Use the built-in Matlab random number generator (the Matlab command is \texttt{rand}) to generate samples of an exponentially-distributed random variable \( Z \). Namely, we want to generate random numbers that are mutually independent and each of them is distributed according to the probability density function \( f_Z(z) = \mu^{-1} \exp(-z/\mu)u(z) \). Note that you can use \texttt{rand}(k,l) to generate a \( k \times l \) array of independent \([0, 1]\)-valued uniformly distributed r.v.’s at once.

1. Show that if \( X \) is a uniformly-distributed r.v. in \([0, 1]\), then the new r.v. \( Z \) defined by \[ Z = -\mu \log(1 - X) \]
is an exponentially-distributed r.v. with parameter \( \mu \).

2. Assuming that \( \mu = 1 \), use the result in (a) to generate \( n = 1000 \) samples of \( f_Z \). Let \( Z = [Z_1, Z_2, \ldots, Z_n] \) denote the array of identically-distributed and mutually independent samples.

3. Compute \( \max_{1 \leq i \leq n}(Z_i) \), \( \min_{1 \leq i \leq n}(Z_i) \), and the arithmetic mean \( \bar{Z} \) of \( Z_1, \ldots, Z_n \). (Use the Matlab functions \texttt{max}, \texttt{min} and \texttt{mean}).

4. Use the histogram generation capability of Matlab to generate an empirical estimate of the density function \( f_Z \) using the above 1000 samples. To achieve this task, follow the procedure outlined below.

(a) Generate the bin array \( B = [0, B_1, \ldots, B_m] \), where \( m = 100 \) is the total number of bins. Use the uniform spacing \( B_{i+1} - B_i = \delta \), where \( \delta = \max_{1 \leq i \leq n}(Z_i)/m. \) (You may use the syntax \( B = [0 : \delta : \max(Z)] \).)

(b) Perform \( H = \texttt{hist}(Z, B) \). Now \( H \) is a histogram array of the array \( Z \) using \( m \) bins, and the \( i \)th bin is centered at \( B_i \). Type \texttt{help hist} to become more familiar with generating histograms in Matlab.

(c) Define \( \hat{f}_Z \triangleq H/(\delta n) \) as an empirical estimate of \( f_Z \).

5. Plot \( \hat{f}_Z \) and \( f_Z \) as functions of the array \( B \). Comment on your results.

6. Now extend the definition of \( \hat{f}_Z \) to the entire real line by assuming that it maintains a constant value over each bin. Note that \( \hat{f}_Z \), as a function of the continuous real variable \( x \), is a piece-wise constant function. Give a rough sketch of \( \hat{f}_Z \) as a function of \( x \).

7. Argue that \[ \int_{-\infty}^{\infty} \hat{f}_Z(x) \, dx = \sum_{i=1}^{m} \hat{f}_Z(B_i)\delta = 1, \]
and hence, \( \hat{f}_Z \) is a valid probability density density

8. Give a concise summary of what you learned from this problem.