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Eigen-Function Expansion of Dyadic Green's Functions

by

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Abstract

This work contains a detailed revision of the treatment of the eigen-function expansion of dyadic Green's functions previously discussed by the author in his book [1]. The singular terms which are missing in the previous work have been amended. Three distinct methods of finding the dyadic Green's function of the electric type are discussed. It is concluded that the method based on the differential equation for the H-field is the simplest. The correct expressions for various dyadic Green's functions are then derived based on this method. In the course of this work, several expansion theorems dealing with a number of dyadic singular functions, or generalized functions, have been found. They are useful in simplifying some integrals encountered in the analysis.

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I. Introduction

In a recent communication [2], we have called attention to an error in the treatment of the eigen-function expansion of dyadic Green's functions described in the author's book [1]. The communication gives a brief description of a revised method which removes the shortcomings found in the previous treatment. Since then a technical report [3], mainly for internal use, has been issued listing the correct expressions for various dyadic Green's functions without detailed analysis. After the issuing of that report Dr. Carl E. Baum has suggested to the author that for the convenience of the workers in this area the material, with detailed derivations, be issued as a note by the Air Force Weapons Laboratory so it can be circulated to various institutions and individuals interested in this subject. This report is, therefore, prepared for that purpose.

II. General Formulation

For clarity we introduce two types of dyadic Green's functions designated by $\overline{\overline{G}}_e$, the electric type, and $\overline{\overline{G}}_m$, the magnetic type, which satisfy the equations

$$\nabla \times \overline{\overline{G}}_e = \overline{\overline{G}}_m \tag{1}$$

$$\nabla \times \overline{\overline{G}}_m = \overline{\overline{I}}\,\delta(\overline{R}-\overline{R}') + k^2 \overline{\overline{G}}_e \tag{2}$$

where

$$\overline{\overline{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$$

$$k^2 = \omega^2 \mu_o \epsilon_o$$

and $\delta(\overline{R}-\overline{R}')$ denotes a three-dimensional delta function which is characterized by

$$\iiint \overline{F}(\overline{R})\,\delta(\overline{R}-\overline{R}')\,dv = \overline{F}(\overline{R}') \quad .$$

These equations are the dyadic version of Maxwell's equations applied to harmonic fields due to infinitesimal current sources. The relationships between $\overline{\overline{G}}_e$, $\overline{\overline{G}}_m$, $\overline{\overline{I}}\delta(\overline{R}-\overline{R}')$ and the corresponding field vectors $\overline{E}$, $\overline{H}$, $\overline{J}$ are

$$\overline{\overline{G}}_e = \overline{E}^{(x)}\hat{x}+\overline{E}^{(y)}\hat{y}+\overline{E}^{(z)}\hat{z} \tag{3}$$

$$\overline{\overline{G}}_m = i\omega\mu_o\left[\overline{H}^{(x)}\hat{x} + \overline{H}^{(y)}\hat{y} + \overline{H}^{(z)}\hat{z}\right] \tag{4}$$

$$\overline{\overline{I}}\delta(\overline{R}-\overline{R}') = i\omega\mu_o\left[\overline{J}^{(x)}\hat{x} + \overline{J}^{(y)}\hat{y} + \overline{J}^{(z)}\hat{z}\right] \tag{5}$$

where $\overline{E}^{(x)}$, $\overline{H}^{(x)}$ represent, respectively, the electric and the magnetic field due to an infinitesimal current source with a current density $\overline{J}^{(x)} = \delta(\overline{R}-\overline{R}')\hat{x}/i\omega\mu_o$ and similarly for the other triads. By eliminating $\overline{\overline{G}}_m$ or $\overline{\overline{G}}_e$ between (1) and (2) we obtain

$$\nabla\times\nabla\times\overline{\overline{G}}_e - k^2\overline{\overline{G}}_e = \overline{\overline{I}}\nabla(\overline{R}-\overline{R}') \tag{6}$$

$$\nabla\times\nabla\times\overline{\overline{G}}_m - k^2\overline{\overline{G}}_m = \nabla\times\left[\overline{\overline{I}}\delta(\overline{R}-\overline{R}')\right] . \tag{7}$$

Equations (6) and (7) differ from each other in the inhomogeneous term. Furthermore, we have

$$\nabla\cdot\overline{\overline{G}}_e = -\frac{1}{k^2}\nabla\cdot\left[\overline{\overline{I}}\delta(\overline{R}-\overline{R}')\right]$$

$$= -\frac{1}{k^2}\nabla\delta(\overline{R}-\overline{R}') \tag{8}$$

and

$$\nabla\cdot\overline{\overline{G}}_m = 0 \quad . \tag{9}$$

Thus, $\overline{\overline{G}}_m$ is a solenoidal dyadic function but $\overline{\overline{G}}_e$ is not.

The dyadic Green's functions, $\overline{\overline{G}}_e$ and $\overline{\overline{G}}_m$, which are solutions to (6) and (7), are classified according to the boundary conditions which they must satisfy on an assigned surface. The functions of the first kind satisfy the Dirichlet boundary condition

$$\hat{n} \times \overline{\overline{G}}_{e1} = 0 \tag{10}$$

$$\hat{n} \times \overline{\overline{G}}_{m1} = 0 \tag{11}$$

and the functions of the second kind satisfy the Neumann boundary condition

$$\hat{n} \times \nabla \times \overline{\overline{G}}_{e2} = 0 \tag{12}$$

$$\hat{n} \times \nabla \times \overline{\overline{G}}_{m2} = 0 \quad . \tag{13}$$

The functions of the third kind satisfy the mixed boundary conditions

$$\left.\begin{aligned} (\hat{n} \times \overline{\overline{G}}_{e3})_{S-} &= (\hat{n} \times \overline{\overline{G}}_{e3})_{S+} \\ \frac{1}{\mu_1} (\hat{n} \times \nabla \times \overline{\overline{G}}_{e3})_{S-} &= \frac{1}{\mu_2} (\hat{n} \times \nabla \times \overline{\overline{G}}_{e3})_{S+} \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} \frac{1}{\mu_1} (\hat{n} \times \overline{\overline{G}}_{m3})_{S-} &= \frac{1}{\mu_2} (\hat{n} \times \overline{\overline{G}}_{m3})_{S+} \\ \frac{1}{k_1^2} (\hat{n} \times \nabla \times \overline{\overline{G}}_{m3})_{S-} &= \frac{1}{k_2^2} (\hat{n} \times \nabla \times \overline{\overline{G}}_{m3})_{S+} \end{aligned}\right\} \tag{15}$$

where S- and S+ denote, respectively, the surfaces approaching from the opposite side of a boundary separating two isotropic media with constitutive constants μ_1, ϵ_1 and μ_2, ϵ_2. In an open region, it is assumed that all these functions, electric or magnetic, satisfy the radiation condition at infinity

$$\lim_{R \to \infty} R \left[\nabla \times \overline{\overline{G}} - i k \hat{R} \times \overline{\overline{G}} \right] = 0 \quad . \tag{16}$$

If the open region is finite, such as the openings of an infinite waveguide, other forms of radiation condition are assumed to be prevailing.

It should be pointed out that in Reference [1] dyadic Green's functions of the first and the second kind are defined only for the electric type. Equation (7) was not mentioned in the previous work. The inclusion of Eq. (7) is necessary in the present treatment. For this reason we need a more precise nomenclature, namely, the electric type and the magnetic type to distinguish these two types of functions.

Because of the relations between $\overline{\overline{G}}_e$ and $\overline{\overline{G}}_m$ as stated by (1) and (2), and the boundary conditions characterizing these various kinds of functions, we have, more specifically

$$\nabla \times \overline{\overline{G}}_{e1} = \overline{\overline{G}}_{m2} \tag{17}$$

$$\nabla \times \overline{\overline{G}}_{m2} = \overline{\overline{I}}\delta(\overline{R}-\overline{R}') + k^2 \overline{\overline{G}}_{e1} \ ; \tag{18}$$

$$\nabla \times \overline{\overline{G}}_{e2} = \overline{\overline{G}}_{m1} \tag{19}$$

$$\nabla \times \overline{\overline{G}}_{m1} = \overline{\overline{I}}\delta(\overline{R}-\overline{R}') + k^2 \overline{\overline{G}}_{e2} \ ; \tag{20}$$

$$\nabla \times \overline{\overline{G}}_{e3} = \overline{\overline{G}}_{m3} \tag{21}$$

$$\nabla \times \overline{\overline{G}}_{m3} = \overline{\overline{I}}\delta(\overline{R}-\overline{R}') + k^2 \overline{\overline{G}}_{e3} \ . \tag{22}$$

In addition to the dyadic Green's functions of the electric and magnetic type, one can introduce a dyadic Green's function of the vector potential type. Since $\nabla \cdot \overline{\overline{G}}_m = 0$, we let

$$\overline{\overline{G}}_m = \nabla \times \overline{\overline{G}}_A \ . \tag{23}$$

In view of Eq. (1), we have

$$\overline{\overline{G}}_e = \overline{\overline{G}}_A - \nabla \overline{\Phi} \tag{24}$$

where $\overline{\Phi}$ denotes a vector function. Substituting Eqs. (23) and (24) into Eq. (2) we obtain

$$\nabla \times \nabla \times \overline{\overline{G}}_A = \overline{\overline{I}}\delta(\overline{R}-\overline{R}') + k^2 (\overline{\overline{G}}_A - \nabla \overline{\Phi})$$

or

$$-\nabla^2\overline{\overline{G}}_A+\nabla\nabla\cdot\overline{\overline{G}}_A=\overline{\overline{I}}\delta(\overline{R}-\overline{R}')+k^2(\overline{\overline{G}}_A-\nabla\overline{\Phi}) \quad . \tag{25}$$

