Circuit and Electromagnetic System Design Notes

Note 44

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Nonuniform-Transmission-Line Transformers for Fast High-Voltage Transients

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Abstract

One technique for raising the voltage in a fast pulse involves the use of a nonuniform-transmission-line transformer, the electrical length of which is large compared to the pulse risetime. The high-frequency/early-time voltage gain is given by the square root of the impedance ratio from end to end of the line (for continuous variation of the characteristic impedance, a well-known formula). This paper examines the undesirable droop after the initial step rise at the transmission-line output, with the idea of minimizing the droop. Two different formulations based on voltage/current variables and renormalized wave variables are discussed. These lead to the case of the exponentially tapered line which is discussed in detail.

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1. Introduction

One technique for raising the voltage in a fast pulse involves propagating the pulse along a transmission line with a characteristic impedance which increases along the line. For this purpose the times in the pulse of interest need to be short compared to the transit time t_ℓ along the line. Stated another way, one needs to know how large t_ℓ needs to be for a given characteristic-impedance taper, and which functional forms of taper allow for the smallest or nearly smallest allowable t_ℓ.

Such a nonuniform (or tapered) transmission line may be incorporated in a pulsed-power system as described in fig. 1.1A. We have some pulsed high-voltage source (such as a Marx generator or resonant transformer). This is put onto a transmission line of characteristic impedance Z_1 through a (relatively) slow transfer switch. One would like to launch a pulse with risetime small compared to t_ℓ onto the nonuniform transmission line. For this purpose, one might have a first peaking switch to convert the relatively slowly rising pulse from the transfer switch to something faster. The nonuniform transmission line (NTL) then transitions (ideally smoothly) the initial characteristic impedance Z_1 to its final value Z_2 which is matched to the load (e.g., an antenna). At the load there might be a final peaking switch (very fast) to achieve yet faster performance and even compensate for some of the nonideal losses and/or dispersion in the tapered transmission line.

For our present analysis the pulse incident on the NTL at $z = 0$ with voltage $V^{(inc)}(t)$ is idealized as a step function

$$V^{(inc)}(t) = V_0\, u(t) \tag{1.1}$$

The Thevenin equivalent source driving the line is then the series combination of resistance of value Z_1 with a voltage source (open circuit voltage)

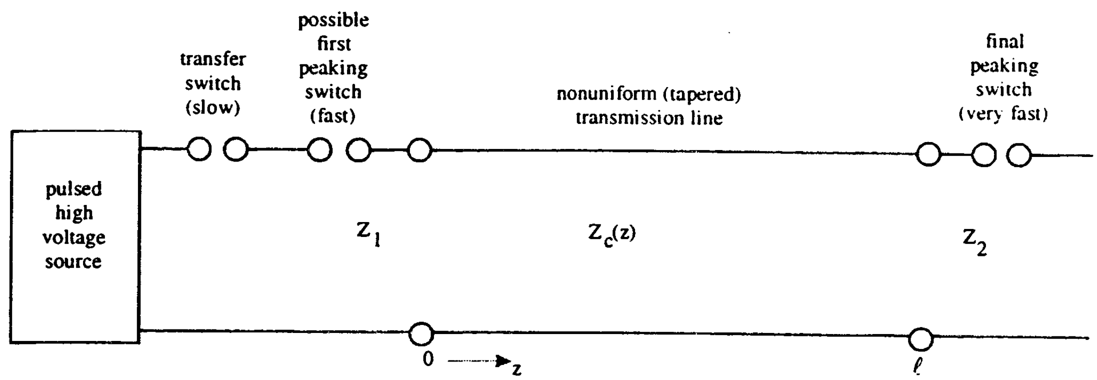
$$V_s(t) = 2V^{(inc)}(t) \tag{1.2}$$

The transmission line has an input impedance

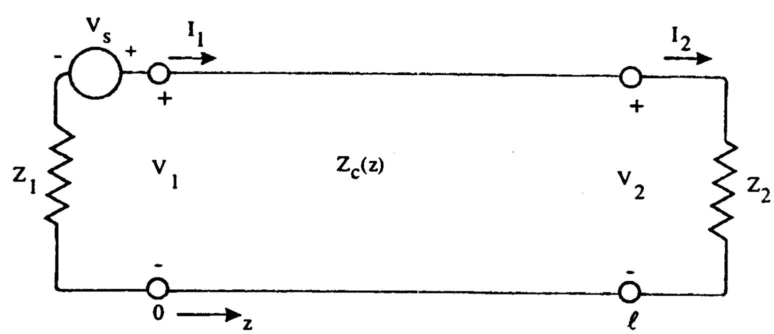
$$\tilde{Z}_{in}(s) = \frac{\tilde{V}_1(s)}{\tilde{I}_1(s)}$$

$$\sim \; \equiv \text{ Laplace transform (two sided) over time t} \tag{1.3}$$

$$s = \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency}$$



A. Pulsed-power system



B. Idealized model

Fig. 1.1 Nonuniform-Transmission-Lin Pulse Transformer.

This parameter will need to be calculated as part of the solution of the NTL problem. There is a transfer function for the line as

$$\tilde{T}_1(s) \equiv e^{st_\ell} \frac{\tilde{V}_2(s)}{\tilde{V}_1(s)} \tag{1.4}$$

where the factor e^{st_ℓ} is introduced for the convenience of removing the delay. More interesting is the transfer function from the voltage *incident* on the NTL as

$$\tilde{T}(s) \equiv e^{st_\ell} \frac{\tilde{V}_2(s)}{\tilde{V}^{(inc)}(s)} = \tilde{T}_1(s) \frac{2\tilde{Z}_{in}(s)}{Z_1 + \tilde{Z}_{in}(s)} \tag{1.5}$$

The telegrapher equations (without sources) are

$$\begin{aligned} \frac{d}{dz} \tilde{V}(z,s) &= -\tilde{Z}'(z,s)\tilde{I}(z,s) \\ \frac{d}{dz} \tilde{I}(z,s) &= -\tilde{Y}'(z,s)\tilde{V}(z,s) \end{aligned} \tag{1.6}$$

For our problem the NTL is assumed lossless with

$$\begin{aligned} \tilde{Z}'(z,s) &= sL'(z) \equiv \text{longitudinal impedance per unit length} \\ \tilde{Y}'(z,s) &= sC'(z) \equiv \text{transverse admittance per unit length} \\ L'(z) &\equiv \text{inductance per unit length} \\ C'(z) &\equiv \text{capacitance per unit length} \end{aligned} \tag{1.7}$$

Furthermore, let us assume (appropriate for pulse power equipment) that the NTL consists of (nearly) perfect conductors in a high-dielectric-strength uniform medium (such as oil or gas) with

$$\begin{aligned} \varepsilon &\equiv \text{permittivity} \\ \mu &\equiv \text{permeability (typically } \mu_0 \text{ of free space)} \\ Z_w &= \left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}} \equiv \text{wave impedance} \\ v_0 &= [\mu\,\varepsilon]^{\frac{1}{2}} \equiv \text{wave speed} \end{aligned} \tag{1.8}$$

This allows us to write some of the parameters in more convenient forms

$$
\begin{aligned}
L'(z) &= \mu f_g(z) \quad , \quad C'(z) = \varepsilon f_g^{-1}(z) \\
Z_c(z) &= \left[\frac{\tilde{Z}'(z,s)}{\tilde{Y}'(z,s)}\right]^{\frac{1}{2}} = Z_w f_g(z) \\
&\equiv \text{NTL characteristic impedance} \\
f_g(z) &\equiv \text{geometric impedance factor} \\
&= \text{positive function of } z \qquad (1.9) \\
t_\ell &= \frac{\ell}{v_0} \equiv \text{transit time (electrical length) of NTL} \\
\gamma &= \frac{s}{v_0} \equiv \text{propagation constant}
\end{aligned}
$$

Note that other forms of an NTL are possible with variable propagation speed, but that these can be converted into the present form by a scaling (change of variable) for the z coordinate.

For later use we introduce normalized parameters

$$
\begin{aligned}
S &\equiv \gamma\ell = s t_\ell \equiv \text{normalized complex frequency} \\
\tau &\equiv \frac{t}{t_\ell} \equiv \text{normalized time} \qquad (1.10)
\end{aligned}
$$

with now a Laplace transform over τ giving a function of S. In this form we are considering the electrical length of the NTL as fixed and attempting to optimize the performance within this constraint. So we rewrite (1.4) and (1.5) in terms of these new variables as

$$
\begin{aligned}
\tilde{T}_1(S) &\equiv e^S \frac{\tilde{V}_2(S)}{\tilde{V}_1(S)} \\
\tilde{T}(S) &\equiv e^S \frac{\tilde{V}_2(S)}{\tilde{V}^{(inc)}(S)} = \tilde{T}_1(S)\, \frac{2\tilde{Z}_{in}(S)}{Z_1 + \tilde{Z}_{in}(S)} \qquad (1.11)
\end{aligned}
$$

The spatial coordinate is also normalized as

$$
\zeta \equiv \frac{z}{\ell} \qquad (1.12)
$$

so that ζ runs from 0_- to 0_+ (allowing for any discontinuities/singularities at the end points of the interval).

In trying to optimize the NTL the constraints on the source and load impedances need to be noted. Some (e.g., [10, 22]) have incorporated frequency-dependent source and load characteristics to compensate for the droop in the pulse at the load after the initial step rise. Much literature has been devoted to the problem of optimizing such NTL transformers, but for power transmission across a broad band of frequencies such as via a Dolph-Tchebycheff taper [8, 9]. In our problem the transmission, including phase, is important. A bibliography of some of the older papers is given in [7]. Much has been done since then. In the present paper we approach the problem using the modern product-integral (or matrizant) approach giving closed-form (product-quadrature) representations [3, 4, 12, 14]. While this has been developed for multiconductor transmission lines it still applies for N=1 giving 2 x 2 matrix equations.

2. High-Frequency Solution for Continuous f_g

With continuous $f_g(z)$ (and bounded derivative) we have the well-known result of power being conserved on the wavefront. In [1] this is developed for N-conductor (plus reference conductor) transmission lines with equal modal speeds. For N = 1 this is

$$\tilde{T}(S) \to \tilde{T}_1(S) \to \left[\frac{Z_2}{Z_1}\right]^{\frac{1}{2}} = \left[\frac{f_g(1)}{f_g(0)}\right] \text{ as } S \to \infty \tag{2.1}$$

$$\tilde{Z}_{in}(S) \to Z_1 \text{ as } S \to \infty$$

Based on this we can define

$$f(\zeta) \equiv \frac{f_g(\zeta)}{f_g(0)} \tag{2.2}$$

and the high-frequency/early-time voltage gain along the line is

$$e^{S\zeta}\,\frac{\tilde{V}(\zeta,S)}{\tilde{V}^{(inc)}(S)} \to e^{S\zeta}\,\frac{\tilde{V}(\zeta,S)}{\tilde{V}_1(S)} \to f^{\frac{1}{2}}(\zeta) \text{ as } S \to \infty \tag{2.3}$$

Then (2.1) can be written for the entire line as

$$\tilde{T}(S) \to \tilde{T}_1(S) \to f^{\frac{1}{2}}(1) \equiv g \text{ as } S \to \infty \tag{2.4}$$

and we can interpret g as the high-frequency/early-time gain of the NTL. In time domain the initial $(\tau = 1_+)$ step response at the matched load (Z_2) is then

$$V_2(1_+) = V_0\, f^{\frac{1}{2}}(\zeta) = gV_0 \tag{2.5}$$

A normalized way to view the problem is to consider the transfer functions in time domain, integrated to give a step response. In this form we have

$$\tilde{R}(S) = \frac{1}{S}\tilde{T}(S) \quad , \quad R(\tau) = \int_{-\infty}^{\tau} T(\tau')d\tau' = \int_{0_-}^{\tau} T(\tau)d\tau'$$

$$\tilde{R}_1(S) = \frac{1}{S}\tilde{T}_1(S) \quad , \quad R_1(\tau) = \int_{-\infty}^{\tau} T_1(\tau')d\tau' = \int_{0_-}^{\tau} T_1(\tau')d\tau' \tag{2.6}$$

In order to determine the response (droop) after the initial step we can consider the asymptotic behavior as $S \to \infty$ in the right half plane (RHP) of the response functions as

$$\frac{1}{g}\tilde{R}(S) = S^{-1} + o\left(S^{-1}\right) \quad \text{as} \quad S \to \infty$$

$$\frac{1}{g}\tilde{R}_1(S) = S^{-2} + o\left(S^{-1}\right) \quad \text{as} \quad S \to \infty \tag{2.7}$$

The $o(S^{-1})$ terms go to zero faster than S^{-1} and give a correction to the step response. For example, if the leading term here is a constant times S^{-1}, this would give a ramp function $\tau u(\tau)$ as the next term in time domain.

