_n \) exist? If so, precisely describe in what stochastic sense it exists and characterize the limit.

b) Calculate the limit \( \lim_{n \to \infty} P\{N_n > n\lambda\} \).

Problem 6. The times between successive bus arrivals can be modeled as an exponential random variables with mean of 1 minute. Suppose that the arrival time of the first bus is also exponentially distributed with mean of 1 minute, and that bus inter-arrival times are also independent. What is the probability that 12 busses arrive in 60 minutes? Justify your approach.

Problem 6. Consider an irreducible birth and death chain on the nonnegative integers with

\[ q_i/p_i = \left( \frac{i}{i+1} \right)^2. \]
Show that the chain is transient.

**Problem 7.** Let $X_1, X_2, \ldots$ be an independent and identically distributed sequence. Assume that $X_1 \in \{1, \ldots, r\}$ and $P\{X_1 = k\} = p_k$, $k = 1, \ldots, r$. Show that

$$-\lim_{n \to \infty} n^{-1} \log (p_{X_1} \ldots p_{X_n}) = -\sum_{i=1}^{r} p_i \log p_i, \quad \text{almost surely.}$$

In information theory, the above limit is called \textit{Shannon's entropy} for a discrete memoryless source with alphabet \{1, \ldots, r\}.

**Problem 8.** Suppose that a sequence $X_n$ converges in distribution to a Gaussian random variable with mean 2 and variance 1. Characterize the distribution function of $\lim_{n \to \infty} X_n^2$ and calculate the probability that this limit is greater than 2. For full credit you must justify your work thoroughly.

**Problem 9.** A standard Brownian motion $B(t)$ is used to drive an LTI system with impulse response $h(t) = e^{-2t} u(t)$.

a) Use basic principles to calculate the mean and the variance of the output. Carry out the required analysis to simplify your answer.

b) Now replace the Brownian-motion input by white Gaussian noise with power-spectral density of 1 W/Hz. Calculate the variance of the output.

c) Find the autocorrelation function and the power spectral density of the output in part (b).

**Problem 10.** Suppose that $\{X_n\}$ is a wide-sense stationary sequence. Prove or disprove, by giving a counter example, the claim

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i = E[X_1], \quad \text{almost surely.}$$

Hint: The answer is in one of the simplest random sequences you can ever think of.

**Problem 11.** Suppose that $X_n$ converges to $X$ in probability. Prove that for any continuous function $f$, $f(X_n)$ also converges to $f(X)$ in probability.

**Problem 12.** Show that a stochastic matrix always have 1 as an eigenvalue.

**Problem 13.** Prove that the random walk on the integers is recurrent. Be precise in stating any theorems that you use.

**Problem 14.** Show that a WSS process is continuous in the mean-square sense if and only if its autocorrelation function is continuous at the origin.

From the text, Chapter 10, solve problems 36, 47(b,d), 48, 51, 58.