Introducing now the gauge condition between $\overline{\overline{G}}_A$ and $\overline{\Phi}$ such that

$$\nabla\cdot\overline{\overline{G}}_A=-k^2\overline{\Phi} \tag{26}$$

we obtain the differential equation for $\overline{\overline{G}}_A$

$$\nabla^2\overline{\overline{G}}_A+k^2\overline{\overline{G}}_A=-\overline{\overline{I}}\delta(\overline{R}-\overline{R}') \quad . \tag{27}$$

As a result of Eqs. (26) and (27), the function $\overline{\Phi}$ satisfies the equation

$$\nabla^2\overline{\Phi}+k^2\overline{\Phi}=\frac{1}{k^2}\nabla\cdot\overline{\overline{I}}\delta(\overline{R}-\overline{R}')$$

$$=\frac{1}{k^2}\nabla\delta(\overline{R}-\overline{R}') \quad . \tag{28}$$

Equation (27) is the dyadic version of the vector Green's function pertaining to the vector potential function while Eq. (28) is the vector version of the scalar Green's function pertaining to the scalar potential for a doublet. As a result of Eqs. (23), (24) and (26), one finds that the relations between $\overline{\overline{G}}_m$, $\overline{\overline{G}}_e$ and $\overline{\overline{G}}_A$ are:

$$\overline{\overline{G}}_m=\nabla x\overline{\overline{G}}_A \tag{29}$$

$$\overline{\overline{G}}_e=(\overline{\overline{I}}+\frac{1}{k^2}\nabla\nabla)\cdot\overline{\overline{G}}_A \quad . \tag{30}$$

From the above discussion, we see that there are three types of dyadic Green's functions $\overline{\overline{G}}_e$, $\overline{\overline{G}}_m$ and $\overline{\overline{G}}_A$. The differential equations defining them are all different. Consequently the eigen-function expansion of these functions are also different, particularly in regard to the sets of functions used in these expansions.

Once the dyadic Green's function of certain type is found it is relatively simple to find the electric or the magnetic field due to any current distribution $\bar{J}(\bar{R}')$. For example, if one is interested in finding the solution for $\bar{E}(\bar{R})$ which satisfies the equation

$$\nabla \times \nabla \times \bar{E} - k^2 \bar{E} = i\omega\mu_o \bar{J} \tag{31}$$

then the proper function to be used is $\bar{\bar{G}}_e$ which satisfies Eq. (6).

To integrate Eq. (31), we can make use of the dyadic Green's identity

$$\iiint \left[\bar{P} \cdot \nabla \times \nabla \times \bar{\bar{Q}} - (\nabla \times \nabla \times \bar{P}) \cdot \bar{\bar{Q}} \right] dv =$$

$$= - \oiint \hat{n} \cdot \left[\bar{P} \times \nabla \times \bar{\bar{Q}} + (\nabla \times \bar{P}) \times \bar{\bar{Q}} \right] dS \quad . \tag{32}$$

The above formula is a dyadic version of the vector Green's theorem described by Stratton [4]. It is obtained by superposing three vector Green's identities with the dyadic function $\bar{\bar{Q}}$ defined by

$$\bar{\bar{Q}} = \bar{Q}^{(x)} \hat{x} + \bar{Q}^{(y)} \hat{y} + \bar{Q}^{(z)} \hat{z}$$

where $\bar{Q}^{(x)}$, $\bar{Q}^{(y)}$ and $\bar{Q}^{(z)}$ denote three independent vector functions each of which appears in the vector Green's theorem. By letting $\bar{P} = \bar{E}$, $\bar{\bar{Q}} = \bar{\bar{G}}_{e1}$ where $\hat{n} \times \bar{\bar{G}}_{e1} = 0$ on S then with the aid of Eqs. (6) and (31) we obtain first

$$\bar{E}(\bar{R}') = i\omega\mu_o \iiint \bar{J}(R) \cdot \bar{\bar{G}}_{e1}(\bar{R}/\bar{R}') \, dv - \iint \hat{n} \cdot \left[\bar{E}(\bar{R}) \times \nabla \times \bar{\bar{G}}_{e1}(\bar{R}/\bar{R}') \right] dS . \tag{33}$$

By interchanging the variables $\bar{R}$ and $\bar{R}'$ and making use of the symmetrical property of $\bar{\bar{G}}_{e1}$, namely,

$$\widetilde{\bar{\bar{G}}_{e1}(\bar{R}'/\bar{R})} = \bar{\bar{G}}_{e1}(\bar{R}/\bar{R}')$$

and

$$\widetilde{\nabla' \times G_{e1}(\bar{R}'/\bar{R})} = \nabla \times \bar{\bar{G}}_{e2}(\bar{R}/\bar{R}')$$

where "$\sim$" denotes the transpose of the dyadic function under the sign we obtain

$$\overline{E}(\overline{R}) = i\omega\mu_o \iiint \overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') \cdot \overline{J}(\overline{R}')\, dv' -$$

$$- \oiint_S \left[\nabla x \overline{\overline{G}}_{e2}(\overline{R}/\overline{R}') \cdot \hat{n} x \overline{E}(\overline{R}') \right] dS' \quad . \tag{34}$$

When the above formulation is applied to problems involving a perfectly conducting scatterer, including waveguides, $\hat{n} \times \overline{E} = 0$ on S we have

$$\overline{E}(\overline{R}) = i\omega\mu_o \iiint \overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') \cdot \overline{J}(\overline{R}')\, dv' \quad . \tag{35}$$

To find $\overline{H}(\overline{R})$ it is simpler to use the relation $\nabla x \overline{E}(\overline{R}) = i\omega\mu_o \overline{H}(\overline{R})$. In the case of Eq. (35) we obtain, in view of Eq. (17),

$$\overline{H}(\overline{R}) = \nabla x \iiint \overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') \cdot \overline{J}(\overline{R}')\, dv' = \iiint \overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') \cdot \overline{J}(\overline{R}')\, dv' \quad . \tag{36}$$

It is perhaps worth mentioning that Eq. (36) can also be derived by letting $\overline{\overline{G}}_{e2} = \overline{\overline{Q}}$ and $\overline{H} = \overline{P}$ in Eq. (32) and simplifying the result. While most of these formulas have been derived in Reference [1], the presentation here emphasizes the distinction between the two types of dyadic Green's functions, the electric and the magnetic, which was not stressed before. In fact, the function $\overline{\overline{G}}_m$ was not introduced previously, at least not explicitly. It is a relatively simple matter to show that expressions (35) and (36) can also be obtained with the aid of $\overline{\overline{G}}_A$. In the subsequent section we will discuss this alternative approach in more detail.

In the remaining sections we shall present the eigen-function expansion of various dyadic Green's functions pertaining to different diffracting bodies. The topics will be arranged in the same order as they appeared in the author's book [1]. Some of the basic formulas, such as the orthogonal properties of vector wave functions, the circulation theorem involving the product of Bessel functions and many other mathematical theorems will not be reviewed. Using the rectangular waveguide as an example we will treat first in great detail the three distinct methods of deriving the eigen-function expansion for $\overline{\overline{G}}_{e1}$. There are the methods of $\overline{\overline{G}}_m$, $\overline{\overline{G}}_e$ and $\overline{\overline{G}}_A$. It will be shown that the method of $\overline{\overline{G}}_m$ is the simplest. This method is then used to derive the complete expression for various dyadic Green's functions pertaining to other structures.

III. Rectangular Waveguide

The rectangular waveguide under consideration is assumed to be perfectly conducting with the dimension and coordinates shown in Fig. 1.

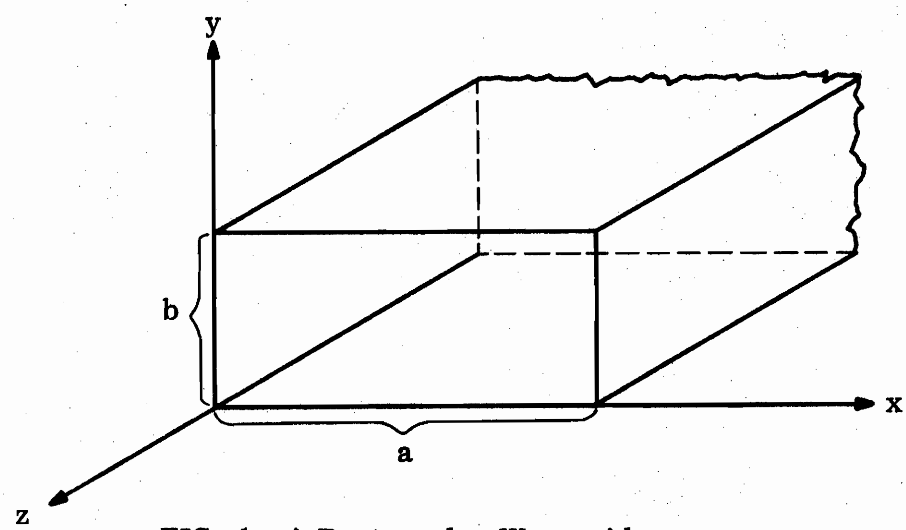

FIG. 1: A Rectangular Waveguide

As has been mentioned previously there are three distinct methods of deriving the eigen-function expansion for $\overline{\overline{G}}_{e1}$. We shall treat them one at a time.