3. Matrix Differential Equation

The telegrapher equations (1.6) can be combined in various ways to give a single first-order matrix differential equation. With our assumptions in Section 1, the telegrapher equations read

$$\begin{aligned} \frac{d}{d\zeta}\tilde{V}(\zeta,S) &= -SZf_g(\zeta)\tilde{I}(\zeta,\ell) \\ \frac{d}{d\zeta}\tilde{I}(\zeta,S) &= -\frac{S}{Z}f_g^{-1}(\zeta)\tilde{V}(\zeta,\ell) \end{aligned} \tag{3.1}$$

Important to this problem is how one may wish to normalize the voltage and/or current for analytic convenience.

One procedure [2] uses a constant impedance, say $Z^{(0)}$, to normalize the current giving

$$\frac{d}{d\zeta}\begin{pmatrix} \tilde{V}(\zeta,S) \\ Z^{(0)}\tilde{I}(\zeta,S) \end{pmatrix} = \begin{pmatrix} 0 & -S\,Z\,Z^{(0)^{-1}}f_g(\zeta) \\ -SZ^{-1}Z^{(0)}f_g^{-1}(\zeta) & 0 \end{pmatrix} \cdot \begin{pmatrix} \tilde{V}(\zeta,S) \\ Z^{(0)}\tilde{I}(\zeta,S) \end{pmatrix} \tag{3.2}$$

Choosing for convenience, $Z^{(0)}$ as Z_1, and using the normalized $f(\zeta)$ previously defined, we have

$$\frac{d}{d\zeta}\begin{pmatrix} \tilde{V}(\zeta,S) \\ Z_1\tilde{I}(\zeta,S) \end{pmatrix} = -S\begin{pmatrix} 0 & f(\zeta) \\ f^{-1}(\zeta) & 0 \end{pmatrix} \cdot \begin{pmatrix} \tilde{V}(\zeta,S) \\ Z_1\tilde{I}(\zeta,S) \end{pmatrix} \tag{3.3}$$

with

$$f(0) = 1 \quad , \quad f(1) = \frac{f_g(1)}{f_g(0)} = g^2 \tag{3.4}$$

and the variation of $f(\zeta)$ for ζ between 0 and 1 to be specified.

This differential equation is solved via the matrizant differential equation

$$\begin{aligned} \frac{d}{d\zeta}\left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) &= \left(\tilde{g}_{\sigma,\sigma'}(\zeta,S)\right) \cdot \left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) \\ \left(\tilde{u}_{\sigma,\sigma'}(\zeta_0,\zeta_0;S)\right) &= \left(1_{\sigma,\sigma'}\right) \quad \text{(boundary condition)} \end{aligned} \tag{3.5}$$

The solution to such a matrix differential equation (the matrizant) is expressed via the product integral [13] as

$$\left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) = \prod_{\zeta_0}^{\zeta} e^{\left(g_{\sigma,\sigma'}(\zeta',s)\right)d\zeta'} \tag{3.6}$$

which is understood as the continued dot multiplication (to the left, noting that matrices do not in general commute) of intervals $\Delta\zeta'$, analogous to the usual sum integral.

Identifying

$$\left(\tilde{g}_{\sigma,\sigma'}(\zeta,s)\right) = -s\begin{pmatrix} 0 & f(\zeta) \\ f^{-1}(\zeta) & 0 \end{pmatrix} \tag{3.7}$$

the solution to (3.3) is expressed as

$$\begin{aligned} \begin{pmatrix} \tilde{V}(\zeta,s) \\ Z_1\tilde{I}(\zeta,s) \end{pmatrix} &= \left(\tilde{u}_{\sigma,\sigma'}(\zeta,0;s)\right) \cdot \begin{pmatrix} \tilde{V}(0,s) \\ Z_1\tilde{I}(0,s) \end{pmatrix} \\ &= \left(\tilde{u}_{\sigma,\sigma'}(\zeta,1;s)\right) \cdot \begin{pmatrix} \tilde{V}(1,s) \\ Z_1\tilde{I}(1,s) \end{pmatrix} \end{aligned} \tag{3.8}$$

Here we see that the solution can be expressed in terms of the boundary conditions at either end of the NTL.

Since we have chosen Z_1 as the normalizing impedance we may expect the left-end boundary conditions to be more convenient. However, we could just as easily have chosen Z_2 as the normalizing impedance. But we can conveniently renormalize the impedance to anywhere along the line via [2]

$$\left(\tilde{U}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) \equiv \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & f(\zeta)f^{-1}(\zeta_0) \end{pmatrix} \cdot \left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) & \text{for } \zeta \geq \zeta_0 \\ \left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & f(\zeta)f^{-1}(\zeta_0) \end{pmatrix} & \text{for } \zeta \leq \zeta_0 \end{cases} \tag{3.9}$$

giving

$$\begin{pmatrix} \tilde{V}(\zeta,s) \\ Z_1(\zeta)\tilde{I}(z,s) \end{pmatrix} = \left(\tilde{U}_{\sigma,\sigma'}(\zeta,0;s)\right) \bullet \begin{pmatrix} \tilde{V}(0,s) \\ Z_1\tilde{I}(1,s) \end{pmatrix} = \left(\tilde{U}_{\sigma,\sigma'}(\zeta,1;s)\right) \bullet \begin{pmatrix} \tilde{V}(1,s) \\ Z_2\tilde{I}(1,s) \end{pmatrix} \tag{3.10}$$

Noting for product integrals that

$$\left(\tilde{u}_{\sigma,\sigma'}(\zeta_0,\zeta;s)\right) = \left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right)^{-1} \tag{3.11}$$

we also have

$$\left(\tilde{U}_{\sigma,\sigma'}(\zeta_0,\zeta;s)\right)^{-1} = \left(\tilde{U}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right)^{-1} \tag{3.12}$$

allowing us to readily go back and forth on the NTL. Applying boundary conditions on both ends requires

$$\begin{aligned} \left(\tilde{U}_{\sigma,\sigma'}(1,0;s)\right) &= \begin{pmatrix} 1 & 0 \\ 0 & f(1) \end{pmatrix} \bullet \left(\tilde{u}_{\sigma,\sigma'}(1,0;s)\right) \\ \left(\tilde{U}_{\sigma,\sigma'}(0,1;s)\right) &= \left(\tilde{u}_{\sigma,\sigma'}(0,1;s)\right) \bullet \begin{pmatrix} 1 & 0 \\ 0 & f^{-1}(1) \end{pmatrix} \end{aligned} \tag{3.13}$$

Applying the boundary condition at $\zeta = 1$, viz,

$$\tilde{V}_2(s) = Z_2\tilde{I}_2(s) \tag{3.14}$$

we have

$$\begin{pmatrix} \tilde{V}_1(s) \\ Z_1\tilde{I}_1(s) \end{pmatrix} = \left(\tilde{U}_{\sigma,\sigma'}(0,1;s)\right) \bullet \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tilde{V}_2(s) \tag{3.15}$$

from which we find the input impedance as

$$\frac{\tilde{Z}_{in}(S)}{Z_1} = \frac{\tilde{V}_1(S)}{Z_1\tilde{I}_1(S)} = \frac{\tilde{U}_{1,1}(0,1;S)+\tilde{U}_{1,2}(0,1;S)}{\tilde{U}_{2,1}(0,1;S)+\tilde{U}_{2,2}(0,1;S)}$$
$$= f(1)\frac{\tilde{u}_{1,1}(0,1;S)+\tilde{u}_{1,2}(0,1;S)}{\tilde{u}_{2,1}(0,1;S)+\tilde{u}_{2,2}(0,1;S)} \tag{3.16}$$

Noting for matrizants that

$$\det\left(\left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;\zeta)\right)\right) = 1 \tag{3.17}$$

we have

$$\left(\tilde{u}_{\sigma,\sigma'}(0,1;S)\right) = \left(\tilde{u}_{\sigma,\sigma'}(0,1;S)\right)^{-1} = \begin{pmatrix} \tilde{u}_{2,2}(1,0;S) & -\tilde{u}_{2,1}(1,0;S) \\ -\tilde{u}_{1,2}(1,0;S) & \tilde{u}_{1,1}(1,0;S) \end{pmatrix} \tag{3.18}$$

giving another representation of the input impedance as

$$\frac{\tilde{Z}_{in}(S)}{Z_1} = f(1)\,\frac{\tilde{u}_{2,2}(1,0;S) - \tilde{u}_{2,1}(1,0;S)}{-\tilde{u}_{1,2}(1,0;S) + \tilde{u}_{1,1}(1,0;S)} \tag{3.19}$$

From (3.15) we also have

$$\tilde{T}_1(S) = e^S\frac{\tilde{V}_2(S)}{\tilde{V}_1(S)} = \left[\tilde{U}_{1,1}(0,1;S) + \tilde{U}_{1,2}(0,1;S)\right]^{-1}$$
$$= e^S\left[\tilde{u}_{1,1}(0,1;S) + \tilde{u}_{1,2}(0,1;S)\right]^{-1}$$
$$= e^S\left[\tilde{u}_{2,2}(1,0;S) - \tilde{u}_{2,1}(1,0;S)\right]^{-1} \tag{3.20}$$

This is extended to

$$\tilde{T}(S) = \tilde{T}_1(S)2\left[1+\frac{Z_1}{\tilde{Z}_{in}(S)}\right]^{-1}$$
$$= 2e^S\left[\tilde{u}_{2,2}(1,0;S) - \tilde{u}_{2,1}(1,0;S)\right]^{-1}\left[1+\frac{Z_1}{\tilde{Z}_{in}(S)}\right]^{-1}$$
$$= 2\,e^S\left[\tilde{u}_{2,2}(1,0;S) - \tilde{u}_{2,1}(1,0;S)+f^{-1}(1)\left[\tilde{u}_{1,1}(1,0;S) - \tilde{u}_{1,2}(1,0;S)\right]\right]^{-1} \tag{3.21}$$

At this point, the results apply to various forms one might choose for $f(\zeta)$. Appendix A applies this to the exponential transmission-line taper.

4. Symmetric Renormalization

The previous section has developed the product integral for the NTL by using a constant (with ζ) normalizing impedance. While this gives a correct result, it does not plainly exhibit the high-frequency/early-time behavior discussed in Section 2. This section develops a form of the matrix equation which brings out the high-frequency/early-time form as a leading term.

