ECE 340: PROBABILISTIC METHODS IN ENGINEERING

SOLUTIONS TO HOMEWORK #5

3.35
a) By the definition of conditional probability,
\[ P(\{X=k\} | \{X > 0\}) = \frac{P(\{X=k\} \cap \{X > 0\})}{P(X > 0)} = \frac{P\{X=k\}}{P(X > 0)} = \frac{P\{X=k\}}{15/16} \]

\[ = \begin{cases} \left(\frac{8}{16}\right) = \frac{8}{15}, & k = 1 \\ \left(\frac{7}{16}\right) = 7/15, & k = 2 \end{cases} \]

Note: we used the fact that \(\{X=k\} \subset \{X>0\}\) for \(k=1, 2\).

b) \[ P(\{X=k\} | \{N_m = 1\}) = \frac{P(\{X=k\} \cap \{N_m = 1\})}{P(N_m = 1)} = \frac{P(\{X=k\} \cap \{N_m = 1\})}{\frac{1}{2}} \]

\[ = \begin{cases} \left(\frac{1}{4} \frac{1}{2} + \frac{1}{2} \frac{1}{2} \right) \frac{1}{2} = \frac{3}{4}, & k = 1 \\ \left(\frac{1}{4} \frac{1}{2} \frac{1}{2} \right) \frac{1}{2} = \frac{1}{4}, & k = 2 \end{cases} \]

3.44
a) \(S=\{1,2,3,4,5\} \quad A=\{\zeta > 3\}\)
Let us assume that all outcomes are equally likely. Now,
\[ P(I_A=0) = P(A^c)=P(1,2,3)=3/5; \]
\[ P(I_A=1) = = P(A)=P(4,5)= 2/5; \]
\[ E[I_A] = 0 \times 3/5 + 1 \times 2/5 = 2/5. \]

b) \(S=[0,1] \quad A=\{0.3 < \zeta < 0.7\}\)
Again let us assume that we have uniform outcomes; that is, if \(E\) is an event, the
\[ P(A)=\text{length}(E). \]
\[ P(I_A=0) = P(A^c)=0.3+0.3 = 0.6 \]
\[ P(I_A=1) = P(A) = 0.4 \]
\[ E[I_A] = 0 \times 0.6 + 1 \times 0.4 = 0.4 \]
c) \( S = \{ \zeta = (x, y): 0 < x < 1, \ 0 < y < 1 \} \) and 
\( A = \{ \zeta = (x, y): 0.25 < x+y < 1.25 \} \)

\[ I_A(\zeta) = 0 \text{ when the pair } (x, y) \text{ is not in the indicated area.} \]
\[ I_B(\zeta) = 0 \text{ when the pair } (x, y) \text{ is in the indicated area.} \]

The probabilities can be calculated by computing the areas
\[ P\{I_A=0\} = \frac{0.25^2}{2} + \frac{(1-0.25)^2}{2} = \frac{5}{16} = 0.3125 \]
\[ P\{I_A=1\} = 1 - P\{I_A=0\} = 1 - \frac{5}{16} = \frac{11}{16} = 0.6875 \]
\[ E[I_A] = 0 \times 0.3125 + 1 \times 0.6875 = 0.6875 \]

\[ d) \ S=(-\infty, \infty) \text{ and } A=\{\zeta \geq a\} \]
\[ P\{I_A=1\} = P(A) \]
\[ P\{I_A=0\} = P(A^c) = 1 - P(A) \]
\[ E[I_A] = 1 \times P(A) = P(A) \]

3.45 A and B are events from a random experiment with sample space \( S \).

a) Show that, \( I_S = 1 \)

From the definition of the indicator function we have that,
\[ I_S(\zeta) = \begin{cases} 
0 & \text{if } \zeta \text{ not in } S \\
1 & \text{if } \zeta \text{ in } S 
\end{cases} \]

However, by the definition of sample space, all the points \( \zeta \) have to be in \( S \), so \( I_S = 1 \) for any \( \zeta \in \Omega \).
Likewise, we show that, \( I_{\emptyset} = 0 \)
From the definition of the indicator function we have that,

$$I_{\emptyset}(\xi) = \begin{cases} 0 & \text{if } \xi \text{ not in } \emptyset \\ 1 & \text{if } \xi \text{ in } \emptyset \end{cases}$$

but by the definition of the null event $\emptyset$, contains no outcomes and hence never occurs, so $I_{\emptyset}=0$ for any $\xi \in \Omega$.

b) $I_{A \cap B} = I_A \cdot I_B$
- $I_{A \cap B}(\xi) = 0$ if $\xi \notin A$ and $\xi \notin B$; which corresponds to $I_A=0$ and $I_B=0$; hence, $I_A \cdot I_B =0$
- or if $\xi \in A$ and $\xi \notin B$, which corresponds to $I_A=1$ and $I_B=0$; hence, $I_A \cdot I_B =0$
- or if $\xi \notin A$ and $\xi \in B$, which corresponds to $I_A=0$ and $I_B=1$; hence, $I_A \cdot I_B =0$.
- $I_{A \cap B}(\xi) = 1$ if $\xi \in A$ and $\xi \in B$, which is corresponds to $I_A=1$ and $I_B=1$; hence, $I_A \cdot I_B =1$.