Method of $\overline{\overline{G}}_m$:

In this method, we start with the equation for $\overline{\overline{G}}_{m2}$ which satisfies the differential equation

$$\nabla \times \nabla \times \overline{\overline{G}}_{m2} - k^2 \overline{\overline{G}}_{m2} = \nabla \times \left[\overline{\overline{I}} \delta (\overline{R} - \overline{R}') \right] \tag{1}$$

and the boundary condition

$$\hat{n} \times \nabla \times \overline{\overline{G}}_{m2} = 0 \quad \text{at } x = 0,\ a;\ y = 0,\ b.$$

The function also satisfies the radiation condition that at the ends of the waveguide, corresponding to $z = \pm\infty$, the guided waves are propagating in an outward direction.

Since $\nabla \cdot \nabla \times \left[\overline{\overline{I}} \delta (\overline{R} - \overline{R}') \right] = 0$, the generalized function $\nabla \times \left[\overline{\overline{I}} \delta (\overline{R} - \overline{R}') \right]$ can be expressed in terms of the vector wave functions $\overline{M}_{omn}(h)$ and $\overline{N}_{emn}(h)$ defined by

$$\overline{M}_{omn}(h) = \nabla \times \left[\psi_{omn}(h)\, \hat{z} \right]$$

$$\overline{N}_{emn}(h) = \frac{1}{K} \nabla \times \nabla \times \left[\psi_{emn}(h)\, \hat{z} \right]$$

where

$$\psi_{\substack{e\\o}mn}(h) = \begin{Bmatrix} \cos \frac{m\pi}{a} x \, \cos \frac{n\pi}{b} y \\ \\ \sin \frac{m\pi}{a} x \, \sin \frac{n\pi}{b} y \end{Bmatrix} e^{ihz}$$

$$K^2 = k_c^2 + h^2$$

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 .$$

Applying the Ohm-Rayleigh method we let

$$\nabla \times \left[\overline{\overline{I}} \delta (\overline{R} - \overline{R}') \right] = \int_{-\infty}^{\infty} \sum_{m,n} \left[\overline{M}_{omn}(h)\, \overline{A}_{omn}(h) + \overline{N}_{emn}(h)\, \overline{B}_{emn}(h) \right] dh \tag{2}$$

where $\overline{A}$ and $\overline{B}$ are two sets of unknown vector coefficients to be determined. By taking the anterior scalar product of Eq. (2) with $\overline{M}_{om'n'}(-h')$ and integrating through the entire volume of the guide we obtain, as a result of the orthogonal property of the vector wave functions,

$$\nabla' \times \overline{M}'_{om'n'}(-h') = \frac{(1+\delta_o)\pi a b k_c'^2}{2} \overline{A}_{om'n'}(h') \qquad (3)$$

where

$$k_c'^2 = \left(\frac{m'\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2$$

$$\delta_o = \begin{cases} 1, & m' \text{ or } n' = 0 \\ 0, & m' \text{ and } n' \neq 0 \end{cases}$$

hence

$$\overline{A}_{omn}(h) = \frac{(2-\delta_o)}{\pi a b k_c^2} \nabla' \times \overline{M}'_{omn}(-h)$$

$$= \frac{(2-\delta_o)K}{\pi a b k_c^2} \overline{N}'_{omn}(-h) \qquad (4)$$

where the primed functions are defined with respect to the primed variables pertaining to $\overline{R}'$. The orthogonal properties of the vector wave functions and their solutions are discussed in Chapter 5 of Reference [1]. In arriving at Eq. (3), we made use of the relation

$$\iiint \overline{M}_{om'n'}(-h') \cdot \nabla \times \left[\overline{\overline{I}} \delta(\overline{R}-\overline{R}') \right] dv =$$

$$= \iiint \left\{ \nabla \cdot \left[\overline{\overline{I}} \delta(\overline{R}-\overline{R}') \times \overline{M}_{om'n'}(-h') \right] + \overline{\overline{I}} \delta(\overline{R}-\overline{R}') \cdot \nabla \times \overline{M}_{om'n'}(-h') \right\} dv$$

$$= \nabla' \times \overline{M}'_{om'n'}(-h') \quad . \qquad (5)$$

Similarly, by taking the scalar product of Eq. (2) with $\overline{N}_{em'n'}(-h')$ and integrating through the entire volume of the guide we obtain

$$\overline{B}_{emn}(h) = \frac{(2-\delta_o)}{\pi a b k_c^2} \nabla' \times \overline{N}'_{emn}(-h) = \frac{(2-\delta_o)K}{\pi a b k_c^2} \overline{M}'_{emn}(-h) \tag{6}$$

thus

$$\nabla \times \left[\overline{\overline{I}}\,\delta(\overline{R}-\overline{R}')\right] = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} K \left[\overline{M}_{omn}(h)\,\overline{N}'_{omn}(-h) + \right.$$
$$\left. + \overline{N}_{emn}(h)\,\overline{M}'_{emn}(-h)\right] dh \tag{7}$$

where

$$C_{mn} = \frac{2-\delta_o}{\pi a b k_c^2} .$$

To determine $\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}')$ we let

$$\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} K \left[a_{omn}\overline{M}_{omn}(h)\,\overline{N}'_{omn}(-h) + \right.$$
$$\left. + b_{emn}\,\overline{N}_{emn}(h)\,\overline{M}'_{emn}(-h)\right] dh . \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (1) we find

$$a_{omn} = b_{emn} = \frac{1}{K^2 - k^2}$$

hence

$$\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \frac{K}{K^2 - k^2} \left[\overline{M}_{omn}(h)\,\overline{N}'_{omn}(-h) + \right.$$
$$\left. + \overline{N}_{emn}(h)\,\overline{M}'_{emn}(-h)\right] dh . \tag{9}$$

The Fourier integration contained in Eq. (9) can be evaluated in closed form with the aid of the residue theorem. This is so because the integrand decays sufficiently fast as h approaches $\pm\infty$. The result gives

$$\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \sum_{m,n} \frac{\pi i k C_{mn}}{k_g} \left[\overline{M}_{omn}(\pm k_g)\, \overline{N}'_{omn}(\mp k_g) + \right.$$

$$\left. + \overline{N}_{emn}(\pm k_g)\, \overline{M}'_{emn}(\mp k_g) \right], \quad z \gtrless z' \tag{10}$$

where

$$k_g = (k^2 - k_c^2)^{1/2} \quad .$$

In principle, once $\overline{\overline{G}}_{m2}$ is known we can find $\overline{\overline{G}}_{e1}$ by means of the relation

$$\nabla x \overline{\overline{G}}_{m2} = \overline{\overline{I}}\delta(\overline{R} - \overline{R}') + k^2 \overline{\overline{G}}_{e1} \tag{11}$$

However, because Eq. (10) represents a discontinuous series, it is rather intricate to evaluate $\nabla x \overline{\overline{G}}_{m2}$ based on Eq. (10). For this reason, it is simpler to determine $\nabla x \overline{\overline{G}}_{m2}$ based on Eq. (9).which yields

$$\nabla x \overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \frac{K^2}{K^2 - k^2} \left[\overline{N}_{omn}(h)\, \overline{N}'_{omn}(-h) + \right.$$

$$\left. + \overline{M}_{emn}(h)\, \overline{M}'_{emn}(-h) \right] dh \tag{12}$$

where we have made use of the relation

$$\nabla x \overline{M}_{omn}(h) = K \overline{N}_{omn}(h)$$

$$\nabla x \overline{N}_{emn}(h) = K \overline{M}_{emn}(h) \quad .$$

The expression for $\nabla x \overline{\overline{G}}_{m2}(\overline{R}/\overline{R}')$ as given by Eq. (12) has a singular term which can be extracted from the expression. For this reason, we split Eq. (12) into two parts as follows:

$$\nabla x \overline{G}_{m2}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \left[\overline{M}_e \overline{M}'_e + \frac{K^2}{h^2} \overline{N}_{ot} \overline{N}'_{ot} \right] dh +$$

$$+ \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \left[\frac{k^2}{K^2 - k^2} \overline{M}_e \overline{M}'_e + \frac{K^2}{K^2 - k^2} \left(\frac{k^2 - k_c^2}{h^2} \overline{N}_{ot} \overline{N}'_{ot} + \right.\right.$$

$$\left.\left. + \overline{N}_{ot} \overline{N}'_{oz} + \overline{N}_{oz} \overline{N}'_{ot} + \overline{N}_{oz} \overline{N}'_{oz} \right) \right] dh \qquad (13)$$

where $\overline{N}_{ot}$ and $\overline{N}_{oz}$ denote, respectively, the transverse part and the longitudinal part of $\overline{N}_o$, that is,

$$\overline{N}_o = \overline{N}_{ot} + \overline{N}_{oz} \;,$$

similarly for $\overline{N}'_o$. We have omitted the subscript 'mn' and the dependence (h) or (−h) for convenience. The splitting can be verified because we can write

$$\frac{K^2}{K^2 - k^2} = 1 + \frac{k^2}{K^2 - k^2}$$

or

$$\frac{K^2}{K^2 - k^2} = \frac{K^2}{h^2} + \frac{K^2}{K^2 - k^2} \cdot \frac{k^2 - k_c^2}{h^2}$$

with

$$K^2 = k_c^2 + h^2 \;.$$

The reason for this decomposition is that the first integral contained in Eq. (13) represents the singular function

$$\overline{\overline{I}}_t \, \delta(\overline{R} - \overline{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \left[\overline{M}_e \overline{M}'_e + \frac{K^2}{h^2} \overline{N}_{ot} \overline{N}'_{ot} \right] dh \qquad (14)$$

where

$$\overline{\overline{I}}_t = \hat{x}\hat{x} + \hat{y}\hat{y} \quad .$$

Equation (14) can be treated as an expansion theorem involving the product of a two-dimensional dyadic idemfactor and a three-dimensional delta function. The theorem can be derived by letting

$$\overline{\overline{I}}_t \delta(\overline{R}-\overline{R}') = \int_{-\infty}^{\infty} \sum_{m,n} \left[\overline{M}_e(h)\, \overline{a}_e(h) + \overline{N}_{ot}(h)\, \overline{b}_{ot}(h) \right] dh \quad .$$

Since $\overline{M}_e$ and $\overline{N}_{ot}$ are orthogonal, the vector coefficients $\overline{a}_e$ and $\overline{b}_{ot}$ can be determined in the same manner as the derivation of Eq. (7).