A previous paper [4] defines a symmetric renormalization procedure for the voltage and current variables which we can use here for the simpler case of $N = 1$. For this purpose we define

$$\tilde{v}(\zeta,s) \equiv f_g^{-\frac{1}{2}}(\zeta)\tilde{V}(\zeta,s) \quad , \quad \tilde{i}(\zeta,s) \equiv Zf_g^{\frac{1}{2}}(\zeta)\tilde{I}(\zeta,s) \tag{4.1}$$

The telegrapher equations (3.1) then become

$$\begin{aligned} \frac{d}{d\zeta}\left[f_g^{\frac{1}{2}}(\zeta)\tilde{v}(\zeta,s)\right] &= -sZf_g^{-\frac{1}{2}}(\zeta)\tilde{i}(\zeta,s) \\ \frac{d}{d\zeta}\left[f_g^{-\frac{1}{2}}(\zeta)\tilde{i}(\zeta,s)\right] &= -\frac{s}{Z}f_g^{\frac{1}{2}}(\zeta)\tilde{v}(\zeta,s) \end{aligned} \tag{4.2}$$

Expanding the derivatives and rearranging we have

$$\begin{aligned} \frac{d\tilde{v}(\zeta,s)}{d\zeta} &= -\frac{1}{2}f_g^{-1}(\zeta)\frac{d f_g(\zeta)}{d\zeta}\tilde{v}(\zeta,s) - s\tilde{i}(\zeta,s) \\ &= -\frac{1}{2}\frac{d\,\ell n\left(f_g(\zeta)\right)}{d\zeta}\tilde{v}(\zeta,s) - s\tilde{i}(\zeta,s) \\ \frac{d\tilde{i}(\zeta,s)}{d\zeta} &= \frac{1}{2}f_g^{-1}(\zeta)\frac{d f_g(\zeta)}{d\zeta}\tilde{i}(\zeta,s) - s\tilde{v}(\zeta,s) \\ &= \frac{1}{2}\frac{d\,\ell n\left(f_g(\zeta)\right)}{d\zeta}\tilde{v}(\zeta,s) - s\tilde{i}(\zeta,s) \end{aligned} \tag{4.3}$$

For convenience define

$$F(\zeta) \equiv \frac{d}{d\zeta} \ell n\left(f_g^{\frac{1}{2}}(\zeta) \right) = \frac{d}{d\zeta} \ell n\left(f^{\frac{1}{2}}(\zeta) \right) \tag{4.4}$$

Note that

$$\begin{aligned} F(0_-) &= F(1_+) = 0 \\ \int_{0_-}^{1_+} F(\zeta) d\zeta &= \ell n\left(f^{\frac{1}{2}}(1) \right) = \ell n\left(g^{\frac{1}{2}} \right) = \frac{1}{2} \ell n(g) \end{aligned} \tag{4.5}$$

Then (4.3) can be combined as

$$\begin{aligned} \frac{d}{d\zeta}\begin{pmatrix} \tilde{v}(\zeta,S) \\ \tilde{i}(\zeta,S) \end{pmatrix} &= -F(\zeta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bullet \begin{pmatrix} \tilde{v}(\zeta,S) \\ \tilde{i}(\zeta,S) \end{pmatrix} - S\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bullet \begin{pmatrix} \tilde{v}(\zeta,S) \\ \tilde{i}(\zeta,S) \end{pmatrix} \\ &= \begin{pmatrix} -F(S) & -S \\ -S & F(\zeta) \end{pmatrix} \bullet \begin{pmatrix} \tilde{v}(\zeta,S) \\ \tilde{i}(\zeta,S) \end{pmatrix} \end{aligned} \tag{4.6}$$

It is now convenient to convert to wave variables [3] as

$$\begin{aligned} \begin{pmatrix} \tilde{v}_+(\zeta,S) \\ \tilde{v}_-(\zeta,S) \end{pmatrix} &= \left(Q_{\sigma,\sigma'}\right) \bullet \begin{pmatrix} \tilde{v}(\zeta,S) \\ \tilde{i}(\zeta,S) \end{pmatrix} \\ \left(Q_{\sigma,\sigma'}\right) &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ , \quad \left(Q_{\sigma,\sigma'}\right)^{-1} = \frac{1}{2}\left(Q_{\sigma,\sigma'}\right) \\ \tilde{v}_\pm(\zeta,S) &= \tilde{v}(\zeta,S) \pm \tilde{i}(\zeta,S) \\ \tilde{v}(\zeta,S) &= \frac{1}{2}\left[\tilde{v}_+(\zeta,S) + \tilde{v}_-(\zeta,S)\right] \ , \quad \tilde{i}(\zeta,S) = \frac{1}{2}\left[\tilde{v}_+(\zeta,S) - \tilde{v}_-(\zeta,S)\right] \end{aligned} \tag{4.7}$$

Let us next define

$$\left(\tilde{\gamma}_{\sigma,\sigma'}(\zeta,S)\right) \equiv \left(Q_{\omega,\sigma'}\right) \bullet \left[-F(\zeta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - S\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \bullet \left(Q_{\sigma,\sigma'}\right)^{-1} \tag{4.8}$$

and note that

$$\left(Q_{\sigma,\sigma'}\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \left(Q_{\sigma,\sigma'}\right)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\left(Q_{\sigma,\sigma'}\right) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \left(Q_{\sigma,\sigma'}\right)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.9}$$

giving

$$\left(\tilde{\gamma}_{\sigma,\sigma'}(\zeta,s)\right) = -s\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - F(\zeta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.10}$$

Then (4.6) is transformed to

$$\frac{d}{d\zeta}\begin{pmatrix} \tilde{\mathrm{v}}_+(\zeta,s) \\ \tilde{\mathrm{v}}_-(\zeta,s) \end{pmatrix} = \left(\tilde{\gamma}_{\sigma,\sigma'}(\zeta,s)\right) \cdot \begin{pmatrix} \tilde{\mathrm{v}}_+(\zeta,s) \\ \tilde{\mathrm{v}}_-(\zeta,s) \end{pmatrix} \tag{4.11}$$

The corresponding matrizant equation is now

$$\frac{d}{d\zeta}\left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) = \left(\gamma_{\sigma,\sigma'}(\zeta,s)\right) \cdot \left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right)$$
$$\left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta_0,\zeta_0;s)\right) = \left(1_{\sigma,\sigma'}\right) \tag{4.12}$$

with product integral solution

$$\left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) = \prod_{\zeta_0}^{\zeta} e^{\left(\tilde{\gamma}_{\sigma,\sigma'}(\zeta',s)\right)d\zeta'} \tag{4.13}$$

The solution to (4.11) is now expressed as

$$\begin{pmatrix} \tilde{\mathrm{v}}_+(\zeta,s) \\ \tilde{\mathrm{v}}_-(\zeta,s) \end{pmatrix} = \left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,0;s)\right) \cdot \begin{pmatrix} \tilde{\mathrm{v}}_+(0,s) \\ \tilde{\mathrm{v}}_-(0,s) \end{pmatrix}$$
$$= \left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,1;s)\right) \cdot \begin{pmatrix} \tilde{\mathrm{v}}_+(1,s) \\ \tilde{\mathrm{v}}_-(1,s) \end{pmatrix} \tag{4.14}$$

in terms of the boundary conditions at either end of the NTL.

Applying the boundary condition at $\zeta = 1$ we have

$$
\begin{aligned}
\tilde{V}_2(S) &= f_g^{\frac{1}{2}}(1)\tilde{v}(1,S) = Z_2\tilde{I}_2(S) = Zf_g(1)\tilde{I}_2(S) = f_g^{\frac{1}{2}}(1)\tilde{i}(1,S) \\
\tilde{v}_\pm(1,S) &= f_g^{\frac{1}{2}}(1)\tilde{V}_2(S) \pm Zf_g^{\frac{1}{2}}(1)\tilde{I}(1,S) \\
&= f_g^{\frac{1}{2}}(1)[1 \pm 1]\tilde{V}_2(S) \\
\tilde{v}_-(1,S) &= 0 \\
\tilde{v}_+(1,S) &= f_g^{-\frac{1}{2}}(1)\tilde{V}_2(S)
\end{aligned}
\tag{4.15}
$$

So we see in this form that there is no reflected wave (conveniently) at $\zeta = 1$. So at $\zeta = 0$ we have

$$
\begin{aligned}
\begin{pmatrix} \tilde{v}_+(0,S) \\ \tilde{v}_-(0,S) \end{pmatrix} &= \left(\tilde{\Gamma}_{\sigma,\sigma'}(0,1;S)\right) \cdot \begin{pmatrix} \tilde{v}_+(1,S) \\ \tilde{v}_-(1,S) \end{pmatrix} = \left(\tilde{\Gamma}_{\sigma,\sigma'}(0,1;S)\right) \cdot \begin{pmatrix} \tilde{v}_+(0,S) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \tilde{\Gamma}_{1,1}(0,1;S) \\ \tilde{\Gamma}_{2,1}(0,1;S) \end{pmatrix} 2 f_g^{\frac{1}{2}}(1)\tilde{V}_2(S) \\
&= \begin{pmatrix} f_g^{-\frac{1}{2}}(0)\tilde{V}_1(0) + Z\, f_g^{\frac{1}{2}}(0)\tilde{I}_1(S) \\ f_g^{-\frac{1}{2}}(0)\tilde{V}_1(0) - Z\, f_g^{\frac{1}{2}}(0)\tilde{I}_1(S) \end{pmatrix}
\end{aligned}
\tag{4.16}
$$

Separating out the voltage and current at $\zeta = 0$ gives

$$
\begin{aligned}
\tilde{V}_1(S) &= f^{-\frac{1}{2}}(1)\left[\tilde{\Gamma}_{1,1}(0,1;S) + \tilde{\Gamma}_{2,1}(0,1;S)\right]\tilde{V}_2(S) \\
Zf_g(0)\tilde{I}_1(S) &= f^{-\frac{1}{2}}(1)\left[\tilde{\Gamma}_{1,1}(0,1;S) - \tilde{\Gamma}_{2,1}(0,1;S)\right]\tilde{V}_2(S)
\end{aligned}
\tag{4.17}
$$

where $f(1)$ is the normalized impedance ratio (Section 2). This leads to the input impedance

$$
\tilde{Z}_{in}(S) = \frac{\tilde{V}_1(S)}{\tilde{I}_1(S)} = Z_1 \frac{\tilde{\Gamma}_{1,1}(0,1;S) + \tilde{\Gamma}_{2,1}(0,1;S)}{\tilde{\Gamma}_{1,1}(0,1;S) - \tilde{\Gamma}_{2,1}(0,1;S)}
\tag{4.18}
$$

The transfer functions are now

$$\tilde{T}_1(S) = e^S \frac{\tilde{V}_2(S)}{\tilde{V}_1(S)} = g\, e^S \left[\tilde{\Gamma}_{1,1}(0,1;S) + \tilde{\Gamma}_{2,1}(0,1;S)\right]^{-1}$$

$$\tilde{T}(S) = e^S \frac{\tilde{V}_2(S)}{\tilde{V}^{(inc)}(S)} = \tilde{T}_1(S)\, 2\left[1 + \frac{Z_1}{\tilde{Z}_{in}(S)}\right] \tag{4.19}$$

$$= g \frac{e^S}{\tilde{\Gamma}_{1,1}(0,1;S)}$$

so $\tilde{T}(S)$ has a conveniently simple form as compared to the form in Section 3.

