Problem 1. (From Hoel, Port, and Stone) Consider an irreducible birth and death chain on the nonnegative integers with
\[ q_i/p_i = \left( \frac{i}{i+1} \right)^2. \]
Show that the chain is transient.

Solution:
For irreducible birth and death chain, the states are transient if and only if
\[ \sum_{m=1}^{\infty} \gamma_m < \infty. \]
Since
\[ \gamma_m = \prod_{i=1}^{m} \frac{q_i}{p_i} = \prod_{i=1}^{m} \left( \frac{i}{i+1} \right)^2 = \left( \frac{1}{m+1} \right)^2 \]
we have
\[ \sum_{m=1}^{\infty} \gamma_m = \sum_{m=1}^{\infty} \left( \frac{1}{m+1} \right)^2 = \frac{\pi^2}{6} - 1 < \infty \]
The chain is transient.

Problem 2. (From Hoel, Port, and Stone) Consider a Markov chain on the nonnegative integers having transition probabilities given by \( p_{i,i+1} = p \) and \( p_{i,0} = 1 - p \), where \( 0 < p < 1 \). Show that this chain has a unique stationary distribution \( \pi \) and find it.

Solution:
\( \forall i,j, \exists n \), such that \( p_{ij}^n > 0 \), therefore the Markov chain is irreducible.

Since the probability to be in origin at time \( n \) is positive for all integer values of \( n \), the state 0 has period 1. Since the Markov chain is irreducible, so does each state \( x > 0 \). Therefore, the Markov chain is aperiodic.

Let
\[ \pi = [\pi_0 \pi_1 \ldots], \]
\[ \Pi = \begin{pmatrix} 1 - p & p & 0 & 0 & 0 & \ldots \\ 1 - p & 0 & p & 0 & 0 & \ldots \\ 1 - p & 0 & 0 & p & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \]

Use the equation \( \pi = \pi \Pi \) to obtain \( \pi_k = p^k \pi_0, k \geq 1 \). Since \( \sum_{k=0}^{\infty} \pi_k = 1 \), the first scalar equation in \( \pi = \pi \Pi \) yields \( \pi_0 = 1 - p \).

This stationary distribution is unique by the relevant theorem in the notes since the chain is irreducible, aperiodic, and a stationary distribution exists.
10.3
\[ C_X(t_1, t_2) = E[(X_{t_1} - EX_{t_1})(X_{t_2} - EX_{t_2})] = E[X_{t_1}X_{t_2} - X_{t_1}EX_{t_2} - X_{t_2}EX_{t_1} + EX_{t_2}EX_{t_1}] = E[X_{t_1}X_{t_2}] - EX_{t_1}EX_{t_2} + EX_{t_1}EX_{t_1} = R_X(t_1, t_2) - m_X(t_1)m_X(t_2) \]
\[ C_{XY}(t_1, t_2) = E[(X_{t_1} - EX_{t_1})(Y_{t_2} - EY_{t_2})] = E[X_{t_1}Y_{t_2} - X_{t_1}EY_{t_2} - Y_{t_2}EX_{t_1} + EY_{t_2}EX_{t_1}] = E[X_{t_1}Y_{t_2}] - EX_{t_1}EY_{t_2} - EY_{t_2}EX_{t_1} + EY_{t_2}EX_{t_1} = E[X_{t_1}Y_{t_2}] - EX_{t_1}EY_{t_2} = R_{XY}(t_1, t_2) - m_X(t_1)m_Y(t_2) \]

10.5
\[ m_X(t) = 0, \text{ for any } t \]
\[ \text{Var}(X_t) = R_X(t, t) - 0 = t \]
Thus, \[ f_X(x) = \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} \right) \]

10.6(a)
\[ m_Y(n) = EY_n = \sum_{i=-\infty}^{\infty} E[h(n-i)X_i] = \sum_{i=-\infty}^{\infty} h(n-i)m_X(i) = \sum_{j=-\infty}^{\infty} h(j)m_X(n-j) \]

10.8
(a) \[ R_Y(t_1, t_2) = E[Y_{t_1}Y_{t_2}] = E[X_{t_1}X_{t_2}\cos(2\pi f_{t_1} + \theta)\cos(2\pi f_{t_2} + \theta)] = \frac{1}{2} E[X_{t_1}X_{t_2}\cos(2\pi f_{t_1} + 2\theta) + \cos(2\pi f_{t_1} - t_2))] = \frac{1}{2} E[X_{t_1}X_{t_2}]\cos(2\pi f_{t_1} - t_2)] = \frac{1}{2} R_X(t_1, t_2)\cos(2\pi f_{t_1} - t_2)] \]
(b) \[ R_{XY}(t_1, t_2) = E[X_{t_1}Y_{t_2}] = E[X_{t_1}Y_{t_2}\cos(2\pi f_{t_2} + \theta)] = 0 \]
(c) Since \[ m_Y(t) = 0 \] and \[ R_Y(t_1, t_2) \] only depends on \[ t_1, t_2 \] only through \[ t_1 - t_2 \] and \[ R_X(t_1, t_2) \] which only through \[ t_1 - t_2 \], we conclude that \[ Y \] is WSS.