In regard to the remaining integral in Eq. (13), it can be shown that its integrand decays sufficiently fast at infinity in the upper or the lower h-plane, hence the method of contour integration can be applied to evaluate this integral. The result gives

$$\nabla \times \overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \overline{\overline{I}}_t(\overline{R}-\overline{R}') + k^2 \overline{\overline{S}}(\overline{R}/\overline{R}') \tag{15}$$

where

$$\overline{\overline{S}}(\overline{R}/\overline{R}') = \frac{i}{ab} \sum_{m,n} \frac{2-\delta_o}{k_g k_c^2} \left[\overline{M}_{emn}(\pm k_g)\, \overline{M}'_{emn}(\mp k_g) + \overline{N}_{omn}(\pm k_g)\, \overline{N}'_{omn}(\mp k_g) \right] z \gtrless z' \quad . \tag{16}$$

It can be shown that Eq. (15) can also be derived by taking the curl of $\overline{\overline{G}}_{m2}$ as given by Eq. (10) but with considerably more labor. Substituting Eq. (15) into Eq. (11) we obtain

$$\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') = \overline{\overline{S}}(\overline{R}/\overline{R}') - \frac{\hat{z}\hat{z}\,\delta(\overline{R}-\overline{R}')}{k^2} \quad . \tag{17}$$

Equation (17) gives the complete expression for $\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}')$. The singular term $-\hat{z}\hat{z}\delta(\overline{R}-\overline{R}')/k^2$ is missing in the old expression for $\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}')$ discussed in Reference [1]. The residue series represented by $\overline{\overline{S}}(\overline{R}/\overline{R}')$ is the same as the one defined by Eq. (8), p. 79 of that reference. The singular term $-\hat{z}\hat{z}\delta(\overline{R}-\overline{R}')/k^2$ vanishes when $\overline{R} \neq \overline{R}'$. When the point of observation lies inside the source region the singular term must be included in evaluating the electric field for an arbitrary current source. It may be of interest to remark that the absence of the singular term in the old formulation was not detected because the test cases used were confined to either fields outside of the source region or current sources without longitudinal component. The lack of a thorough check hindered the old work for many years until the error was discovered by P-O. Brundell.

<u>Method of $\overline{\overline{G}}_e$:</u>

In this method we start with the equation for $\overline{\overline{G}}_{e1}$

$$\nabla \times \nabla \times \overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') - k^2 \overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') = \overline{\overline{I}}\delta(\overline{R}-\overline{R}') \quad . \tag{18}$$

Since $\nabla \cdot \left[\overline{\overline{I}}\delta(\overline{R}-\overline{R}')\right] \neq 0$, the function $\overline{\overline{I}}\delta(\overline{R}-\overline{R}')$ must be expressed not only in terms of the soldnoidal vector wave functions $\overline{M}_{emn}$ and $\overline{N}_{omn}$ but also in terms of the nonsolenoidal vector wave functions $\overline{L}_{omn}$ defined by

$$\overline{L}_{omn}(h) = \nabla\psi_{omn}(h)$$

with

$$\psi_{omn}(h) = \sin\frac{m\pi}{a}x \, \sin\frac{n\pi}{b}y \, e^{ihz} \quad .$$

It should be pointed out that $\overline{L}_{omn}$ is not a solution for the homogeneous vector wave equation $\nabla \times \nabla \times \overline{F} - K^2\overline{F} = 0$ but it is a solution for $\nabla^2\overline{A} + K^2\overline{A} = 0$. The inclusion of $\overline{L}_{omn}$ is necessary because both $\overline{\overline{I}}\delta(\overline{R}-\overline{R}')$ and $\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}')$ are nonsolenoidal functions. The fact that $\overline{L}_{omn}$ is not a solution for the homogeneous vector wave equation makes the extension of the Ohm-Rayleigh method from the scalar case to the vector case not in the sense of one-to-one correspondence. To expand $\overline{\overline{I}}\delta(\overline{R}-\overline{R}')$, we let

$$\bar{\bar{I}}\delta(\bar{R}-\bar{R}') = \int_{-\infty}^{\infty} \sum_{m,n} \Big[\bar{L}_{omn}(h)\,\bar{A}_{omn}(h) + \bar{M}_{emn}(h)\,\bar{B}_{emn}(h) + \bar{N}_{omn}(h)\,\bar{C}_{omn}(h) \Big] dh \tag{19}$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$ are three sets of coefficients to be determined. It can be verified that the functions $\bar{L}$, $\bar{M}$ and $\bar{N}$ are orthogonal in the spatial domain and in the h-domain. In the spatial domain alone $\bar{L}_{omn}$ is not orthogonal to $\bar{N}_{omn}$, that is,

$$\iiint \bar{N}_{omn}(h) \cdot \bar{L}_{omn}(-h')\, dv \neq 0 ,$$

but

$$\iiiint \bar{N}_{omn}(h) \cdot \bar{L}_{omn}(-h')\, dv\, dh = 0 .$$

This characteristic is similar to the spherical case mentioned by Stratton [5] who has examined the orthogonal property of the spherical vector wave functions in the spatial domain but not in the eigen-value domain.

As a result of the orthogonal property of the three sets of vector wave functions in the spatial and h-domain we can readily determine the coefficients $\bar{A}$, $\bar{B}$, $\bar{C}$ in Eq. (19). They are

$$\bar{A}_{omn}(h) = \frac{2-\delta_o}{\pi a b K^2} \bar{L}'_{omn}(-h)$$

$$\bar{B}_{emn}(h) = \frac{2-\delta_o}{\pi a b k_c^2} \bar{M}'_{emn}(-h)$$

$$\bar{C}_{omn}(h) = \frac{2-\delta_o}{\pi a b k_c^2} \bar{N}'_{omn}(-h)$$

hence

$$\bar{\bar{I}}\delta(\bar{R}-\bar{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \left[\frac{k_c^2}{K^2} \bar{L}_{omn}(h)\, \bar{L}'_{omn}(-h) + \right.$$

$$\left. + \bar{M}_{emn}(h)\, \bar{M}'_{emn}(-h) + \bar{N}_{omn}(h)\, \bar{N}'_{omn}(-h) \right] dh \qquad (20)$$

where

$$C_{mn} = \frac{2-\delta_o}{\pi a b k_c^2}\,, \qquad \delta_o = \begin{cases} 1, & n \text{ or } m = 0 \\ 0, & n \text{ and } m \neq 0 \end{cases}$$

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad .$$

We now let

$$\bar{\bar{G}}_{e1}(\bar{R}/\bar{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \left[\alpha \frac{k_c^2}{K^2} \bar{L}_o \bar{L}'_o + \beta \bar{M}_e \bar{M}'_e + \nu \bar{N}_o \bar{N}'_o \right] dh \qquad (21)$$

where the subscript 'mn' and the functional dependence (h) or (-h) have been omitted. Substituting Eqs. (20) and (21) into (18), we find

$$\alpha = -\frac{1}{k^2}\,, \qquad \beta = \nu = \frac{1}{K^2 - k^2} \quad .$$

Thus, the complete expression for $\bar{\bar{G}}_{e1}(\bar{R}/\bar{R}')$ is given by

$$\bar{\bar{G}}_{e1}(\bar{R}/\bar{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \left[-\frac{k_c^2}{k^2 K^2} \bar{L}_o \bar{L}'_o + \frac{1}{K^2 - k^2} (\bar{M}_e \bar{M}'_e + \bar{N}_o \bar{N}'_o) \right] dh \qquad (22)$$

The fact that Eq. (22) is an alternative representation of Eq. (17) can be proved as follows. If we write

$$\bar{L}_o = \bar{L}_{ot} + \bar{L}_{oz}$$

$$\bar{N}_o = \bar{N}_{ot} + \bar{N}_{oz}$$

then from the definition of $\overline{L}_o$ and $\overline{N}_o$, one finds

$$\overline{L}_{ot} = \frac{-iK}{h}\overline{N}_{ot}\ , \qquad \overline{L}'_{ot} = \frac{iK}{h}\overline{N}'_{ot}$$

$$\overline{L}_{oz} = \frac{ihK}{k_c^2}\overline{N}_{oz}\ , \qquad \overline{L}'_{oz} = \frac{-ihK}{k_c^2}\overline{N}'_{oz}\ .$$

