1. Recall the biased-coin hypothesis testing problem discussed in class. Determine the Neyman-Pearson test for the case when two coin flips are observed. Plot the receiver-operating-characteristic (ROC) curve for this test.

2. Show that we can always find a test for which the detection and false-alarm probabilities are equal. What does this tell you about the behavior the ROC curve in general?

3. Show that the ROC is convex. Namely, if \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) are points on the ROC, where \(\alpha_1 < \alpha_2\), then for any \(0 \leq \lambda \leq 1\), the detection probability corresponding to false-alarm probability \(\lambda \alpha_1 + (1-\lambda)\alpha_2\) is greater than or equal to \(\lambda \beta_1 + (1-\lambda)\beta_2\). Hint: Consider the test \(\delta = \lambda \delta_1 + (1-\lambda)\delta_2\).

4. Consider the following hypothesis testing problem consisting of a deterministic signal embedded in Cauchy noise:

\[
H_0 : Y_i = N_i, \quad N_i \sim \frac{1}{\pi(1 + x^2)} \\
H_1 : Y_i = 1 + N_i,
\]

where the samples are iid and \(i = 1, 2, 3, 4\). A linear detector is used in this case, i.e., \(\sum_{i=1}^{4} Y_i\) is compared to a threshold \(\tau\). Choose \(\tau\) so that \(P_F = 0.1\). What is \(P_D\)? Redo the problem using 8 samples.

5. Consider the previous problem and assume \(i = 1\). Determine the Neyman-Pearson test (assume \(\alpha = 0.05\)) and determine the detection probability. Compare your results to the linear test.

6. Use Matlab to simulate the detector in Problem 4 for both sample-size cases. Empirically verify the correctness of your theoretical false alarm and detection probabilities. Neatly document your results. Experiment with several experiment run lengths until your empirical results stabilize. Next, verify that the Neyman-Pearson decision rule outperforms the test in Problem 4 for the case \(i = 1\).

7. Plot the ROC for a Neyman-Pearson detector for the following hypothesis testing problem:

\[
H_0 : Y \sim N(0, 1) \\
H_1 : Y \sim N(0, 2).
\]

8. Show that the Bayers risk is a convex function of the \(\pi_0\), the prior probability that hypothesis \(H_0\) is true.