Solutions to Homework 3

Problem 1.61  For clarity, let us rename the signal described in the textbook from $x$ to $x_\Delta$. Now differentiate $x$ and observe that the derivative is zero outside the interval $(-\Delta/2, -\Delta/2)$, and it is $\Delta^{-1}$ over this interval. Clearly, $\lim_{\Delta \to 0} x_\Delta(t) = 0$ for any $t \neq 0$. At the same time, the integral of $x_\Delta(t)$ over the interval $(-\infty, \infty)$ is always unity. These two properties are those that define a delta function.

Problem 1.64  The systems that follow have input $x(t)$ or $x[n]$ and output $y(t)$ or $y[n]$. For each system, determine whether it is (i) memoryless, (ii) stable, (iii) causal, and (v) time-invariant.

(a) $y(t) = \cos(x(t))$

Solution:

(i) Is the system memoryless? Yes, since $y(t)$ only depends on the present value of $x(t)$.

(ii) Is the system stable? Yes, since $|y(t)| \leq 1$ (property of the cosine function).

(iii) Is the system causal? Yes, since it is memoryless, it only depends on the present input (For a system to be causal, its present output must not depend on future values of the input).

(v) Is the system time-invariant? Yes, since $y(t + \tau) = \cos(x(t + \tau))$, for any $t$ and $\tau$.

(b) $y[n] = 2x[n]u[n]$

Solution:

(i) Is the system memoryless? Yes, since $y[n]$ only depends on the present value of $x[n]$.

(ii) Is the system stable? Yes, since $|y(t)| = 2|x[n]|u[n] \leq 2|x[n]|$. Hence, if $x[n]$ is bounded by $M$ ($|x[n]| \leq M$), then $y[n]$ is bounded by $2M$.

(iii) Is the system causal? Yes, since it is memoryless, it only depends on the present input.

(v) Is the system time-invariant? No. The time-invariance condition does not hold, because the signal that is being multiplied by $x[n]$ varies with time.
(d) \( y(t) = \int_{-\infty}^{t/2} x(\tau) d\tau \)

Solution:

(i) Is the system memoryless? No, since the integral is evaluated on the input over all the time from \(-\infty\) to \(t/2\).

(ii) Is the system stable? No. A simple counter example is when the input signal is \( x(t) \equiv 1 \), which is obviously bounded, while the output \( y(t) \) is not finite, since the integral of 1 from \(-\infty\) to \(t/2\) is not finite.

(iii) Is the system causal? No. For negative \( t \), the output depends on all values of \( x(t) \), from \(-\infty\) to \(t/2\), which is greater than \( t \). Hence, the output depends on future values of \( x(t) \).

(iv) Is the system time-invariant? No. The output \( y_d(t) \) for a time-shifted version of the input \( x(t - d) \) is

\[
y_d(t) = \int_{-\infty}^{t/2} x(\tau - d) d\tau
= \int_{-\infty}^{t/2-d} x(s) ds
= \int_{-\infty}^{(t-2d)/2} x(s) ds
= y(t - 2d).
\]

Therefore, it does not obey the time-invariance condition.

(f) \( y(t) = \frac{d}{dt} x(t) \)

Solution:

(i) Is the system memoryless? No, since the derivative of a function at a specific point \( t_o \) cannot be determined just from the knowledge of the value of the function on \( t_o \). (ex. you cannot determine the derivative of \( x(t) \) at \( t = 2 \), if you only know that \( x(2) = 10 \)).

(ii) Is the system stable? No. A counter example is when \( x(t) = \sqrt{1 - t^2}, -1 < t < 1 \), and it is zero otherwise. \( x(t) \) is bounded, but its derivative, which is given by

\[
\frac{dx(t)}{dt} = -\frac{t}{\sqrt{1-t^2}},
\]

goes to \(-\infty\), when \( t \) approaches 1.
(iii) Is the system causal? Yes, since the derivative can be determined from the expression
\[ \frac{dx(t)}{dt} = \lim_{h \to 0^+} \frac{x(t) - x(t - h)}{h}, \]
which only depends on past values of \( x(t) \).