Equation (22) can, therefore, be written in the form

$$\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty}\sum_{m,n} C_{mn}\left\{\frac{1}{K^2-k^2}\overline{M}_e\overline{M}'_e + \right.$$

$$\left. + \frac{K^2}{k^2(K^2-k^2)}\left[\frac{k^2-k_c^2}{h^2}\overline{N}_{ot}\overline{N}'_{ot} + \overline{N}_{ot}\overline{N}'_{oz} + \overline{N}_{oz}\overline{N}'_{ot} + \frac{k^2-h^2}{k_c^2}\overline{N}_{oz}\overline{N}'_{oz}\right]\right\}dh \qquad (23)$$

The singular term in Eq. (23) is contained in the component $\overline{N}_{oz}\overline{N}'_{oz}$. From Eq. (20), we have

$$\hat{z}\hat{z}\delta(\overline{R}-\overline{R}') = \int_{-\infty}^{\infty}\sum_{m,n} C_{mn}\left[\frac{k_c^2}{K^2}\overline{L}_{oz}\overline{L}'_{oz} + \overline{N}_{oz}\overline{N}'_{oz}\right]dh =$$

$$= \int_{-\infty}^{\infty}\sum_{m,n} C_{mn}\frac{K^2}{k_c^2}\overline{N}_{oz}\overline{N}'_{oz}\,dh\ . \qquad (24)$$

Thus, Eq. (23) can be split into

$$\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty}\sum_{m,n} C_{mn}\left\{\frac{1}{K^2-k^2}\overline{M}_e\overline{M}'_e + \right.$$

$$\left. + \frac{K^2}{k^2(K^2-k^2)}\left[\frac{k^2-k_c^2}{h^2}\overline{N}_{ot}\overline{N}'_{ot} + \overline{N}_{ot}\overline{N}'_{oz} + \overline{N}_{oz}\overline{N}'_{ot} + \overline{N}_{oz}\overline{N}'_{oz}\right]\right\}dh -$$

$$-\int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \frac{K^2}{k^2 k_c^2} \overline{N}_{oz} \overline{N}'_{oz} \, dh \quad . \tag{25}$$

The first integral which is of the same form as the second integral in Eq. (13) can be evaluated in a closed form by the method of contour integration. Thus, $\overline{\overline{G}}_{e1}$ is given by

$$\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') = \overline{\overline{S}}(\overline{R}/\overline{R}') - \frac{1}{k^2} \hat{z}\hat{z}\, \delta(\overline{R}-\overline{R}') \tag{26}$$

where $\overline{\overline{S}}(\overline{R}/\overline{R}')$ is defined by Eq. (16). It is seen from the above analysis that the method of $\overline{\overline{G}}_{e1}$ is considerably more complicated than the method of $\overline{\overline{G}}_m$. In the first place we have to include the function $\overline{L}_{omn}$ in the synthesis but as far as the final result is concerned the function disappears.

Method of $\overline{\overline{G}}_A$

In this method we start with the differential equation for $\overline{\overline{G}}_A$

$$\nabla^2 \overline{\overline{G}}_A(\overline{R}/\overline{R}') + k^2 \overline{\overline{G}}_A(\overline{R}/\overline{R}') = -\overline{\overline{I}}\delta(\overline{R}-\overline{R}') \quad . \tag{27}$$

The Fourier integral representation of $\overline{\overline{I}}\delta(\overline{R}-\overline{R}')$ is given by Eq. (20). To find $\overline{\overline{G}}_A$ we let

$$\overline{\overline{G}}_A(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \left[a \frac{k_c^2}{K^2} \overline{L}_o \overline{L}'_o + b \overline{M}_e \overline{M}'_e + c \overline{N}_o \overline{N}'_o \right] dh \quad . \tag{28}$$

Substituting Eqs. (20) and (28) into Eq. (27), we obtain

$$a = b = c = \frac{1}{K^2 - k^2} \tag{29}$$

hence,

$$\bar{\bar{G}}_A(\bar{R}/\bar{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \frac{1}{K^2-k^2}\left[\frac{k_c^2}{K^2}\bar{L}_o\bar{L}_o' + \bar{M}_e\bar{M}_e' + \bar{N}_o\bar{N}_o'\right] dh \quad . \tag{30}$$

Substituting the above expression into Eq. (II-30)* and simplifying the result we obtain

$$\bar{\bar{G}}_{e1}(\bar{R}/\bar{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \left[-\frac{k_c^2}{k^2K^2}\bar{L}_o\bar{L}_o' + \frac{1}{K^2-k^2}(\bar{M}_e\bar{M}_e' + \bar{N}_o\bar{N}_o')\right] dh \tag{31}$$

which is the same as Eq. (22). The rest follows the same procedure as before.

It should be pointed out that the use of $\bar{L}$, $\bar{M}$ and $\bar{N}$ to synthesize $\bar{\bar{G}}_A$ or $\bar{\bar{G}}_{e1}$ is not the only approach to find the solution for these dyadic Green's functions. When Brundell [6] first found the missing term $-\hat{z}\hat{z}\delta(\bar{R}-\bar{R}')/k^2$ in our old work he used three functions defined by

$$\bar{l}_{omn}(h) = ih\hat{z}\psi_{omn}(h) \tag{32}$$

$$\bar{m}_{emn}(h) = \nabla_t\psi_{emn}(h) \times \hat{z} \tag{33}$$

$$\bar{n}_{omn}(h) = \nabla_t\psi_{omn}(h) \tag{34}$$

where
$$\psi_{\substack{e\\o}mn}(h) = \left\{\begin{matrix} \cos\frac{m\pi}{a}x \, \cos\frac{n\pi}{b}y \\ \\ \sin\frac{m\pi}{a}x \, \sin\frac{n\pi}{b}y \end{matrix}\right\} e^{ihz} \quad .$$

These three functions have been used previously by Morse and Feshbach [7] in their treatment of the dyadic Green's function $\bar{\bar{G}}_A$ and recently by Collin [8] in his investigation of the completeness problem. For comparison we shall give a brief sketch of the analysis based on these sets of functions. Since the three sets of functions $\bar{l}_o$, $\bar{m}_e$ and $\bar{n}_o$ are orthogonal in the spatial domain, it can be verified

*(II-30) denotes Eq. 30 in Section II.

$$\bar{\bar{I}}\delta(\bar{R}-\bar{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \left[\frac{k_c^2}{h^2} \bar{l}_o \bar{l}'_o + \bar{m}_e \bar{m}'_e + \bar{n}_o \bar{n}'_o \right] dh \tag{35}$$

and subsequently we find

$$\bar{\bar{G}}_A(\bar{R}/\bar{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \frac{1}{K^2-k^2} \left[\frac{k_c^2}{h^2} \bar{l}_o \bar{l}'_o + \bar{m}_e \bar{m}'_e + \bar{n}_o \bar{n}'_o \right] dh \quad . \tag{36}$$

Substituting Eq. (36) into Eq. (II-30) and making use of the relations

$$\nabla\nabla \cdot \bar{l}_o = -h^2(\bar{l}_o + \bar{n}_o)$$

$$\nabla\nabla \cdot \bar{n}_o = -k_c^2(\bar{l}_o + \bar{n}_o)$$

$$\nabla \cdot \bar{m}_c = 0$$

we obtain

$$\bar{\bar{G}}_{e1}(\bar{R}/\bar{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \frac{1}{K^2-k^2} \left[\frac{k_c^2(k^2-h^2)}{h^2 k^2} \bar{l}_o \bar{l}'_o - \frac{k_c^2}{k^2}(\bar{n}_o \bar{l}'_o + \bar{l}_o \bar{n}'_o) + \frac{k^2-k_c^2}{k^2} \bar{n}_o \bar{n}'_o + \bar{m}_e \bar{m}'_e \right] dh \quad . \tag{37}$$

Now the functions $\bar{l}_o$, $\bar{m}_e$ and $\bar{n}_o$ are related to vector wave functions $\bar{M}_e$ and $\bar{N}_o$ as follows:

$$\bar{l}_o = \frac{ihK}{k_c^2} \bar{N}_{oz} \tag{38}$$

$$\bar{m}_e = \bar{M}_e \tag{39}$$

$$\bar{n}_o = \frac{-iK}{h} \bar{N}_{ot} \tag{40}$$

Expressing Eq. (37) in terms of $\overline{N}_{oz}$, $\overline{N}_{ot}$ and $\overline{M}_e$ we find that the resultant representation for $\overline{\overline{G}}_{e1}$ is identical to Eq. (23). It is evident that if we start with the sets $\overline{l}_o$, $\overline{m}_e$, $\overline{n}_o$ it is not obvious that Eq. (37) can be readily converted into the form given by Eq. (26).

By comparing these different methods, it seems quite convincing that the method of $\overline{\overline{G}}_m$ is by far the simplest. In the following sections we will apply this method to derive the complete expressions for various dyadic Green's functions pertaining to different diffracting bodies. The topics will be arranged in the same order as they appear in Reference [1].