Now consider the product integral in (4.13). Write the integrand as

$$\left(\tilde{\gamma}_{\sigma,\sigma'}(\zeta,S)\right) = \left(\tilde{a}_{\sigma,\sigma'}(S)\right) + \left(b_{\sigma,\sigma'}(\zeta)\right)$$

$$\left(\tilde{a}_{\sigma,\sigma'}(S)\right) = -S\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.20}$$

$$\left(b_{\sigma,\sigma'}(\zeta)\right) = -F(\zeta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and apply the sum rule of the product integral [4, 13] to give

$$\left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) = \left(\tilde{A}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) \cdot \prod_{\zeta_0}^{\zeta} e^{\left(\tilde{A}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right)^{-1} \cdot \left(b_{\sigma,\sigma'}(\zeta')\right) \cdot \left(\tilde{A}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) d\zeta'}$$

$$\left(\tilde{A}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) = \prod_{\zeta_0}^{\zeta} e^{-\zeta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} d\zeta'}$$

$$= e^{-\int_{\zeta_0}^{\zeta} S d\zeta'} = e^{-S[\zeta-\zeta_0]\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$= \cosh\left(S[\zeta-\zeta_0]\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \sinh\left(S[\zeta-\zeta_0]\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.21}$$

$$= \begin{pmatrix} e^{-S[\zeta-\zeta_0]} & 0 \\ 0 & e^{S[\zeta-\zeta_0]} \end{pmatrix}$$

This last result is readily found from the series representation of the matrix exponential. Next we have

$$\left(\tilde{B}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) \equiv \left(\tilde{A}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right)^{-1} \cdot \left(b_{\sigma,\sigma'}(\zeta)\right) \cdot \left(\tilde{A}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right)$$

$$
\begin{aligned}
&= -F(\zeta)\begin{pmatrix} e^{S[\zeta-\zeta_0]} & 0 \\ 0 & e^{-S[\zeta-\zeta_0]} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{-S[\zeta-\zeta_0]} & 0 \\ 0 & e^{S[\zeta-\zeta_0]} \end{pmatrix} \\
&= -F(\zeta)\begin{pmatrix} 0 & e^{2S[\zeta-\zeta_0]} \\ e^{-2S[\zeta-\zeta_0]} & 0 \end{pmatrix}
\end{aligned} \tag{/4.22}
$$

giving

$$
\left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) = \begin{pmatrix} e^{-S[\zeta-\zeta_0]} & 0 \\ 0 & e^{S[\zeta-\zeta_0]} \end{pmatrix} \cdot \prod_{\zeta_0}^{\zeta} e^{\left(\tilde{B}_{\sigma,\sigma'}(\zeta',\zeta_0;S)\right)d\zeta'} \tag{4.23}
$$

Now we see the first matrix does give directly the high-frequency/early-time response of the NTL with the 1,1 term giving the leading term for $\tilde{v}_+$ and the 2,2 term giving similarly for $\tilde{v}_-$. Note in (4.1) the scaling of voltage as $f_g^{\frac{1}{2}}$ times these wave variables.

For the remaining product integral we have the well-known series representation [3] of the matrizant as

$$
\begin{aligned}
\left(\tilde{X}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) &= \prod_{\zeta_0}^{\zeta} e^{\left(\tilde{B}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right)d\zeta'} \\
&= \sum_{m=0}^{\infty}\left(\tilde{X}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right)_m \\
\left(\tilde{X}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right)_0 &= \left(1_{\sigma,\sigma'}\right) \\
\left(\tilde{X}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right)_{m+1} &= \int_{\zeta_0}^{\zeta}\left(\tilde{B}_{\sigma,\sigma'}(\zeta',\zeta_0;S)\right) \cdot \left(\tilde{X}_{\sigma,\sigma'}(\zeta',\zeta_0;S)\right)_m d\zeta' \\
\left(\tilde{X}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right)_m &= \int_{\zeta_0}^{\zeta}\left(\tilde{B}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) \cdot \int_{\zeta_0}^{\zeta_1}\left(\tilde{B}_{\sigma,\sigma'}(\zeta_2,\zeta_0;S)\right) \cdot \int_{\zeta_0}^{\zeta_2} \cdots \\
&\qquad \int_{\zeta_0}^{\zeta_{m-1}}\left(\tilde{B}_{\sigma,\sigma'}(\zeta_m,\zeta_0;S)\right) d\zeta_m d\zeta_{m-1} \cdots d\zeta_1
\end{aligned} \tag{4.24}
$$

The leading term being the identity, then if the rest are sufficiently small, the$\left(\tilde{A}_{\sigma,\sigma'}\right)$ term gives the result which is the high-frequency result of Section 2.

The second term ($m = 1$) is

$$\left(\tilde{X}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right)_1 = -\int_{\zeta_0}^{\zeta} F(\zeta')\begin{pmatrix} 0 & e^{2S[\zeta' \bullet \zeta_0]} \\ e^{-2S[\zeta' \bullet \zeta_0]} & 0 \end{pmatrix} d\zeta' \tag{4.25}$$

Integrating by parts gives

$$\begin{aligned} \left(\tilde{X}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right)_1 &= -\frac{F(\zeta')}{2S}\begin{pmatrix} 0 & e^{2S[\zeta'-\zeta_0]} \\ -e^{-2S[\zeta'-\zeta_0]} & 0 \end{pmatrix}\Bigg|_{\zeta_0}^{\zeta} \\ &\quad + \frac{1}{2S}\int_{\zeta_0}^{\zeta} \frac{dF(z')}{d\zeta'}\begin{pmatrix} 0 & e^{2S[\zeta'-\zeta_0]} \\ -e^{-2S[\zeta'-\zeta_0]} & 0 \end{pmatrix} d\zeta' \\ &= -\frac{F(\zeta)}{2S}\begin{pmatrix} 0 & e^{2S[\zeta'-\zeta_0]} \\ -e^{-2S[\zeta'-\zeta_0]} & 0 \end{pmatrix} + \frac{F(\zeta_0)}{2S}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &\quad + \frac{1}{2S}\int_{\zeta_0}^{\zeta} \frac{dF(\zeta')}{d\zeta'}\begin{pmatrix} 0 & e^{2S[\zeta'-\zeta_0]} \\ -e^{-2S[\zeta'-\zeta_0]} & 0 \end{pmatrix} d\zeta' \end{aligned} \tag{4.26}$$

assuming that the derivative of $F(\zeta)$ exists, or is at least integrable.

If we set $\zeta_0 = 1_+$ and $\zeta = 0_-$ we have (for use in (4.18) and (4.19))

$$\left(\tilde{\Gamma}_{\sigma,\sigma'}(0_-,1_+;s)\right) = \begin{pmatrix} e^{S} & 0 \\ 0 & e^{-S} \end{pmatrix} \cdot \left(\tilde{X}_{\sigma,\sigma'}(0_-,1_+;S)\right) \tag{4.27}$$

From (4.26) we have

$$\left(\tilde{X}_{\sigma,\sigma'}(0_-,1_+;S)\right)_1 = -\frac{1}{2S}\int_{0_-}^{1_+} \frac{dF(\zeta')}{d\zeta'}\begin{pmatrix} 0 & e^{2S[\zeta'-1]} \\ -e^{-2S[\zeta'-1]} & 0 \end{pmatrix} d\zeta' \tag{4.28}$$

The integration by parts has brought out a leading S^{-1} coefficient which in time domain is a term proportional to $\tau u(\tau)$ (or the droop term) in response to a step input. If $F(\zeta)$ is discontinuous at either $\zeta = 0$ or $\zeta = 1$ (discontinuity in the derivative of $f(z)$) then $dF(\zeta)/d\zeta$ has a delta function there and we can write

$$\left(\mathrm{X}_{\sigma,\sigma'}(0_-,1_+;S)\right)_1 = -\frac{1}{2S}\left[\left[F(1_+) - F(1_+)\right]\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right.$$
$$\left. + \left[F(0_+) - F(0_-)\right]\begin{pmatrix}0 & e^{-2S}\\ -e^{2S} & 0\end{pmatrix}\right] + \mathrm{o}\left(S^{-1}\right) \tag{4.29}$$
$$+ -\frac{1}{2S}\left[F(0_+)\begin{pmatrix}0 & e^{-2S}\\ -e^{2S} & 0\end{pmatrix} - F(1_-)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]$$
$$+ \mathrm{o}\left(S^{-1}\right) \text{ as } S \to \infty$$

The order symbol is actually $\mathrm{O}(S^{-2})$ if we impose boundedness of $dF(\zeta)/d\zeta$ elsewhere in the interval. This result also shows that if we make

$$F(0_+) = F(1) = 0 \tag{4.30}$$

so that $d[\ell n(f(\zeta))]/d\zeta$ is continuous throughout the interval (including the end points), then $(\tilde{\mathrm{X}}_{\sigma,\sigma'})_1$ is $\mathrm{O}(S^{-2})$. Integrating (4.28) again by parts gives

$$\left(\tilde{\mathrm{X}}_{\sigma,\sigma'}(0_-,1_+;S)\right)_1 = -\frac{1}{[2S]^2}\frac{dF(\zeta')}{d\zeta'}\begin{pmatrix}0 & e^{2S\zeta'}\\ e^{-2S\zeta'} & 0\end{pmatrix}\Bigg|_{0_-}^{1_+}$$
$$+ \frac{1}{[2S]^2}\int_{0_-}^{1_+}\frac{d^2F(\zeta')}{d\zeta'^2}\begin{pmatrix}0 & e^{2S\zeta'}\\ e^{-2S\zeta'} & 0\end{pmatrix}d\zeta' \tag{4.31}$$
$$= \frac{1}{[2S]^2}\int_{0_-}^{1_+}\frac{d^2F(\zeta')}{d\zeta'^2}\begin{pmatrix}0 & e^{2S\zeta'}\\ e^{-2S\zeta'} & 0\end{pmatrix}d\zeta'$$

now assuming that $d^2F(\zeta')/d\zeta'^2$ is integrable. If $dF(\zeta')/d\zeta'$ is discontinuous, say at the end points (noting that all derivatives are zero outside the interval) then this integral is evaluated as

$$\left(\tilde{\mathrm{X}}_{\sigma,\sigma'}(0_-,1_+;S)\right)_1 = \frac{1}{[2S]^2}\left[\frac{dF(\zeta)}{d\zeta}\bigg|_{\zeta=0_+}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \frac{dF(\zeta)}{d\zeta}\bigg|_{\zeta=0_-}\begin{pmatrix}0 & e^{2S}\\ e^{-2S} & 0\end{pmatrix}\right] + \mathrm{o}\left(S^{-2}\right) \text{ as } S \to \infty \tag{4.32}$$

This process can be extended to higher orders by bounding successive derivatives of $F(\zeta)$, thereby making the high-frequency form of $(\tilde{X}_{\sigma,\sigma'})_1$ go to arbitrarily large negative powers of S. This says that by controlling the derivatives of $F(\zeta)$, one can make this term contribute negligibly to the droop, if by droop we mean S^{-1} terms. Of course, since the $(\tilde{X}_{\sigma,\sigma'})_1$ term is not identically zero, there is, in general, some deviation from unity (in time domain) after the initial rise, but the above asymptotic expansion is not adequate to describe it. With all derivatives existing, some other more elaborate description may be appropriate. However, higher order $(\tilde{X}_{\sigma,\sigma'})_m$ also need to be considered.