To show that,

$I_{A \cup B} = I_A + I_B - I_{A \cap B}$ notice that
- $I_{A \cup B}(\xi) = 0$ only if $\xi \notin A$ and $\xi \notin B$, which is corresponds to $I_A=0$ and $I_B=0$, and $I_A + I_B - I_{A \cup B} = 0$
- $I_{A \cup B}(\xi) = 1$ if $\xi \in A$ and $\xi \in B$, which in this case makes to $I_A=1$ and $I_B=1$ and $I_{A \cup B}=1$, and $I_A + I_B - I_{A \cup B} = 1$
- or if $\xi \in A$ and $\xi \notin B$, which corresponds to $I_A=1$ and $I_B=0$ and $I_{A \cup B}=0$, so $I_A + I_B - I_{A \cup B} = 1$
- or if $\xi \notin A$ and $\xi \in B$, which is equivalent to $I_A=0$ and $I_B=1$ and $I_{A \cup B}=0$, so $I_A + I_B - I_{A \cup B} = 1$

c) The expected values are

$E[I_3] = 0p_i(0) + 1p_i(1) = p_i(1) = 1$

$E[I_3] = p_i(1) = 0$

$E[I_{A \cap B}] = p_i(1) = P(A \cap B)$

$E[I_{A \cup B}] = p_i(1) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$
3.48
The pmf of a binomial random variable is given as

\[ p_X(k) = P[X = k] = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, ..., n \]

The pmfs of various values of n and p are given in the following figures

for n = 4 and p = 0.1

![Graph for n = 4 and p = 0.1](image1)

for n = 5 and p = 0.1

![Graph for n = 5 and p = 0.1](image2)

for n = 4 and p = 0.5

![Graph for n = 4 and p = 0.5](image3)
3.49

a) Let $I_k$ denote the outcome of the $k$th Bernoulli trial. The probability that the single event occurred in the $k$th trial is

$$P(I_k = 1 | X = 1) = \frac{P(I_k = 1 \cap \{ I_j = 0 \text{ for all } j \neq k \})}{P(X = 1)}$$

$$= \frac{P((0, 0, \ldots, 1, 0, \ldots))}{P(X = 1)}$$
\[ p_1 - p! = \frac{1}{n} \]

Thus, the single event is equally likely to have occurred in any of the \( n \) trials.

b) The probability that the two successes occurred in trials \( j \) and \( k \) is

\[
P(\{l_j = 1, l_k = 1 \mid X = 2\}) = \frac{P(\{l_j = 1, l_k = 1\} \cap \{l_m = 0 \text{ for all } m \neq j, k\})}{P(X = 2)} = 1 \]

\[
\frac{\binom{n}{2} (1 - p)^{n-2}}{p^2 (1 - p)^n} = \frac{1}{\binom{n}{2}}
\]

c) If \( X=k \), then the locations of the successes are randomly selected from the \( \binom{n}{k} \) possible permutations of possible locations. The concept of completely at random can be interpreted as the concept of ‘uniformly distributed’ as can be seen from parts a) and b).

3.52

\[ p=0.01 \quad N=\text{number of error-free characters until the first error.} \]

a) \( P(\{N=k\})=(1-p)^kp \quad k=0,1,2,... \)

b) \[ E[N] = \sum_{k=0}^{\infty} k(1-p)^kp = (1-p)p \sum_{k=0}^{\infty} k(1-p)^{k-1} = (1-p)p \frac{1}{(1-(1-p))^2} = \frac{1-p}{p} \quad \text{by Eq. (3.14)} \]

c) \[ 0.99 = P(N > k_0) = \sum_{k=k_0+1}^{\infty} (1-p)p^k = p(1-p)^{k_0+1} \sum_{k=0}^{\infty} (1-p)^k = (1-p)^{1001} \Rightarrow p = 1 - 0.99^{1001} = 1.004 \times 10^{-5} \]

3.56 Let's denote the random variable \( X \) as the cost of repair of the audio player in one year period, and denote \( c \) as the charge for the one year warranty.