IV. Cylindrical Waveguide

The eigen-function expansion of $\nabla \times \left[\overline{\overline{I}}\,\delta(\overline{R}-\overline{R}')\right]$, analogous to Eq. (III-7) for the rectangular waveguide, is given by

$$\nabla \times \left[\overline{\overline{I}}\,\delta(\overline{R}-\overline{R}')\right] = \int_{-\infty}^{\infty} \left[\sum_{n,\lambda} C_\lambda K_\lambda \overline{M}_{\substack{e\\o}n\lambda}(h)\, \overline{N}'_{\substack{e\\o}n\lambda}(-h) + \sum_{n,\mu} C_\mu K_\mu \overline{N}_{\substack{e\\o}n\mu}(h)\, \overline{M}'_{\substack{e\\o}n\mu}(-h) \right] dh \qquad (1)$$

where

$$\overline{M}_{\substack{e\\o}n\lambda}(h) = \nabla \times \left[\psi_{\substack{e\\o}n\lambda}(h)\,\hat{z}\right]$$

$$\overline{N}_{\substack{e\\o}n\lambda}(h) = \frac{1}{K_\lambda} \nabla \times \nabla \times \left[\psi_{\substack{e\\o}n\lambda}(h)\,\hat{z}\right]$$

$$\psi_{\substack{e\\o}n\lambda}(h) = J_n(\lambda r) \begin{matrix}\cos\\ \sin\end{matrix} n\phi\, e^{ihz}$$

$$J_n(\lambda r) = 0 \quad \text{at} \quad r = a; \quad K_\lambda^2 = \lambda^2 + h^2$$

$$\overline{M}_{\substack{e\\o}n\mu}(h) = \nabla \times \left[\psi_{\substack{e\\o}n\mu}(h)\,\hat{z} \right]$$

$$\overline{N}_{\substack{e\\o}n\mu}(h) = \frac{1}{K_\mu} \nabla \times \nabla \times \left[\psi_{\substack{e\\o}n\mu}(h)\,\hat{z} \right]$$

$$\psi_{\substack{e\\o}n\mu}(h) = J_n(\mu r) \begin{matrix}\cos\\ \sin\end{matrix} n\phi\, e^{ihz}$$

$$\frac{\partial J_n(\mu r)}{\partial r} = 0 \quad \text{at} \quad r = a; \; K_\mu^2 = \mu^2 + h^2$$

$$C_\lambda = \frac{2-\delta_o}{4\pi^2 \lambda^2 I_\lambda}$$

$$I_\lambda = \int_0^a J_n^2(\lambda r)\, r\, dr = \frac{a^2}{2\lambda^2} \left[\frac{\partial J_n(\lambda r)}{\partial r} \right]^2_{r=a}$$

$$C_\mu = \frac{2-\delta_o}{4\pi^2 \mu^2 I_\mu}$$

$$I_\mu = \int_0^a J_n^2(\mu r)\, r\, dr = \frac{a^2}{2\mu^2} \left(\mu^2 - \frac{n^2}{a^2} \right) J_n^2(\mu a) \quad .$$

The origin of these functions or coefficients is found in Reference [1].

The function $\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}')$ can then be expanded in a similar manner which yields

$$\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} \left[\sum_{n,\lambda} \frac{C_\lambda K_\lambda}{K_\lambda^2 - k^2} \overline{M}_{\substack{e\\o}n\lambda}(h)\, \overline{N}'_{\substack{e\\o}n\lambda}(-h) + \sum_{n,\mu} \frac{C_\mu K_\mu}{K_\mu^2 - k^2} \overline{N}_{\substack{e\\o}n\mu}(h)\, \overline{M}'_{\substack{e\\o}n\mu}(-h) \right] dh \qquad (2)$$

hence,

$$\nabla \times \bar{\bar{G}}_{m2}(\bar{R}/\bar{R}') = \int_{-\infty}^{\infty} \left[\sum_{n,\lambda} \frac{C_\lambda K_\lambda^2}{K_\lambda^2 - k^2} \bar{N}_{\substack{e\\o}n\lambda}(h)\, \bar{N}'_{\substack{e\\o}n\lambda}(-h) + \right.$$

$$\left. + \sum_{n,\mu} \frac{C_\mu K_\mu^2}{K_\mu^2 - k^2} \bar{M}_{\substack{e\\o}n\mu}(h)\, \bar{M}'_{\substack{e\\o}n\mu}(-h) \right] dh \ . \qquad (3)$$

The singular term contained in Eq. (3) is represented by

$$\bar{\bar{I}}_t\, \delta(\bar{R} - \bar{R}') = \int_{-\infty}^{\infty} \left[\sum_{n,\lambda} C_\lambda \left(\frac{K_\lambda}{h} \right)^2 \bar{N}_{\substack{e\\o}n\lambda t}(h)\, \bar{N}'_{\substack{e\\o}n\lambda t}(-h) + \right.$$

$$\left. + \sum_{n,\mu} C_\mu \bar{M}_{\substack{e\\o}n\mu}(h)\, \bar{M}'_{\substack{e\\o}n\mu}(-h) \right] dh \qquad (4)$$

where $\bar{N}_{\substack{e\\o}n\lambda t}$ denotes the transversal part of $\bar{N}_{\substack{e\\o}n\lambda}$. By removing this singular term from Eq. (3), we can evaluate the remaining terms by means of contour integration. The result is given by

$$\nabla \times \bar{\bar{G}}_{m2}(\bar{R}/\bar{R}') = \bar{\bar{I}}_t\, \delta(\bar{R} - \bar{R}') + k^2 \bar{\bar{S}}(\bar{R}/\bar{R}') \qquad (5)$$

where

$$\bar{\bar{S}}(\bar{R}/\bar{R}') = \frac{i}{4\pi} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (2 - \delta_o) \cdot$$

$$\cdot \left[\frac{1}{\lambda^2 k_\lambda I_\lambda} \bar{N}_{\substack{e\\o}n\lambda}(\pm k_\lambda)\, \bar{N}'_{\substack{e\\o}n\lambda}(\mp k_\lambda) + \frac{1}{\mu^2 k_\mu I_\mu} \bar{M}_{\substack{e\\o}n\mu}(\pm k_\mu)\, \bar{M}'_{\substack{e\\o}n\mu}(\mp k_\mu) \right], \quad z \gtrless z' \ ,$$

$$k_\lambda = \sqrt{k^2 - \lambda^2} \ , \qquad k_\mu = \sqrt{k^2 - \mu^2} \ .$$

Hence

$$\bar{\bar{G}}_{e1}(\bar{R}/\bar{R}') = \frac{1}{k^2}\left[\nabla \times \bar{\bar{G}}_{m2}(\bar{R}/\bar{R}') - \bar{\bar{I}}\delta(\bar{R}-\bar{R}')\right]$$

$$= \bar{\bar{S}}\,(\bar{R}/\bar{R}') - \frac{1}{k^2}\hat{z}\hat{z}\,\delta\,(\bar{R}-\bar{R}') \qquad (7)$$

The residue series $\bar{\bar{S}}(\bar{R}/\bar{R}')$ is the same as the one described by Eq. (5), p. 89 of Reference [1] while the term $-\hat{z}\hat{z}\,\delta\,(\bar{R}-\bar{R}')/k^2$ was missing in the old work for $\bar{\bar{G}}_{e1}$.

V. Eigen-function Expansion of Free-space Dyadic Green's Functions Using Cylindrical Vector Wave Functions

Using the cylindrical vector wave functions defined by

$$\bar{M}_{\substack{e\\o}n\lambda}(h) = \nabla \times \left[\psi_{\substack{e\\o}n\lambda}(h)\ \hat{z}\right]$$

$$\bar{N}_{\substack{e\\o}n\lambda}(h) = \frac{1}{K}\nabla \times \bar{M}_{\substack{e\\o}n\lambda}(h)$$

where $K^2 = h^2 + \lambda^2$ with both h and λ being continuous eigen values, we can expand $\nabla \times \left[\bar{\bar{I}}\delta\,(\bar{R}-\bar{R}')\right]$ as follows:

$$\nabla \times \left[\bar{\bar{I}}\delta(\bar{R}-\bar{R}')\right] = \int_{-\infty}^{\infty} dh \int_{0}^{\infty} d\lambda \sum_{n} C_\lambda K \cdot$$

$$\cdot \left[\bar{M}_{\substack{e\\o}n\lambda}(h)\,\bar{N}'_{\substack{e\\o}n\lambda}(-h) + \bar{N}_{\substack{e\\o}n\lambda}(h)\,\bar{M}'_{\substack{e\\o}n\lambda}(-h)\right] \qquad (1)$$

$$C_\lambda = \frac{2-\delta_0}{4\pi^2\lambda} \quad .$$

Denoting the free-space dyadic Green's function of the magnetic type by $\bar{\bar{G}}_{m0}(\bar{R}/\bar{R}')$, we find

$$\overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} dh \int_{0}^{\infty} d\lambda \sum_{n} \frac{C_\lambda K}{K^2 - k^2} \cdot$$

$$\cdot \left[\overline{M}_{\substack{e\\o}n\lambda}(h)\, \overline{N}'_{\substack{e\\o}n\lambda}(-h) + \overline{N}_{\substack{e\\o}n\lambda}(h)\, \overline{M}'_{\substack{e\\o}n\lambda}(-h) \right] , \qquad (2)$$

and

$$\nabla \times \overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} dh \int_{0}^{\infty} d\lambda \sum_{n} \frac{C_\lambda K^2}{K^2 - k^2} \cdot$$

$$\cdot \left[\overline{N}_{\substack{e\\o}n\lambda}(h)\, \overline{N}'_{\substack{e\\o}n\lambda}(-h) + \overline{M}_{\substack{e\\o}n\lambda}(h)\, \overline{M}'_{\substack{e\\o}n\lambda}(-h) \right] . \qquad (3)$$