Writing out the first two terms in (4.27) gives

$$\begin{aligned}\left(\tilde{\Gamma}_{\sigma,\sigma'}(0,1;s)\right) &= e^S\begin{pmatrix}1 & 0\\ 0 & e^{-2S}\end{pmatrix}\cdot\left[\left(1_{\sigma,\sigma'}\right)+\left(\tilde{X}_{\sigma,\sigma'}(0_-,1_+;S)\right)_1 + \text{higher order terms } (m\geq 2)\right]\\ &= e^S\left[\begin{pmatrix}1 & \tilde{X}_{1,2;1}(0_-,1_+;S)\\ e^{-2S}\tilde{X}_{2,1;1}(0_-,1_+;S) & e^{-2S}\end{pmatrix} + \text{higher order terms } (m\geq 2)\right]\end{aligned} \tag{4.33}$$

Substituting from (4.29) this becomes

$$\begin{aligned}\left(\tilde{\Gamma}_{\sigma,\sigma'}(0,1;s)\right) = e^S\Bigg[&\begin{pmatrix}1 & -\frac{1}{2S}\left[F(0_+)e^{-2S}+F(1_-)\right]\\ \frac{1}{2S}\left[F(0_+)+F(1_-)e^{-2S}\right] & e^{-2S}\end{pmatrix}\\ &+ O\left(S^{-2}\right) + \text{higher order terms } (m\geq 2)\Bigg]\end{aligned} \tag{4.34}$$

From (4.19) we have

$$\begin{aligned}\tilde{T}_1(S) &= g\left[1+\frac{1}{2S}F(0_+)+O\left(S^{-2}\right) \text{ in RHP} + \text{higher order terms } (m\geq 2)\right]^{-1}\\ &= g\left[1-\frac{1}{2S}F(0_+)+O\left(S^{-2}\right) \text{ in RHP} + \text{higher order terms } (m\geq 2)\right]\end{aligned} \tag{4.35}$$

Note that the delay term e^{-2S} is neglecting, arriving in time domain at $\tau = 2$. Here we see a droop term

$$-\frac{1}{2S}F(0_+) \rightarrow -\frac{\tau}{2}u(\tau)F(0_+) \tag{4.36}$$

which is proportional to the discontinuity in the logarithmic derivative of $f_g(z)$ at $z=0$. There are also the terms for $m \geq 2$ which we expect to be generally smaller. From (4.19) we also have

$$\tilde{T}(S) = g\left[1+O\left(S^{-2}\right) \text{ in RHP+ higher order terms } (m \geq 2)\right] \tag{4.37}$$

the $m = 1$ term having no contribution.

Now let us consider the $m = 2$ term which is

$$\begin{aligned}
&\left(\tilde{X}_{\sigma,\sigma'}(0_-,1_+;s)\right)_2 \\
&= \int_{0_-}^{1_+} F(\zeta')\begin{pmatrix} 0 & e^{2S[\zeta'-1]} \\ e^{-2S[\zeta'-1]} & 0 \end{pmatrix} \cdot \int_{\zeta_-}^{1_+} F(\zeta'')\begin{pmatrix} 0 & e^{2S[\zeta''-1]} \\ e^{-2S[\zeta''-1]} & 0 \end{pmatrix} d\zeta'' d\zeta' \\
&= \int_{0_-}^{1_+} F(\zeta') = \int_{\zeta'}^{1_+} F(\zeta'')\begin{pmatrix} e^{-2S[\zeta''-\zeta']} & 0 \\ 0 & e^{2S[\zeta''-\zeta']} \end{pmatrix} d\zeta'' d\zeta'
\end{aligned} \tag{4.38}$$

Again we integrate by parts over ζ'' giving

$$\begin{aligned}
&\int_{\zeta'}^{1_+} F(\zeta'')\begin{pmatrix} e^{-2S[\zeta''-\zeta']} & 0 \\ 0 & e^{2S[\zeta''-\zeta']} \end{pmatrix} d\zeta'' \\
&= \frac{F(\zeta'')}{2S}\begin{pmatrix} -e^{-2S[\zeta''-\zeta']} & 0 \\ 0 & e^{2S[\zeta''-\zeta']} \end{pmatrix}\Bigg|_{\zeta'}^{1_+} \\
&\quad -\frac{1}{2S}\int_{\zeta'}^{1_+} \frac{dF(\zeta'')}{d\zeta''}\begin{pmatrix} -e^{-2S[\zeta''-\zeta']} & 0 \\ 0 & e^{2S[\zeta''-\zeta']} \end{pmatrix} d\zeta'' \\
&= \frac{F(\zeta')}{2S}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{2S}\int_{\zeta'}^{1_+} \frac{dF(\zeta'')}{d\zeta''}\begin{pmatrix} -e^{-2S[\zeta''-\zeta']} & 0 \\ 0 & e^{2S[\zeta'-\zeta']} \end{pmatrix} d\zeta'' \quad .
\end{aligned} \tag{4.39}$$

Integrating the remaining integral by parts gives

$$\int_{\zeta'}^{1_+} \frac{dF(\zeta'')}{d\zeta''}\begin{pmatrix} -e^{-2S[\zeta''-\zeta']} & 0 \\ 0 & e^{2S[\zeta''-\zeta']} \end{pmatrix} d\zeta''$$

$$
\begin{aligned}
&= \frac{1}{2S}\frac{dF(\zeta'')}{d\zeta''}\begin{pmatrix} e^{-2S[\zeta''-\zeta']} & 0 \\ 0 & e^{2S[\zeta''-\zeta']}\end{pmatrix}\Bigg]_{\zeta'}^{1_+} \\
&\quad -\frac{1}{2S}\int_{\zeta'}^{1_+}\frac{d^2F(\zeta'')}{d\zeta''^2}\begin{pmatrix} e^{-2S[\zeta''-\zeta']} & 0 \\ 0 & e^{2S[\zeta''-\zeta']}\end{pmatrix}d\zeta'' \\
&= -\frac{1}{2S}\frac{dF(\zeta')}{d\zeta'}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \frac{1}{2S}\int_{\zeta'}^{1_+}\frac{d^2F(\zeta'')}{d\zeta''^2}\begin{pmatrix} e^{-2S[\zeta''-\zeta']} & 0 \\ 0 & e^{2S[\zeta''-\zeta']}\end{pmatrix}d\zeta''
\end{aligned} \tag{4.40}
$$

where now we assume that $dF(\zeta)/d\zeta$ has at most step discontinuities throughout the interval. This makes the above integral $O(S^{-1})$. Substituting in (4.38) we have

$$
\left(\tilde{X}_{\sigma,\sigma'}(0_-,1_+;S)\right)_2 = \frac{1}{2S}\int_{0_-}^{1_+} F^2(\zeta')d\zeta'\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + O\left(S^{-2}\right) \text{ as } S\to\infty \tag{4.41}
$$

Now we see that (4.41) provides a correction to $\tilde{T}(s)$ in (4.19) as

$$
\begin{aligned}
\tilde{T}(S) &= g\left[1+\frac{1}{2S}\int_{0_-}^{1_+}F^2(\zeta')d\zeta' + O\left(S^{-2}\right) + \text{higher order terms } (m\ge 3)\right] \\
&= g\left[1-\frac{1}{2S}\int_{0_-}^{1_+}F^2(\zeta')d\zeta' + O\left(S^{-2}\right) + \text{higher order terms } (m\ge 3)\right]
\end{aligned} \tag{4.42}
$$

Here we see a droop term

$$
-\frac{1}{2S}\int_{0_-}^{1_+}F^2(\zeta')d\zeta' \to -\frac{\tau}{2}u(\tau)\int_{0_-}^{1_+}F^2(\zeta')d\zeta' \tag{4.43}
$$

So now we see an interesting term to minimize, viz, the constant

$$
C_1 \equiv \int_{0_-}^{1_+}F^2(\zeta')d\zeta' \ , \quad F(\zeta) = \frac{d}{d\zeta}\ell n\left(f^{\frac{1}{2}}(\zeta)\right) \tag{4.44}
$$

However, we cannot just make $F(\zeta) = 0$ over the entire interval or we have no NTL transformer. We have

$$F(0_-) = \frac{d}{d\zeta}\ell n\left(f^{\frac{1}{2}}(\zeta)\right)\Bigg|_{\zeta=0_-} = 0$$

$$F(1_+) = \frac{d}{d\zeta}\ell n\left(f^{\frac{1}{2}}(\zeta)\right)\Bigg|_{\zeta=1_+} = 0 \tag{4.45}$$

$$f^{\frac{1}{2}}(1) = \left[\frac{f_g(1)}{f_g(0)}\right]^{\frac{1}{2}} = g \equiv \text{voltage gain of NTL}$$

This, in turn, implies

$$C_2 = \int_{0_-}^{1_+} \frac{d}{d\zeta'}\ell n\left(f^{\frac{1}{2}}(\zeta')\right)d\zeta' = \ell n\left(\left[\frac{f_g(1_+)}{f_g(0_-)}\right]^{\frac{1}{2}}\right) = \ell n(g) = G \tag{4.46}$$

as a constraint on $F(\zeta)$. A monotone increasing $f(\zeta)$ on the NTL makes $F(\zeta)$ positive on the interval. The question then becomes one of minimizing the 2-norm of $F(\zeta)$ with its average equal to G on the interval. One readily finds that the solution with this as the value throughout the interval is

$$F(\zeta) = \ell n(g) = \frac{d}{d\zeta}\ell n\left(f^{\frac{1}{2}}(z)\right)$$

$$\frac{1}{2}\ell n(f(s)) = \ell n(g)\zeta \tag{4.47}$$

$$f(\zeta) = \frac{f_g(\zeta)}{f_g(0_-)} = e^{2\zeta \ell n(g)} = g^{2\zeta}$$

$$f(1_+) = g^2 \quad , \quad f(0_-) = 1$$

This is an exponential transmission line. (See Appendix A.) that this choice minimizes C_1 as

$$C_1 = \ell n^2(g) = G^2 \tag{4.48}$$

can be seen by writing $F(\zeta)$ as $\ell n(g)+\Delta F$ where ΔF has average value zero on the interval (thereby retaining C_2 as in (4.45)). In integrating F^2 as in (4.43) the cross term $2\ell n(g)\Delta F$ integrates to zero, but the $[\Delta F]^2$ term is positive unless $\Delta F = 0$ throughout the interval.

The result of (4.47) needs to be tempered by the assumption in (4.40) that $dF/d\zeta$ has at most step discontinuities, implying that F is continuous with bounded derivative. Since F is zero outside the interval this means that near the end points $F(\zeta)$ needs a smooth transition between 0 and $\ell n(g)$. Of course, one can allow step discontinuities there, but this brings in other contributions to the S^{-1} term.

With the foregoing results, we now have an estimate of the droop time τ_d (normalized) via

$$\begin{aligned} \tilde{T}(S) &= g\left[1 - \frac{1}{S\tau_d} + O\left(S^{-2}\right) + \text{higher order terms } (m \geq 3)\right] \\ \tau_d &= 2\left[\int_{0_-}^{1_+} F^2(\zeta')d\zeta'\right]^{-2} = 2G^{-2} \end{aligned} \tag{4.49}$$

Curiously enough, this matches the exact result for the exponential NTL (Appendix A).

Returning to consideration of $\tilde{T}_1(S)$ we now have the additional contribution to (4.33) from the $m = 2$ term, giving from (4.19) the result

$$\begin{aligned} \tilde{T}_1(S) &= g\left[1 + \frac{1}{2S}\left[F(0_+) + \int_{0_-}^{1_+} F^2(\zeta')d\zeta'\right] + O\left(S^{-2}\right) + \text{higher order terms } (m \geq 3)\right]^{-1} \\ &= g\left[1 - \frac{1}{2S}\left[F(0_+) + \int_{0_-}^{1_+} F^2(\zeta')d\zeta'\right] + O\left(S^{-2}\right) + \text{higher order terms } (m \geq 3)\right] \\ &= g\left[1 - \frac{1}{S\tau_{d1}} + O\left(S^{-2}\right) + \text{higher order terms } (m \geq 3)\right] \\ \tau_{d1} &= 2\left[F(0_+) + \int_{0_-}^{1_+} F^2(\zeta')d\zeta'\right]^{-1} \\ &= 2\left[G + G^2\right]^{-1} \end{aligned} \tag{4.50}$$

Again consulting Appendix A, this agrees with the exact result for the exponential NTL.

5. Concluding Remarks

An NTL pulse transformer can be combined with various pulse-power equipment to increase the pulse voltage. However, we need a fast pulse for effective voltage increase, thereby requiring fast switches. The high-frequency/early-time limiting form of the voltage gain g is given by a well-known formula. This is followed, for a step excitation, by a droop term which one would like to minimize. An exponential NTL is an important case for small droop (large droop time). This is solved two different ways involving voltage/current variables and renormalized wave variables.

An interesting benefit of this analysis has been the expression of some product integrals in nontrivial closed form.

