Text problems: Chapter 7: 26, 27, 34

**Problem 1.** Show that \( \lim_{n \to \infty} X_n \) exists if and only if \( \lim_{n \to \infty} X_n = \overline{\lim}_{n \to \infty} X_n \).

**Problem 2.** Show that \( \lim_{n \to \infty} X_n = \lim_{n \to \infty}(X_n) \).

**Problem 3.** Suppose that \( W_k \) is a sequence of random variables. Prove that \( (\sup_{k \geq n} W_k)^{-1}(a, \infty) = \bigcup_{k=n}^{\infty} W_k^{-1}(a, \infty) \). What does this result tell us about \( \sup_{k \geq n} W_k \)?

**Problem 4.** Referring to the proof of Theorem 8 in the notes: where did we use the fact that \( \mathbb{P} \) is a finite measure?

**Problem 5.** Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \( X \in L_1(\Omega, \mathcal{F}, \mathbb{P}) \). Let \( \mathcal{D} \) be a sub \( \sigma \)-algebra. For each \( n \geq 1 \), we define \( X_n \) as \( X \) when \(|X| \leq n\) and \( n \) otherwise. Note that \( X_n \in L_2(\Omega, \mathcal{F}, \mathbb{P}) \) (since it is bounded), so we can talk about \( E[X_n|\mathcal{D}] \) as a projection of \( X_n \) onto \( L_2(\Omega, \mathcal{D}, \mathbb{P}) \), which we call \( Z_n \).

a) Show that \( Z_n \) is Cauchy in \( L_1 \).

b) We argued in the notes, using the completeness of \( L_1 \), that there exists \( Z \in L_1(\Omega, \mathcal{D}, \mathbb{P}) \) such that \( \lim_{n \to \infty} E[|Z_n - Z|] = 0 \). Further, \( Z_n \) has a subsequence, \( Z_{n_k} \), that converges almost surely to \( Z \). We took this \( Z \) as a candidate for \( E[X|\mathcal{D}] \). Show that \( Z \) satisfies \( E[ZY] = E[XY] \) for any bounded, \( \mathcal{D} \)-measurable \( Y \).

**Problem 6.** Recall Lindeberg’s Central Limit Theorem discussed in class. Show that the Lindeberg condition is automatically satisfied if the sequence is identically distributed. (Whenever applicable, you must rigorously justify passing limits into expectations.)

**Problem 7.** Let \( X \) be a random variable with \( E[|X|] < \infty \). Show that \( \lim_{n \to \infty} E[XI_{\{|X|>n\}}] = 0 \). Justify all your steps. Hint: Refer to class notes.