For cylindrical problems, we remove the λ-integration in Eq. (3). In this case, the singular term contained in Eq. (3) is represented by

$$\hat{z}\hat{z}\,\delta(\overline{R} - \overline{R}') = \int_{-\infty}^{\infty} dh \int_{0}^{\infty} d\lambda \sum_{n} C_\lambda \left(\frac{K^2}{\lambda^2} \right) \overline{N}_{\substack{e\\o}n\lambda z}(h)\, \overline{N}'_{\substack{e\\o}n\lambda z}(-h) \qquad (4)$$

where $\overline{N}_{\substack{e\\o}n\lambda z}(h)$ denotes the longitudinal component of $\overline{N}_{\substack{e\\o}n\lambda}(h)$. By removing the singular term from Eq. (3), we can evaluate the remaining terms by means of contour integral. The result is given by

$$\nabla \times \overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = k^2\, \overline{\overline{S}}_h(\overline{R}/\overline{R}') + \hat{z}\hat{z}\,\delta(\overline{R} - \overline{R}') \qquad (5)$$

where

$$\overline{\overline{S}}_h(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} dh \sum_n \frac{i(2-\delta_o)}{8\pi\eta^2} \cdot$$

$$\left\{ \begin{array}{l} \overline{N}^{(1)}_{\substack{e\\o}n\eta}(h)\,\overline{N}'_{\substack{e\\o}n\eta}(-h) + \overline{M}^{(1)}_{\substack{e\\o}n\eta}(h)\,\overline{M}'_{\substack{e\\o}n\eta}(-h) \\ \overline{N}_{\substack{e\\o}n\eta}(h)\,\overline{N}'^{(1)}_{\substack{e\\o}n\eta}(-h) + \overline{M}_{\substack{e\\o}n\eta}(h)\,\overline{M}'^{(1)}_{\substack{e\\o}n\eta}(-h) \end{array} \right\} \quad r \gtrless r' \quad . \qquad (6)$$

Functions with superscript (1) are defined with respect to the Hankel function of the first kind and $\eta = \sqrt{k^2 - h^2}$.

In view of Eq. (II-2), we have

$$\overline{\overline{G}}_{eo}(\overline{R}/\overline{R}') = \frac{1}{k^2}\left[\nabla x \overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') - \overline{\overline{I}}\delta(\overline{R}-\overline{R}')\right]$$

$$= \overline{\overline{S}}_h(\overline{R}/\overline{R}') - \frac{1}{k^2}\overline{\overline{I}}_t\delta(\overline{R}-\overline{R}') \quad . \qquad (7)$$

The integral-residue series given by Eq. (6) is the same as Eq. (5), p. 96 of Reference [1].

For flat earth, we remove the h-integration in Eq. (3). The singular term contained in Eq. (3) is represented by

$$\overline{\overline{I}}_t\delta(\overline{R}-\overline{R}') = \int_{-\infty}^{\infty} dh \int_0^{\infty} d\lambda \sum_n \left[\frac{K^2}{h^2}\overline{N}_{\substack{e\\o}n\lambda t}(h)\,\overline{N}'_{\substack{e\\o}n\lambda t}(-h) + \overline{M}_{\substack{e\\o}n\lambda}(h)\,\overline{M}'_{\substack{e\\o}n\lambda}(-h)\right] \qquad (8)$$

where $\overline{N}_{\substack{e\\o}n\lambda t}(h)$ denotes the transversal component of $\overline{N}_{\substack{e\\o}n\lambda}(h)$. After removing the singular term from Eq. (3), we can evaluate the remaining terms by means of contour integration in the h-plane which yields

$$\nabla x \overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = k^2\overline{\overline{S}}_\lambda(\overline{R}/\overline{R}') + \overline{\overline{I}}_t\delta(\overline{R}-\overline{R}') \qquad (9)$$

hence

$$\overline{\overline{G}}_{eo}(\overline{R}/\overline{R}') = \frac{1}{k^2}\left[\nabla \times \overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') - \overline{\overline{I}}\delta(\overline{R}-\overline{R}')\right]$$

$$= \overline{\overline{S}}_{\lambda}(\overline{R}/\overline{R}') - \frac{1}{k^2}\hat{z}\hat{z}\,\delta(\overline{R}-\overline{R}') \qquad (10)$$

where

$$\overline{\overline{S}}_{\lambda}(\overline{R}/\overline{R}') = \frac{i}{4\pi}\int_0^{\infty} d\lambda \sum_n \frac{2-\delta_0}{\lambda h_1} \cdot$$

$$\cdot \begin{Bmatrix} \overline{M}_{\substack{e\\o}n\lambda}(h_1)\,\overline{M}'_{\substack{e\\o}n\lambda}(-h_1) + \overline{N}_{\substack{e\\o}n\lambda}(h_1)\,\overline{N}'_{\substack{e\\o}n\lambda}(-h_1) \\ \overline{M}_{\substack{e\\o}n\lambda}(-h_1)\,\overline{M}'_{\substack{e\\o}n\lambda}(h_1) + \overline{N}_{\substack{e\\o}n\lambda}(-h_1)\,\overline{N}'_{\substack{e\\o}n\lambda}(h_1) \end{Bmatrix} \quad z \gtrless z' \qquad (11)$$

and $h_1 = \sqrt{k^2-\lambda^2}$. Equation (11) is the same as Eq. (1), p. 103 found in Reference [1]. Once $\overline{\overline{G}}_{eo}$ is known the dyadic Green's functions of other kinds can be found by the method of scattering superposition.

VI. Perfectly Conducting Elliptic Cylinder

To find $\overline{\overline{G}}_{e1}$ or $\overline{\overline{G}}_{m2}$ for perfectly conducting elliptic cylinder we need the eigen-function expansion for $\overline{\overline{G}}_{eo}$ or $\overline{\overline{G}}_{mo}$ first. Following the same procedure as before for the circular cylindrical case, we find

$$\overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} dh \int_0^{\infty} d\lambda \sum_m \frac{K}{\pi^2\lambda(K^2-k^2)\, I_{\substack{e\\o}m\lambda}} \cdot$$

$$\cdot \left[\overline{M}_{\substack{e\\o}m\lambda}(h)\,\overline{N}'_{\substack{e\\o}m\lambda}(-h) + \overline{N}_{\substack{e\\o}m\lambda}(h)\,\overline{M}'_{\substack{e\\o}m\lambda}(-h)\right] \qquad (1)$$

where

$$\overline{M}_{\substack{e\\o}m\lambda}(h) = \nabla \times \left[\psi_{\substack{e\\o}m\lambda}(h)\,\hat{z} \right]$$

$$\overline{N}_{\substack{e\\o}m\lambda}(h) = \frac{1}{K} \nabla \times \overline{M}_{\substack{e\\o}m\lambda}(h)$$

$$\psi_{\substack{e\\o}m\lambda}(h) = S_{\substack{e\\o}m\lambda}(v)\, R_{\substack{e\\o}m\lambda}(u)\, e^{ihz}$$

$$K^2 = h^2 + \lambda^2 \quad .$$

$S_{\substack{e\\o}m\lambda}(h)$ and $R_{\substack{e\\o}m\lambda}(h)$ denote, respectively, the angular and the radial elliptical cylinder wave functions, and

$$I_{\substack{e\\o}m\lambda} = \int_0^{2\pi} S^2_{\substack{e\\o}m\lambda}(v)\, dv \quad .$$

From Eq. (1), we obtain

$$\nabla \times \overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} dh \int_0^{\infty} d\lambda \sum_m \frac{K^2}{\pi^2 \lambda\, (K^2 - k^2)\, I_{\substack{e\\o}n\lambda}} \cdot$$

$$\cdot \left[\overline{N}_{\substack{e\\o}m\lambda}(h)\, \overline{N}'_{\substack{e\\o}m\lambda}(-h) + \overline{M}_{\substack{e\\o}m\lambda}(h)\, \overline{M}'_{\substack{e\\o}m\lambda}(-h) \right] \quad . \tag{2}$$

For problems involving perfectly conducting elliptical cylinder, we remove the λ-integration. The singular term contained in Eq. (2) is represented by

$$\hat{z}\hat{z}\,\delta(\overline{R} - \overline{R}') = \int_{-\infty}^{\infty} dh \int_0^{\infty} d\lambda \sum_m \frac{K^2}{\pi^2 \lambda^3 I_{\substack{e\\o}n\lambda}} \overline{N}_{\substack{e\\o}m\lambda z}(h)\, \overline{N}'_{\substack{e\\o}m\lambda z}(-h) \tag{3}$$

After removing this term from Eq. (2), we can evaluate the remaining terms by contour integration which yields

$$\nabla \times \overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = k^2 \overline{\overline{S}}(\overline{R}/\overline{R}') + \hat{z}\hat{z}\,\delta(\overline{R}-\overline{R}') \tag{4}$$

and

$$\begin{aligned}\overline{\overline{G}}_{eo}(\overline{R}/\overline{R}') &= \frac{1}{k^2}\left[\nabla \times \overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') - \overline{\overline{I}}\,\delta(\overline{R}-\overline{R}')\right] \\ &= \overline{\overline{S}}(\overline{R}/\overline{R}') - \overline{\overline{I}}_t\,\delta(\overline{R}-\overline{R}')/k^2\end{aligned} \tag{5}$$

where

$$\overline{\overline{S}}(\overline{R}/\overline{R}') = \frac{i}{2\pi}\int_{-\infty}^{\infty} dh \sum_m \frac{1}{\eta^2 I_{\substack{e\\o}m\eta}} \cdot$$

$$\cdot \begin{Bmatrix} \overline{M}^{(1)}_{\substack{e\\o}m\eta}(h)\,\overline{M}'_{\substack{e\\o}m\eta}(-h) + \overline{N}^{(1)}_{\substack{e\\o}m\eta}(h)\,\overline{N}'_{\substack{e\\o}m\eta}(-h) \\ \overline{M}_{\substack{e\\o}m\eta}(h)\,\overline{M}'^{(1)}_{\substack{e\\o}m\eta}(-h) + \overline{N}_{\substack{e\\o}m\eta}(h)\,\overline{N}'^{(1)}_{\substack{e\\o}m\eta}(-h) \end{Bmatrix} \quad u \gtrless u' \tag{6}$$

$$\eta = \sqrt{k^2 - h^2}\ .$$

The functions with subscript (1) are defined with respect to the radial functions of the first kind as described by Eqs. (29-14) and (29-15), p. 114 of Reference [1].