Appendix A: The Exponential Transmission Line

As an example of the analysis in Section 3, we have the exponential transmission line defined by

$$\frac{f_g(\zeta)}{f_g(\zeta_0)} = e^{2\alpha[z-z_0]} \tag{A.1}$$

which for conditions defined at both NTL ends gives

$$\begin{aligned} f(\zeta) &= \frac{f_g(\zeta)}{f_g(0)} = e^{2\alpha z} = e^{2\alpha \ell \zeta}, \quad f(\ell) = e^{2\alpha\ell} = e^{2G} \\ G &\equiv \alpha\ell = \frac{1}{2}\ell n\big(f(1)\big) = \ell n(g) \end{aligned} \tag{A.2}$$

This case is also treated in the literature (e.g., [5, 6]).

The solution for the matrizant for

$$\big(\tilde{g}_{\sigma,\sigma'}(\zeta,S)\big) = -S\begin{pmatrix} 0 & e^{2G\zeta} \\ e^{-2G\zeta} & 0 \end{pmatrix} \tag{A.3}$$

is given by [2(Appendix B)]

$$\begin{aligned} \big(\tilde{U}_{\sigma,\sigma'}(\zeta,0;S)\big) &= \begin{pmatrix} 1 & 0 \\ 0 & e^{2G\zeta} \end{pmatrix} \cdot \big(u_{\sigma,\sigma'}(\zeta,0;S)\big) \\ &= e^{G\zeta} \begin{pmatrix} \left[\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\zeta\right) - \frac{G}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\zeta\right)\right] & -\frac{S}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\zeta\right) \\ -\frac{S}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\zeta\right) & \left[\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\zeta\right) + \frac{G}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\zeta\right)\right] \end{pmatrix} \end{aligned} \tag{A.4}$$

Stated another way we have a closed-form product integral as

$$
\left(\tilde{u}_{\sigma,\sigma'}(\zeta,0;S)\right) = \prod_{0}^{\zeta} e^{-S\begin{pmatrix} 0 & e^{2G\zeta'} \\ e^{-2G\zeta'} & 0 \end{pmatrix} d\zeta'}
$$

$$
= \begin{pmatrix} e^{G\zeta}\left[\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\zeta\right) - \dfrac{G}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\zeta\right)\right] & -e^{G\zeta}\dfrac{S}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\zeta\right) \\ -e^{-G\zeta}\dfrac{S}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\zeta\right) & e^{-G\zeta}\left[\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\zeta\right) + \dfrac{G}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\zeta\right)\right] \end{pmatrix} \tag{A.5}
$$

Using matrizant identities this is extended to

$$
\begin{aligned}
\left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) &= \prod_{\zeta_0}^{\zeta} e^{-S\begin{pmatrix} 0 & e^{2G\zeta'} \\ e^{-2G'} & 0 \end{pmatrix} d\zeta'} \\
&= \left(\tilde{u}_{\sigma,\sigma'}(\zeta,0;S)\right) \cdot \left(\tilde{u}(\zeta_0,0;S)\right)^{-1} \\
&= \left(\tilde{u}_{\sigma,\sigma'}(\zeta,0;S)\right) \cdot \left(\tilde{u}(0,\zeta_0;S)\right)
\end{aligned} \tag{A.6}
$$

We also have from the similarity rule of the product integral [3, 13]

$$
\begin{aligned}
\left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) &= \prod_{\zeta_0}^{\zeta} e^{-S\begin{pmatrix} 0 & e^{2G\zeta'} \\ e^{-2G\zeta'} & 0 \end{pmatrix} d\zeta'} \\
&= \prod_{0}^{\zeta-\zeta_0} e^{-S\begin{pmatrix} 0 & e^{2G\zeta'}e^{-2G\zeta_0} \\ e^{-2G\zeta'}e^{-2G\zeta_0} & 0 \end{pmatrix} d\zeta'} \\
&= \prod_{0}^{\zeta-\zeta_0} e^{\begin{pmatrix} e^{G\zeta_0} & 0 \\ 0 & e^{-G\zeta_0} \end{pmatrix} \cdot \left[-S\begin{pmatrix} 0 & e^{2G\zeta'} \\ e^{-2G\zeta'} & 0 \end{pmatrix}\right] \cdot \begin{pmatrix} e^{-G\zeta_0} & 0 \\ 0 & e^{G\zeta_0} \end{pmatrix} d\zeta'} \\
&= \begin{pmatrix} e^{G\zeta_0} & 0 \\ 0 & e^{-G\zeta_0} \end{pmatrix} \cdot \left[\prod_{0}^{\zeta-\zeta_0} e^{-S\begin{pmatrix} 0 & e^{2G\zeta'} \\ e^{-2G\zeta'} & 0 \end{pmatrix} de\zeta'} \right] \cdot \begin{pmatrix} e^{-G\zeta_0} & 0 \\ 0 & e^{G\zeta_0} \end{pmatrix} \\
&= \begin{pmatrix} e^{G\zeta_0} & 0 \\ 0 & e^{-G\zeta_0} \end{pmatrix} \cdot \left(\tilde{u}_{\sigma,\sigma'}(\zeta-\zeta_0,0;S)\right) \cdot \begin{pmatrix} e^{-G\zeta_0} & 0 \\ 0 & e^{G\zeta_0} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & e^{-2G\zeta_0} \end{pmatrix} \cdot \left(\tilde{u}_{\sigma,\sigma'}(\zeta-\zeta_0,0;S)\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & e^{2G\zeta_0} \end{pmatrix} \qquad \text{(A.7)}
\end{aligned}
$$

So, with the inclusion of the simple pre- and post-multiplying matrices the (A.5) matrix can be used with ζ replaced by $\zeta - \zeta_0$.

With these results we can fill in the parameters in Section 3. The input impedance is

$$
\begin{aligned}
\frac{\tilde{Z}_{in}(S)}{Z_1} &= \frac{\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right) + \dfrac{S+G}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right)}{\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right) + \dfrac{S-G}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right)} \\
&= \frac{1 + \dfrac{S+G}{\left[S^2+G^2\right]^{\frac{1}{2}}}\tanh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right)}{1 + \dfrac{S-G}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right)}
\end{aligned}
$$

$$\frac{\tilde{Z}_{in}(S)}{Z_1} \to \begin{cases} e^{2G} = f(\ell) = \dfrac{Z_2}{Z_1} \text{ as } S \to 0 \\ 1 \text{ as } S \to 0 \text{ in RHP} \end{cases} \tag{A.8}$$

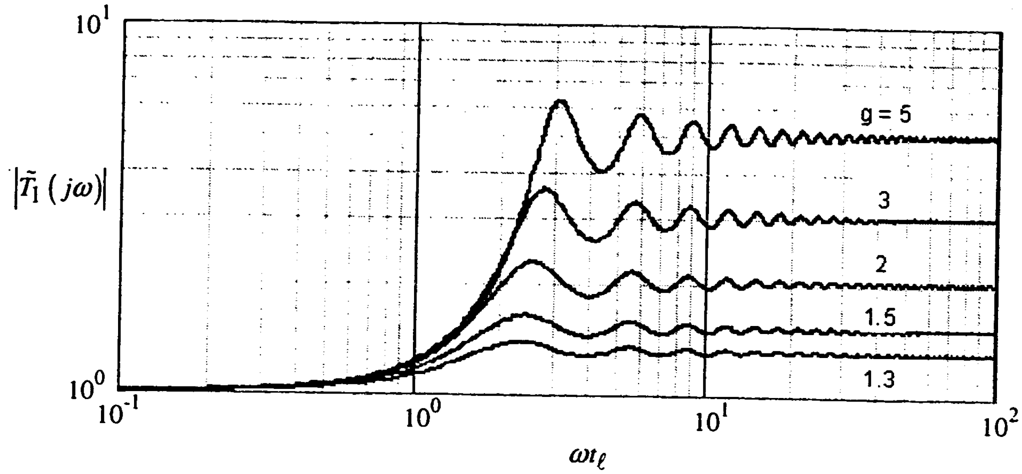
The transfer functions are

$$\begin{aligned}
\tilde{T}_1(s) &= e^{S+G}\left[\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right)+\frac{S+G}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right)\right]^{-1} \\
\tilde{T}_1(s) &\to \begin{cases} 1 \text{ as } S \to 0 \\ e^{G} = g \text{ as } S \to \infty \text{ in RHP} \end{cases} \\
\tilde{T}(s) &= e^{S+G}\left[\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right)+\frac{S}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right)\right]^{-1} \\
\tilde{T}(s) &\to \begin{cases} 2\left[1+e^{-2G}\right]^{-1} = 2\left[1+f^{-1}(\ell)\right]^{-1} = \dfrac{2Z_2}{Z_1+Z_2} \text{ as } s \to 0 \\ e^{G} = g \text{ as } S \to \infty \text{ in RHP} \end{cases}
\end{aligned} \tag{A.9}$$

Here we note that the formulae behave as we expect in the low- and high-frequency limits.

These two transfer functions are plotted in Fig. A.1. Here we see that they asymptotically approach g in the high frequency limit, while $\tilde{T}_1$ approaches 1 at low frequencies and $\tilde{T}$ goes to something greater than one there. (Note for g = 1, both $\tilde{T}_1$ and $\tilde{T}$ are 1.0 for all frequencies.) The transition from low to high frequencies lies in the region of $\omega t_\ell = 1$ with larger values of g moving the transition to slightly higher frequencies. This comes from what one can call the cutoff frequency at which $\omega t_\ell = G = \ell n(g)$. Note the oscillatory behavior above this transition. It is noticeably smaller for $\tilde{T}$ as compared to $\tilde{T}_1$. This would seem to be due to the termination in the local characteristic impedance at the beginning of the NTL in the case of $\tilde{T}$, giving smaller high-frequency reflections there.

Having the early-time behavior from the $S \to \infty$ behavior in (A.9) let us now go on to consider the droop after the initial rise. For this purpose, we ned to consider the next term in the expansion for large S, for which we have



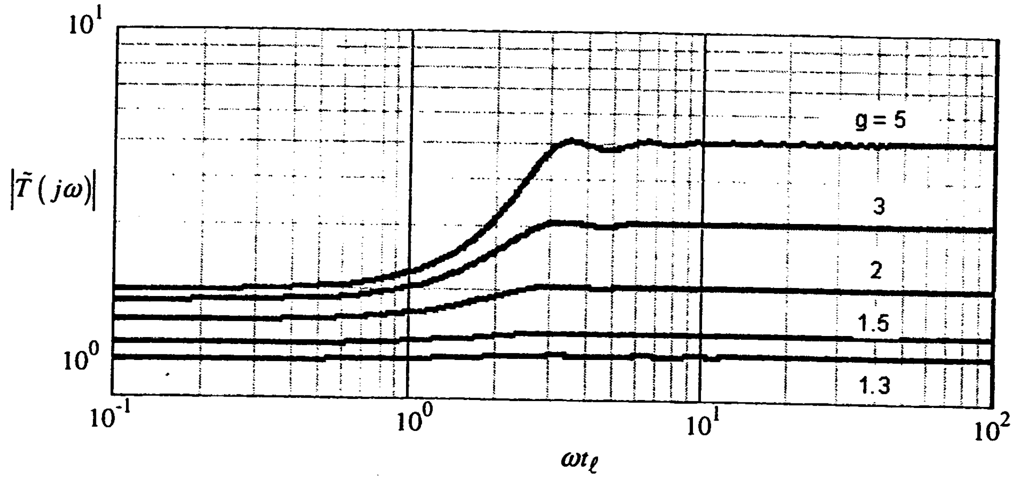


Fig. A.1. Transfer Functions Versus Frequency with g as a Parameter

$$\left[S^2+G^2\right]^{\frac{1}{2}} = S\left[1+\left[\frac{G}{S}\right]^2\right]^{\frac{1}{2}} + S\left[1+\frac{1}{2}\left[\frac{G}{S}\right]^2+O\left(\left[\frac{G}{S}\right]^4\right)\right]$$

$$\left[S^2+G^2\right]^{-\frac{1}{2}} = S^{-1}\left[1-\frac{1}{2}\left[\frac{G}{S}\right]^2+O\left(\left[\frac{G}{S}\right]^4\right)\right]$$

$$e^{\left[S^2+G^2\right]^{\frac{1}{2}}} = e^S\, e^{\frac{G^2}{2S}+O\left(S^{-3}\right)} \tag{A.10}$$

$$= e^S\left[1+\frac{G^2}{2s}+O\left(S^{-2}\right)\right]$$

$$e^{-\left[S^2+G^2\right]^{\frac{1}{2}}} = e^{-S}\left[1+O\left(S^{-1}\right)\right]$$

as $S\to\infty$ in RHP

Note that e^{-S} terms are delays for reflections and do not contribute to the early-time response.