(v) Is the system time-invariant? Yes. The derivative of a time-shifted signal is
\[ y_d(t) = \frac{d}{dt}[x(t - d)] = \frac{dx}{dt}(t - d) \frac{d}{dt}(t - d) = \frac{dx}{dt}(t - d) = y(t - d). \]

(i) \( y(t) = x(2 - t) \)

(i) Is the system memoryless? No, since the output depends on the value of the input at a time-instant other than \( t \).

(ii) Is the system stable? Yes, since \( |y(t)| \leq M \), if \( |x(t)| \leq M \).

(iii) Is the system causal? No. For negative \( t \), \( 2 - t \) is positive, therefore the output depends on the future.

(v) Is the system time-invariant? No.
\[ y_d(t) = x(2 - t - d)) = x(2 - (t + d)) = y(t + d). \]

**Problem 1.71**

(a) Yes. Consider the system defined by the rule \( O(i)(t) = \int_{-\infty}^{t} \frac{1}{M(s)} f(s) \, ds \),
where \( M(t) = 50e^{-0.01t}u(t) \). Show that this system is linear but it is time variant. Can you give an example of a physical system that can be modeled by the above system?

(b) The equation for this circuit is: \( i(t)R(t) + v_2(t) = v_i(t) \). Assume that \( v_2(\infty) = 0 \). Since \( i(t) = cv'_2(t) \), we can rewrite the circuit equation as \( v'_2(t) + \frac{1}{RC}v_2(t) = \frac{1}{RC}v_i(t) \). Following class notes on linearity of ODE’s, show that a system represent by a differential equation with time varying coefficients is still linear.

**Problem 1.76** A linear system \( H \) has the input-output pairs depicted in Fig. 1.76(a) (in the book). Answer the following questions, and explain your answers:
Could this system be causal?

Solution: No. The system is linear, therefore, for an input $x(t) \equiv 0$, the output should be $y(t) \equiv 0$. This is true because, by the homogeneity property, when $x(t) = 0 \cdot f(t)$, the output must be $y(t) = 0 \cdot H(f(t)) \equiv 0$.

If the system is causal, it doesn’t know anything about the future. Therefore, if for some input $x(\tau) = 0$, for $\tau < t$, then the output must be $y(\tau) = 0$, for $\tau < t$. Because, as for what the system knows, $x(t)$ could be 0 for every $t$.

On the figure, we notice that $y_2(t) = 1$ for $t \in (0, 1)$, while $x_2(t) = 0$ for $t \in (-\infty, 1)$. This contradicts the conclusions discussed before. Therefore, the system cannot be causal.

Problem 1.77 A discrete-time system is both linear and time-invariant. Suppose the output due to an input $x[n] = \delta[n]$ is given in Fig. 1.77(a) (in the book).

(a) Find the output due to an input $x[n] = \delta[n - 1]$

Solution: Let’s call the signal in Fig. 1.77(a) $h[n]$. Since the system is time-invariant, for $x[n] = \delta[n - 1]$,

$$y[n] = h[n - 1].$$

(b) Find the output due to an input $x[n] = 2\delta[n] - \delta[n - 2]$.

Solution: Now we use the linearity $x[n] = 2\delta[n] - \delta[n - 2]$ \Rightarrow

$$y[n] = 2h[n] - h[n - 2].$$
(c) Find the output due to the input depicted in Fig. 1.77(b).

Solution: Now, we can see that

\[ x[n] = \delta[n + 1] - \delta[n] + 2\delta[n - 1]. \]

Therefore,

\[ y[n] = h[n + 1] - h[n] + 2h[n - 1]. \]

Problem 1.93  (a) The solution of a linear differential equation is given by

\[ x(t) = 10e^{-t} - 5e^{-0.5t}. \]
Using MATLAB, plot $x(t)$ versus $t$ for $t = 0:0.01:5$.

Solution: A possible MATLAB code to do that is:

```matlab
t = 0 : 0.01 : 5;
x = 10 * exp(-t) - 5 * exp(-0.5 * t);
plot(t,x);
```