The pmf of r.v. \( X \) can be found as

\[ P(X = x) = \binom{12}{x} \left( \frac{1}{12} \right)^{x/20} \left( 1 - \frac{1}{12} \right)^{12-x/20} \quad \text{for } x = 0, 20, 40, 60, ..., 240. \]

Thus,

\[ P(X = 0) = \binom{12}{0} \left( \frac{1}{12} \right)^{0/20} \left( 1 - \frac{1}{12} \right)^{12-0/20} = 0.351995628014137 \]
\[ P(X = 20) = 0.383995230560877 \]
If now we want the probability of losing money on a player is 1% or less, we want
\[ P[X > c] \leq 0.01. \] And the question now is 'what value should the c take such that the
previous inequality is satisfied?'

Note that \( P[X>c] \) is monotonically decreasing as c increases and
\[ P[X>60] = 0.013830430605021 > 0.01; \]
and \( P[X>c] = 0.013830430605021 > 0.01 \) for any \( c \) in the interval of (60,80);
\[ P[X>80] = 0.001929751971936 \leq 0.01; \]
\[ P[X>c] \leq P[X>80] = 0.001929751971936 \leq 0.01 \) for any \( c \geq 80. \)

So the manufacturer should charge at least 80 dollars to make sure the probability of
losing money on a player is 1% or less (excluding the initial cost of $50.).

The average cost (of repair) per player is \( E[X] = \sum_x x P[X = x] = 20 \) dollars.
This expectation can be calculated by the following matlab code:
```matlab
for x = 0:20:240
    cost((x+20)/20) = x*(nchoosek(12,x/20))*((1/12)^(x/20))*((11/12)^(12-(x/20)));
end
>> sum(cost)
ans =
19.999999999999996
>>
```

3.59 (Please also see Example 3.30 on page 122)
a) We know that the average is 6000 requests per minute, which is equivalent to 0.1
requests per millisecond, which corresponds to having \( \alpha = \lambda t = 0.1t \) in the Poisson
distribution (where \( t \) is in milliseconds). The probability of having no requests in a 100-
ms period is
\[
P(N=0) = \frac{(\lambda t)^0}{0!}e^{-\lambda t} = e^{-\lambda t} = e^{-0.1\times100} = e^{-10} = 45.399 \times 10^{-6}
\]
b) The probability that there are between 5 and 10 requests in a 100-ms period

\[ P\{5 \leq N \leq 10\} = \sum_{k=5}^{10} \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \sum_{k=5}^{10} \frac{(0.1 \times 100)^k}{k!} e^{-0.1 \times 100} = 0.5538 \]

3.62 Solution:
The Poisson r.v. has the following pmf:

\[ P[N = k] = p_N(k) = \frac{\alpha^k}{k!} e^{-\alpha}, \text{ for } k = 0, 1, 2, \ldots \]

\[ \frac{p_k}{p_{k-1}} = \frac{\left(\frac{\alpha^k}{k!} e^{-\alpha}\right)}{\left(\frac{\alpha^{k-1}}{(k-1)!} e^{-\alpha}\right)} = \frac{\alpha}{k} \]

If \( \alpha < 1 \) then \( \frac{p_k}{p_{k-1}} = \frac{\alpha}{k} < 1 \) for \( k \geq 1 \)

\[ \therefore p_k \text{ decreases as } k \text{ increases from 0} \]

\[ \therefore p_k \text{ attains its maximum at } k = 0 \]

If \( \alpha > 1 \) then (Note that we denote by \([x]\) the largest integer that is smaller than or equal to \( x \).)

If \( 0 \leq k \leq [\alpha] < \alpha \), then \( \frac{p_k}{p_{k-1}} = \frac{\alpha}{k} > 1 \)

Hence, \( p_k \) increases from \( k=0 \) to \( k=[\alpha] \)

If \( [\alpha] \leq \alpha \leq k \), \( \frac{p_k}{p_{k-1}} = \frac{\alpha}{k} < 1 \)

and \( p_k \) decreases as \( k \) increases beyond \([\alpha]\)

\[ \therefore p_k \text{ attains its maximum at } k_{\text{max}} = [\alpha] \]

If \( \alpha = [\alpha] \) then for \( k = [\alpha] \)

\[ \frac{p_k}{p_{k-1}} = 1, \text{ and hence, } p_{k_{\text{max}}} = p_{k_{\text{max}}-1} \]

3.68 Solution:

\[ E[X] = \sum_{k=1}^{L} k P\{X = k\} = \sum_{k=1}^{L} k \frac{1}{L} = \frac{1}{L} \sum_{k=1}^{L} k = \frac{L(L + 1)}{2L} = \frac{L + 1}{2} \]

\[ \alpha_X^2 = E[X^2] - E[X]^2 = \sum_{k=1}^{L} k^2 \frac{1}{L} - \left(\frac{L + 1}{2}\right)^2 = \frac{L(L + 1)(2L + 1)}{6L} - \frac{(L + 1)^2}{4} = \frac{L^2 - 1}{12} \]