Consider first $\tilde{T}(s)$ for which we have the step-response form as

$$\tilde{R}(s) = \frac{1}{S}\tilde{T}(s)$$

$$= \frac{1}{S}e^{S+G}\left[e^S\left[1+\frac{G^2}{2S}+O\left(S^{-2}\right)\right]\left[1-\frac{1}{4}\left[\frac{G}{2S}\right]^2+O\left(S^{-4}\right)\right]\right]^{-1}$$

$$= \frac{1}{S}e^G\left[1+\frac{G^2}{2S}+O\left(S^{-2}\right)\right]^{-1}$$

$$= \frac{1}{S}g\left[1-\frac{G^2}{2S}+O\left(S^{-2}\right)\right] \tag{A.11}$$

$$= g\left[\frac{1}{S}-\frac{G^2}{2S^2}+O\left(S^{-3}\right)\right]$$

$$= g\left[\frac{1}{S}-\frac{G^2}{\tau_d S^2}+O\left(S^{-3}\right)\right] \quad \text{as } S\to\infty \text{ in RHP}$$

where the normalized droop time is

$$\tau_d = \frac{2}{G^2} = 2\ell n^{-2}(g) \tag{A.12}$$

Note that as the high-frequency gain g increases then τ_d decreases. In time domain (normalized) we have

$$R(\tau) = g\left[1-\frac{\tau}{\tau_d}+O\left(\tau^2\right)\right]u(\tau) \text{ as } \rightarrow 0 \tag{A.13}$$

For small droop we need $\tau << \tau_d$. Note that

$$\tau = \tau_d \frac{\tau}{\tau_d} = 2\ell n^{-2}(g)\frac{\tau}{\tau_d} \tag{A.14}$$

For an acceptabily small τ / τ_d (say 5% or whatever) this tells us that τ needs to be of a certain smallness, or equivalently that the NTL needs to be of some sufficiently large electrical length.

For comparison we have

$$\begin{aligned}
\tilde{R}_1(s) &= \frac{1}{s}\tilde{T}_1(s) \\
&= \frac{1}{S}e^{S+G}\left[e^G\left[1+\frac{G^2}{2S}+O\left(S^{-2}\right)\right]\left[1-\frac{1}{4}\left[\frac{G}{S}\right]^2+\frac{G}{2S}+O\left(S^{-3}\right)\right]\right]^{-1} \\
&= \frac{1}{S}e^G\left[1-\frac{G}{2S}+\frac{G^2}{2S}+O\left(S^{-2}\right)\right]^{-1} \\
&= \frac{1}{S}g\left[1-\frac{1}{2S}\left[G+G^2\right]+O\left(S^{-2}\right)\right] \\
&= g\left[\frac{1}{S}-\frac{G+G^2}{2S}+O\left(S^{-2}\right)\right] \\
&= g\left[\frac{1}{S}-\frac{1}{\tau_{d1}S^2}+O\left(S^{-3}\right)\right] \text{ as } S\rightarrow\infty \text{ in RHP}
\end{aligned} \qquad \text{A.15)}$$

where

$$\tau_{d1} = 2\left[G+G^2\right]^{-1} = 2\left[\ell n(g)+\ell_n^2(g)\right]^{-1} \tag{A.16}$$

In time domain this becomes

$$R_1(\tau) = g\left[1-\frac{\tau}{\tau_d}+O\left(\tau^2\right)\right]u(t) \text{ as } t \to 0 \tag{A.17}$$

Examining τ_{d1} we find

$$\tau_{d1}^{-1} = \tau_d^{-1} + \frac{1}{2}\ell n(g) \tag{A.18}$$

This extra term makes $\tau_d > \tau_{d1}$. The improvement in τ_d over τ_{d1} is associated with the positive reflection of $\tilde{V}^{(inc)}$ at $\zeta = 0$, giving a transmission coefficient with increased magnitude for $\tau > 0$. This, in turn, compensates for the lessened droop (increase in τ_d) for retarded time a little positive.

For some sample numbers, consider an NTL with

$$\begin{aligned} f(1) &= 4 && \text{(fourfold impedance increase)} \\ g &= f^{\frac{1}{2}}(1) = 2 && \text{(early-time voltage increase)} \\ \tau_d &\simeq 4.163 \\ \tau_{d1} &\simeq 1.704 \end{aligned} \tag{A.18}$$

So we see that there is some significant improvement (increase) in the droop time in going from a $\tilde{V}_1$ reference to a $\tilde{V}^{(\mathrm{inc})}$ reference.

Appendix B. Alternate Product Integral for the Exponential Transmission Line

In Section 4 we have a product integral ((4.11) and (4.13)) as

$$\left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) = \prod_{\zeta_0}^{\zeta} e^{\left(\tilde{\gamma}_{\sigma,\sigma'}(\zeta',s)\right)d\zeta'}$$
$$\left(\tilde{\gamma}_{\sigma,\sigma'}(\zeta,s)\right) = -s\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - F(\zeta)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{B.1}$$

which applies to quite general NTLs, depending on the choice of

$$F(\zeta) = \frac{d}{d\zeta}\ell n\left(f^{\frac{1}{2}}(\zeta)\right)$$
$$f^{\frac{1}{2}}(0_-) = 1 \quad , \quad f^{\frac{1}{2}}(1_+) = g \equiv \text{high-frequency voltage gain} \tag{B.2}$$

In Appendix A we treat the exponential transmission line using the known solution of the telegrapher equations placed in product-integral form. Such an NTL has

$$f(\zeta) = G = \ell n(g) \tag{B.3}$$

which is a constant. In this case $(\tilde{\gamma}_{\sigma,\sigma'})$ is a constant (with respect to ζ) giving

$$\left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) = \prod_{\zeta_0}^{\zeta} e^{\begin{pmatrix}-s & -G\\ -G & s\end{pmatrix}d\zeta'} = e^{\int_{\zeta_0}^{\zeta}\begin{pmatrix}-s & -G\\ -G & s\end{pmatrix}d\zeta'}$$
$$= e^{\begin{pmatrix}-s & -G\\ -G & s\end{pmatrix}[\zeta-\zeta_0]} \tag{B.4}$$

which a deceptively simple result. One can, of course, use the series definition of the matrix exponential as

$$e^{\begin{pmatrix}-s & -G\\ -G & s\end{pmatrix}[\zeta-\zeta_0]} = \sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix}-s & -G\\ -G & s\end{pmatrix}^n[\zeta-\zeta_0]^n \tag{B.5}$$

Note also that it is a function of $\zeta - \zeta_0$.

This product integral described propagation on the NTL from Section 4 as

$$\begin{pmatrix} \tilde{v}_+(\zeta,s) \\ \tilde{v}_-(\zeta,s) \end{pmatrix} = \left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) \cdot \begin{pmatrix} \tilde{v}_+(\zeta_0,s) \\ \tilde{v}_-(\zeta_0,s) \end{pmatrix} \tag{B.6}$$

Converting these wave variables to the normalized voltage/current variables used in Section 4 and Appendix A we have

$$\begin{aligned}
\begin{pmatrix} \tilde{v}(\zeta,s) \\ \tilde{i}(\zeta,s) \end{pmatrix} &= \left(Q_{\sigma,\sigma'}\right)^{-1} \cdot \begin{pmatrix} \tilde{v}_+(\zeta,s) \\ \tilde{v}_-(\zeta,s) \end{pmatrix} \\
\left(Q_{\sigma,\sigma'}\right)^{-1} &= \frac{1}{2}\left(Q_{\sigma,\sigma'}\right) \quad , \quad \left(Q_{\sigma,\sigma'}\right) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
\begin{pmatrix} \tilde{V}(\zeta,s) \\ Z_1\tilde{I}(\zeta,s) \end{pmatrix} &= \begin{pmatrix} f_g^{\frac{1}{2}}(\zeta) & 0 \\ 0 & \dfrac{1}{f_g(0) f_g^{\frac{1}{2}}(\zeta)} \end{pmatrix} \cdot \begin{pmatrix} \tilde{v}(\zeta,s) \\ \tilde{i}(\zeta,s) \end{pmatrix} \\
&= f_g^{\frac{1}{2}}(0) \begin{pmatrix} f_g^{\frac{1}{2}}(\zeta) & 0 \\ 0 & f_g^{-\frac{1}{2}}(\zeta) \end{pmatrix} \cdot \begin{pmatrix} \tilde{v}(\zeta,s) \\ \tilde{i}(\zeta,s) \end{pmatrix} \\
f^{\frac{1}{2}}(\zeta) &= e^{G\zeta}
\end{aligned} \tag{B.7}$$

Then utilizing the product integral in Appendix A we have

$$\begin{aligned}
\begin{pmatrix} \tilde{V}(\zeta,s) \\ Z_1\tilde{I}(\zeta,s) \end{pmatrix} &= \left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) \cdot \begin{pmatrix} \tilde{V}(\zeta,s) \\ Z_1\tilde{I}(\zeta,s) \end{pmatrix} \\
\left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) &= \prod_{\zeta_0}^{\zeta} e^{-s\begin{pmatrix} 0 & e^{2G\zeta'} \\ e^{-2G\zeta'} & 0 \end{pmatrix} d\zeta'}
\end{aligned} \tag{B.8}$$

where $(\tilde{u}_{\sigma,\sigma'})$ is written out in (A.5) and (A.6). With (B.6) and (B.7) we then have

$$
\begin{pmatrix} \tilde{V}(\zeta,s) \\ Z_1\tilde{I}(\zeta,s) \end{pmatrix} =
\begin{pmatrix} e^{G\zeta} & 0 \\ 0 & e^{-G\zeta} \end{pmatrix} \cdot \left(Q_{\sigma,\sigma'}\right) \cdot \left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) \cdot \left(Q_{\sigma,\sigma'}\right)^{-1} \cdot \begin{pmatrix} e^{-G\zeta_0} & 0 \\ 0 & e^{G\zeta_0} \end{pmatrix} \cdot \begin{pmatrix} \tilde{V}(\zeta_0,s) \\ Z_1\tilde{I}(\zeta_0,s) \end{pmatrix} \tag{B.9}
$$

