Problem 1:
Suppose that the density function of an observed quantity is $f(x) = 0.5\theta e^{-\theta|x|}$, where $\theta$ is a random parameter whose probability density is $f_{\theta}(\theta) = \theta^{-1}, 1 \leq \theta \leq e$, and zero otherwise.

a) Find the MAP estimate of $\theta$ once $X$ is observed. Is it biased?

b) Determine the MMSE estimate. Is it biased?

Problem 2:
Suppose you receive the signal $R(t) = Bs(t - A) + N(t)$ over $0 \leq t \leq T$. Here, the unknown $A$ is uniformly distributed in $[0, T]$ and $B$ is zero-mean Gaussian with variance $\sigma^2$. Moreover, $N(\cdot)$ is white noise. Assume that $s(\cdot)$ is a known signal. Find the MAP estimate $(\hat{A}_{MAP}, \hat{B}_{MAP})$ of $(A, B)$.

Problem 3:
A known signal $s(t)$, $0 \leq t \leq T$, is transmitted over a channel of unknown, non-negative gain $A$ and additive Gaussian noise $N(t)$. Suppose that $\int_0^T s^2(t) \, dt = E$ and the correlation function of the noise is $R_N(t, s) = (N_0/2)\delta(t - s)$.

a) What is the ML estimate of $A$?

b) What is the bias in the ML estimate?

c) Is the ML estimate asymptotically unbiased?

Problem 4:
A signal is transmitted through a murky medium, and the amplitude of the signal is inversely proportional to the murkiness level $M$. Specifically, the output observation is $R(t) = M^{-1}s(t) + N(t), 0 \leq t \leq T$. Assume $\int_0^T s^2(t) \, dt = E$.

a) What is the ML estimate of $M$?

b) Assuming $M$ is zero-mean Gaussian with variance $\sigma^2_M$, find the MAP estimate of $M$.

c) Show that the MAP estimate converges to the ML estimate as $\sigma^2_M \to \infty$.

Interpret this result.