12.6
The stationary distribution of the chain is:
\[
\begin{align*}
\pi_0 &= \frac{1}{2} \pi_0 + \frac{1}{4} \pi_1 + \frac{1}{2} \pi_2 \\
\pi_1 &= \frac{1}{2} \pi_0 + \frac{1}{2} \pi_2 \\
\pi_0 + \pi_1 + \pi_2 &= 1
\end{align*}
\implies \begin{align*}
\pi_0 &= \frac{5}{12} \\
\pi_1 &= \frac{1}{3} \\
\pi_2 &= \frac{1}{4}
\end{align*}
\]
17. From the Fourier transform table, \( S_X(f) = \sqrt{2\pi} e^{-(2\pi f)^2/2} \).

18. From the Fourier transform table, \( S_X(f) = \pi e^{-2\pi |f|} \).

25. First note that since \( R_X(\tau) = e^{-\tau^2/2} \), \( S_X(f) = \sqrt{2\pi} e^{-(2\pi f)^2/2} \).

(a) \( S_{XY}(f) = H(f)^* S_X(f) = [e^{-(2\pi f)^2/2}]^* \sqrt{2\pi} e^{-(2\pi f)^2/2} = \sqrt{2\pi} e^{-(2\pi f)^2} \).

(b) Writing

\[
S_{XY}(f) = \frac{1}{\sqrt{2}} \sqrt{2\pi} \sqrt{2} e^{-(\sqrt{2})^2 (2\pi f)^2/2},
\]

we have from the transform table that

\[
R_{XY}(\tau) = \frac{1}{\sqrt{2}} e^{-(\tau/\sqrt{2})^2/2} = \frac{1}{\sqrt{2}} e^{-\tau^2/4}.
\]

(c) Write

\[
E[X_1 Y_2] = R_{XY}(t_1 - t_2) = \frac{1}{\sqrt{2}} e^{-(t_1 - t_2)^2/4}.
\]

(d) \( S_Y(f) = |H(f)|^2 S_X(f) = e^{-(2\pi f)^2} \cdot \sqrt{2\pi} e^{-(2\pi f)^2/2} = \sqrt{2\pi} e^{-(2\pi f)^2} \).

(c) Writing

\[
S_Y(f) = \frac{1}{\sqrt{3}} \sqrt{2\pi} \sqrt{3} e^{-(\sqrt{3})^2 (2\pi f)^2/2},
\]

we have from the transform table that

\[
R_Y(\tau) = \frac{1}{\sqrt{3}} e^{-(\tau/\sqrt{3})^2/2} = \frac{1}{\sqrt{3}} e^{-\tau^2/6}.
\]
48. Write
\[ E[X^2] = R_X(0) = \left| \int_{-\infty}^{\infty} S_X(f) e^{j2\pi \tau f} df \right|_{\tau=0} = \int_{-\infty}^{\infty} S_X(f) df - \int_{-W}^{W} 1 df = 2W. \]

49. (a) \( e^{-f^2} \) is not even.
(b) \( e^{-f^2} \cos(f) \) is not nonnegative.
(c) \( (1 - f^2)/(1 + f^4) \) is not nonnegative.
(d) \( 1/(1 + jf^2) \) is not real valued.

50. (a) Since \( \sin \tau \) is odd, it is NOT a valid correlation function.
(b) Since the Fourier transform of \( \cos \tau \) is \( [\delta(f-1) + \delta(f+1)]/2 \), which is real, even, and nonnegative, \( \cos \tau \) IS a valid correlation function.
(c) Since the Fourier transform of \( e^{-\tau^2/2} \) is \( \sqrt{2\pi} e^{-(2\pi f)^2/2} \), which is real, even, and nonnegative, \( e^{-\tau^2/2} \) IS a valid correlation function.
(d) Since the Fourier transform of \( e^{-||\tau||} \) is \( 2/[1 + (2\pi f)^2] \), which is real, even, and nonnegative, \( e^{-||\tau||} \) IS a valid correlation function.
(e) Since the value of \( \tau^2 e^{-|\tau|} \) at \( \tau = 0 \) is less than the value for other values of \( \tau \), \( \tau^2 e^{-|\tau|} \) is NOT a valid correlation function.
(f) Since the Fourier transform of \( I_{[-T,T]}(\tau) \) is \( (2T)\sin(2\pi T f)/(2\pi Tf) \) is not nonnegative, \( I_{[-T,T]}(\tau) \) is NOT a valid correlation function.

53. To begin, write
\[ R(\tau) = \int_{-\infty}^{\infty} S(f)e^{j2\pi \tau f} df = \int_{-\infty}^{\infty} S'(f)[\cos(2\pi f \tau) - j\sin(2\pi f \tau)] df. \]

Since \( S \) is real and even, the integral of \( S(f)\sin(2\pi f \tau) \) is zero, and we have
\[ R(\tau) = \int_{-\infty}^{\infty} S(f)\cos(2\pi f \tau) df, \]
which is a real and even function of \( \tau \). Finally,
\[ |R(\tau)| = \left| \int_{-\infty}^{\infty} S(f)e^{j2\pi \tau f} df \right| \leq \int_{-\infty}^{\infty} |S(f)e^{j2\pi \tau f}| df \]
\[ = \int_{-\infty}^{\infty} |S(f)| df = \int_{-\infty}^{\infty} S(f) df = R(0). \]