Thus we can identify

$$
\begin{aligned}
&\left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) \\
&\quad = \begin{pmatrix} e^{G\zeta} & 0 \\ 0 & e^{-G\zeta} \end{pmatrix} \cdot \left(Q_{\sigma,\sigma'}\right) \cdot \left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) \cdot \left(Q_{\sigma,\sigma'}\right)^{-1} \cdot \begin{pmatrix} e^{-G\zeta_0} & 0 \\ 0 & e^{G\zeta_0} \end{pmatrix} \\
&\left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) \\
&\quad = \left(Q_{\sigma,\sigma'}\right)^{-1} \cdot \begin{pmatrix} e^{-G\zeta} & 0 \\ 0 & e^{G\zeta} \end{pmatrix} \cdot \left(\tilde{u}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) \cdot \begin{pmatrix} e^{G\zeta_0} & 0 \\ 0 & e^{-G\zeta_0} \end{pmatrix} \cdot \left(Q_{\sigma,\sigma'}\right) \\
&\quad = \left(Q_{\sigma,\sigma'}\right)^{-1} \cdot \begin{pmatrix} e^{-G[\zeta-\zeta_0]} & 0 \\ 0 & e^{G[\zeta-\zeta_0]} \end{pmatrix} \cdot \left(\tilde{u}_{\sigma,\sigma'}(\zeta-\zeta_0,0;s)\right) \cdot \left(Q_{\sigma,\sigma'}\right)
\end{aligned} \tag{B.10}
$$

using (A.7). We can readily verify

$$
\begin{aligned}
&\det\left(\left(\Gamma_{\sigma,\sigma}(\zeta,\zeta_0;s)\right)\right) \\
&\quad = \det\left(\left(Q_{\sigma,\sigma'}\right)^{-1}\right) \det\left(\begin{pmatrix} e^{-G[\zeta-\zeta_0]} & 0 \\ 0 & e^{G[\zeta-\zeta_0]} \end{pmatrix}\right) \det\left(\left(\tilde{u}_{\sigma,\sigma'}(\zeta-\zeta_0;s)\right) \det\left(\left(Q_{\sigma,\sigma'}\right)\right)\right) \\
&\quad = \det\left(\begin{pmatrix} e^{-G[\zeta-\zeta_0]} & 0 \\ 0 & e^{G[\zeta-\zeta_0]} \end{pmatrix}\right) \det\left(\left(\tilde{u}_{\sigma,\sigma'}(\zeta-\zeta_0;s)\right)\right) \\
&\quad = \det\left(\left(\tilde{u}_{\sigma,\sigma'}(\zeta-\zeta_0;s)\right)\right) = 1
\end{aligned} \tag{B.11}
$$

as is required of a matrizant. Using (A.5) we have

$$
\begin{aligned}
&\left(\tilde{X}_{\sigma,\sigma'}(\zeta,\zeta_0;s)\right) \\
&\quad \equiv \begin{pmatrix} e^{-G[\zeta-\zeta_0]} & 0 \\ 0 & e^{G[\zeta-\zeta_0]} \end{pmatrix} \cdot \left(\tilde{u}_{\sigma,\sigma'}(\zeta-\zeta_0,s)\right)
\end{aligned}
$$

$$
= \begin{bmatrix} \left[\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}} [\zeta-\zeta_0] \right) - \dfrac{G}{\left[S^2+G^2\right]^{\frac{1}{2}}} \sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}} [\zeta-\zeta_0] \right) \right] & - \dfrac{S}{\left[S^2+G^2\right]^{\frac{1}{2}}} \sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}} [\zeta-\zeta_0] \right) \\ - \dfrac{S}{\left[S^2+G^2\right]^{\frac{1}{2}}} \sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}} [\zeta-\zeta_0] \right) & \left[\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}} [\zeta-\zeta_0] \right) - \dfrac{G}{\left[S^2+G^2\right]^{\frac{1}{2}}} \sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}} [\zeta-\zeta_0] \right) \right] \end{bmatrix} \tag{B.12}
$$

from which we see that $(\tilde{\Gamma}_{\sigma,\sigma'})$ is a function of $\zeta-\zeta_0$ as required in (B.4). Continuing the development

$$
\left(\tilde{\Gamma}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) = e^{\begin{pmatrix} -S & -G \\ -G & S \end{pmatrix}[\zeta-\zeta_0]}
$$

$$
= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \bullet \left(\tilde{X}_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) \bullet \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
$$

$$
= \begin{bmatrix} \left[\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}} [\zeta-\zeta_0] \right) - \dfrac{S}{\left[S^2+G^2\right]^{\frac{1}{2}}} \sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}} [\zeta-\zeta_0] \right) \right] & - \dfrac{G}{\left[S^2+G^2\right]^{\frac{1}{2}}} \sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}} [\zeta-\zeta_0] \right) \\ - \dfrac{G}{\left[S^2+G^2\right]^{\frac{1}{2}}} \sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}} [\zeta-\zeta_0] \right) & \left[\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}} [\zeta-\zeta_0]^{\frac{1}{2}} \right) + \dfrac{S}{\left[S^2+G^2\right]^{\frac{1}{2}}} \sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}} [\zeta-\zeta_0] \right) \right] \end{bmatrix} \tag{B.13}
$$

which merely interchanges the positions of S and G from (B.12). Note also that this matrizant is a symmetric matrix.

From (4.19) the transfer functions are now

$$\tilde{T}_1(S) = g\, e^{S}\left[\tilde{\Gamma}_{1,1}(0,1;S) + \tilde{\Gamma}_{2,1}(0,1;S)\right]^{-1}$$
$$= e^{S+G}\left[\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right) + \frac{S}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right)\right]^{-1}$$
$$\tilde{T}(S) = g\, e^{S}\left[\tilde{\Gamma}_{1,1}^{-1}(0,1;S)\right] \tag{B.14}$$
$$= e^{S+G}\left[\cosh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right) + \frac{S}{\left[S^2+G^2\right]^{\frac{1}{2}}}\sinh\left(\left[S^2+G^2\right]^{\frac{1}{2}}\right)\right]^{-1}$$

which agrees with the previous result in (A.9)

The reader may note that the result for $(\tilde{\Gamma}_{\sigma,\sigma'})$ can also be obtained by diagonalizing $\begin{pmatrix} -S & -G \\ -G & S \end{pmatrix}$ and applying the similarity rule. This is a direct technique, not relying on the solutions of a second-order differential equation for constructing the matrizant as in (A.5) from [2].

Noting zero trace and $-S^2 - G^2$ for the determinant, we have eigenvalues

$$\lambda_+ + \lambda_- = 0 \ , \quad \lambda_+\lambda_- = -\left[S^2+G^2\right]$$
$$\lambda_+ = -\lambda_- = -\left[S^2+G^2\right]^{\frac{1}{2}} \tag{B.15}$$

The corresponding eigenvectors $(x_n^{(\pm)})$ and $(x_n^{(-)})$ are found from

$$\begin{pmatrix} -S & -G \\ -G & S \end{pmatrix} \bullet \begin{pmatrix} x_1^{(\pm)} \\ x_2^{(\pm)} \end{pmatrix} = \lambda_\pm \begin{pmatrix} x_1^{(\pm)} \\ x_2^{(\pm)} \end{pmatrix} = \begin{pmatrix} x_1^{(\pm)} \\ x_2^{(\pm)} \end{pmatrix} \bullet \begin{pmatrix} -S & -G \\ -G & S \end{pmatrix} \tag{B.16}$$

noting that the matrix is symmetric (equal to its transpose). Requiring the eigenvectors to be unit vectors we have

$$
\begin{aligned}
x_1^{(+)} &= \left[2 + \frac{2S^2}{G^2} + \frac{2S}{G^2}\left[S^2 + G^2\right]^{\frac{1}{2}}\right]^{-\frac{1}{2}} = x_2^{(-)} \\
x_2^{(+)} &= \left[2 + \frac{2S^2}{G^2} + \frac{2S}{G^2}\left[S^2 + G^2\right]^{\frac{1}{2}}\right]^{-\frac{1}{2}} = -x_1^{(-)}
\end{aligned}
\tag{B.17}
$$

with

$$
\begin{aligned}
& x_1^{(\pm)^2} + x_2^{(\pm)^2} = 1 \\
& \begin{pmatrix} x_1^{(+)} \\ x_2^{(+)} \end{pmatrix} \cdot \begin{pmatrix} x_1^{(-)} \\ x_2^{(-)} \end{pmatrix} = 0 = x_1^{(+)} x_1^{(-)} + x_2^{(+)} x_2^{(-)}
\end{aligned}
\tag{B.18}
$$

and convenient combinations

$$
\begin{aligned}
& x_1^{(+)} x_2^{(+)} = x_1^{(-)} x_2^{(-)} = -\frac{G}{2\lambda_+} \\
& x_1^{(+)^2} - x_2^{(+)^2} = x_1^{(-)^2} - x_2^{(-)^2} = -\frac{S}{\lambda_+} \\
& \begin{pmatrix} x_1^{(+)} & x_1^{(-)} \\ x_2^{(+)} & x_2^{(-)} \end{pmatrix}^{-1} = \begin{pmatrix} x_1^{(+)} & x_2^{(+)} \\ x_1^{(-)} & x_2^{(-)} \end{pmatrix}
\end{aligned}
\tag{B.19}
$$

Thus we construct

$$
\begin{aligned}
\begin{pmatrix} -S & -G \\ -G & S \end{pmatrix} &= \begin{pmatrix} x_1^{(+)} & x_1^{(-)} \\ x_2^{(+)} & x_2^{(-)} \end{pmatrix} \cdot \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \cdot \begin{pmatrix} x_1^{(+)} & x_2^{(+)} \\ x_1^{(-)} & x_2^{(-)} \end{pmatrix} \\
&= \begin{pmatrix} x_1^{(+)} & -x_2^{(+)} \\ x_2^{(+)} & x_1^{(+)} \end{pmatrix} \cdot \begin{pmatrix} \lambda_+ & 0 \\ 0 & -\lambda_+ \end{pmatrix} \cdot \begin{pmatrix} x_1^{(+)} & x_2^{(+)} \\ -x_2^{(+)} & x_1^{(+)} \end{pmatrix}
\end{aligned}
\tag{B.20}
$$

Inserting this in the product integral and applying the similarity rule we have

$$
\begin{aligned}
\left(\Gamma_{\sigma,\sigma'}(\zeta,\zeta_0;S)\right) &= \prod_{\zeta_0}^{\zeta} e^{\begin{pmatrix} -S & -G \\ -G & S \end{pmatrix} d\zeta'} = e^{\begin{pmatrix} -S & -G \\ -G & S \end{pmatrix}[\zeta-\zeta_0]} \\
&= e^{\begin{pmatrix} x_1^{(+)} & x_1^{(-)} \\ x_2^{(+)} & x_2^{(-)} \end{pmatrix} \cdot \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \cdot \begin{pmatrix} x_1^{(+)} & x_2^{(+)} \\ x_1^{(-)} & x_2^{(-)} \end{pmatrix}[\zeta-\zeta_0]} \\
&= \begin{pmatrix} x_1^{(+)} & x_1^{(-)} \\ x_2^{(+)} & x_2^{(-)} \end{pmatrix} \cdot e^{\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}[\zeta-\zeta_0]} \cdot \begin{pmatrix} x_1^{(+)} & x_2^{(+)} \\ x_1^{(-)} & x_2^{(-)} \end{pmatrix} \\
&= \begin{pmatrix} x_1^{(+)} & x_1^{(-)} \\ x_2^{(+)} & x_2^{(-)} \end{pmatrix} \cdot \begin{pmatrix} e^{\lambda_+[\zeta-\zeta_0]} & 0 \\ 0 & e^{\lambda_-[\zeta-\zeta_0]} \end{pmatrix} \cdot \begin{pmatrix} x_1^{(+)} & x_2^{(+)} \\ x_1^{(-)} & x_2^{(-)} \end{pmatrix} \\
&= \begin{pmatrix} x_1^{(+)} & -x_2^{(+)} \\ x_2^{(+)} & x_1^{(+)} \end{pmatrix} \cdot \begin{pmatrix} e^{\lambda_+[\zeta-\zeta_0]} & 0 \\ 0 & e^{-\lambda_+[\zeta-\zeta_0]} \end{pmatrix} \cdot \begin{pmatrix} x_1^{(+)} & x_2^{(+)} \\ -x_2^{(+)} & x_1^{(+)} \end{pmatrix}
\end{aligned}
\tag{B.21}
$$

Carrying out the matrix dot multiplications the result in (B.13) is obtained, completing the derivation without resort to the solutions of a second-order differential equation.

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