VII. Perfectly Conducting Wedge and Half-Sheet

As explained in Reference [1], it is more expedient to determine directly $\overline{\overline{G}}_{e1}$ or $\overline{\overline{G}}_{m2}$ for a perfectly conducting wedge without going through the expansion for the free-space dyadic Green's function. The expansion for $\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}')$ is found to be given by

$$\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} dh \int_{0}^{\infty} d\lambda \sum_{\nu} C_{\lambda} \frac{K}{K^2 - k^2} \cdot$$

$$\cdot \left[\overline{N}_{e\nu\lambda}(h)\, \overline{M}'_{e\nu\lambda}(-h) + \overline{M}_{o\nu\lambda}(h)\, \overline{N}'_{o\nu\lambda}(-h) \right] \tag{1}$$

where

$$\overline{M}_{\substack{e\\o}\nu\lambda}(h) = \nabla \times \left[\psi_{\substack{e\\o}\nu\lambda}(h)\, \hat{z} \right]$$

$$\overline{N}_{\substack{e\\o}\nu\lambda}(h) = \frac{1}{K} \nabla \times \overline{M}_{\substack{e\\o}\nu\lambda}(h)$$

$$\psi_{\substack{e\\o}\nu\lambda}(h) = J_{\nu}(\lambda r) \begin{matrix}\cos\\ \sin\end{matrix} \nu\phi\, e^{ihz}$$

$$\nu = n/(2 - \frac{\phi_o}{\pi}), \quad n = 0, 1, 2, \ldots$$

ϕ_o = angular span of the conducting wedge

$$K^2 = h^2 + \lambda^2$$

$$C_{\lambda} = \frac{2 - \delta_o}{2\pi\,(2\pi - \phi_o)\,\lambda} \quad .$$

From Eq. (1), we obtain

$$\nabla \times \overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} dh \int_{0}^{\infty} d\lambda \sum_{\nu} C_{\lambda} \frac{K^2}{K^2 - k^2} \cdot$$

$$\cdot \left[\overline{M}_{e\nu\lambda}(h)\, \overline{M}'_{e\nu\lambda}(-h) + \overline{N}_{o\nu\lambda}(h)\, \overline{N}'_{o\nu\lambda}(-h) \right] \quad . \tag{2}$$

The singular term contained in Eq. (2) is represented by

$$\hat{z}\hat{z}\delta(\bar{R}-\bar{R}') = \int_{-\infty}^{\infty} dh \int_{0}^{\infty} d\lambda \sum_{\nu} C_{\lambda} \frac{K^2}{\lambda^2} \bar{N}_{o\nu\lambda z}(h)\, \bar{N}'_{o\nu\lambda z}(-h) \tag{3}$$

where $\bar{N}_{o\nu\lambda z}$ denotes the longitudinal component of $\bar{N}_{o\nu\lambda}$. Removing this term from Eq. (2), we can evaluate the remaining terms by contour integration in the λ-plane which yields

$$\nabla\times\bar{\bar{G}}_{m2}(\bar{R}/\bar{R}') = k^2\bar{\bar{S}}(\bar{R}/\bar{R}') + \hat{z}\hat{z}\delta(\bar{R}-\bar{R}') \tag{4}$$

hence

$$\bar{\bar{G}}_{e1}(\bar{R}/\bar{R}') = \frac{1}{k^2}\left[\nabla\times\bar{\bar{G}}_{m2}(\bar{R}/\bar{R}') - \bar{\bar{I}}\delta(\bar{R}-\bar{R}')\right]$$

$$= \bar{\bar{S}}(\bar{R}/\bar{R}') - \frac{1}{k^2}\bar{\bar{I}}_t\delta(\bar{R}-\bar{R}') \tag{5}$$

where

$$\bar{\bar{S}}(\bar{R}/\bar{R}') = \int_{-\infty}^{\infty} dh \sum_{\nu} \frac{i(2-\delta_o)}{4(2\pi-\phi_o)\eta^2} \cdot$$

$$\cdot \begin{Bmatrix} \bar{M}^{(1)}_{e\nu\eta}(h)\,\bar{M}'_{e\nu\eta}(-h) + \bar{N}^{(1)}_{o\nu\eta}\,\bar{N}'_{o\nu\eta}(-h) \\ \bar{M}_{e\nu\eta}(h)\,\bar{M}'^{(1)}_{e\nu\eta}(-h) + \bar{N}_{o\nu\eta}(h)\,\bar{N}'^{(1)}_{o\nu\eta}(-h) \end{Bmatrix}, \quad r \gtrless r' \tag{6}$$

and

$$\eta = \sqrt{k^2-h^2}\,.$$

The expression for $\bar{\bar{S}}(\bar{R}/\bar{R}')$ is the same as Eq. (31-9), p. 123, of Reference [1]. Vector wave functions with subscript (1) are defined with respect to the Hankel function of the first kind, that is,

$$\bar{M}^{(1)}_{e\nu\eta}(h) = \nabla\times\left[H^{(1)}_{\nu}(\eta r)\cos\nu\phi\, e^{ihz}\,\hat{z}\right]$$

$$\bar{N}^{(1)}_{o\nu\eta}(h) = \frac{1}{k}\nabla\times\nabla\times\left[H^{(1)}_{\nu}(\eta r)\sin\nu\phi\, e^{ihz}\,\hat{z}\right].$$

VIII. Eigen-function Expansion of Free-Space Dyadic Green's Functions Using Spherical Vector Wave Functions

For spherical problems, we introduce the spherical vector wave functions defined by

$$\overline{M}_{\substack{e\\o}mn}(K) = \nabla x \left[\psi_{\substack{e\\o}mn}(K)\,\overline{R}\right]$$

$$\overline{N}_{\substack{e\\o}mn}(K) = \frac{1}{K}\,\nabla x \overline{M}_{\substack{e\\o}mn}(K)$$

where

$$\psi_{\substack{e\\o}mn}(K) = j_n(KR)\,\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\, P_n^m(\cos\theta)$$

and $j_n(KR)$ denotes the spherical Bessel function of order n. The eigen-function expansion of $\overline{\overline{G}}_{mo}(\overline{R}/\overline{R}')$ can be shown to be

$$\overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = \int_0^\infty dK \sum_{n=1}^{\infty}\sum_{m=0}^{n} C_{mn}\,\frac{K^3}{K^2-k^2}\,\cdot$$

$$\cdot\left[\overline{N}_{\substack{e\\o}mn}(K)\,\overline{M}'_{\substack{e\\o}mn}(K) + \overline{M}_{\substack{e\\o}mn}(K)\,\overline{N}'_{\substack{e\\o}mn}(K)\right] \qquad (1)$$

where

$$C_{mn} = \frac{2-\delta_o}{2\pi^2}\,\frac{(2n+1)}{n(n+1)}\,\frac{(n-m)!}{(n+m)!}$$

$$\delta_o = \begin{cases}1, & m = 0\\ 0, & m \neq 0\end{cases}$$

hence

$$\nabla x\overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = \int_0^\infty dK \sum_{n=1}^{\infty}\sum_{m=0}^{n} C_{mn}\,\frac{K^4}{K^2-k^2}\,\cdot$$

$$\cdot\left[\overline{M}_{\substack{e\\o}mn}(K)\,\overline{M}'_{\substack{e\\o}mn}(K) + \overline{N}_{\substack{e\\o}mn}(K)\,\overline{N}'_{\substack{e\\o}mn}(K)\right]. \qquad (2)$$

The singular term contained in Eq. (2) is represented by

$$\bar{\bar{s}}_1(\bar{R}/\bar{R}') = \int_0^\infty dK \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} K^2 \bar{M}_{\substack{e\\o}mn}(K)\, \bar{M}'_{\substack{e\\o}mn}(K) \quad . \tag{3}$$

Thus, Eq. (2) can be written in the form

$$\nabla x \bar{\bar{G}}_{mo}(\bar{R}/\bar{R}') = \int_0^\infty dK \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} K^2 \bar{M}_{\substack{e\\o}mn}(K)\, \bar{M}'_{\substack{e\\o}mn}(K) +$$

$$+ \int_0^\infty dK \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} \left[\frac{k^2 K^2}{K^2 - k^2} \bar{M}_{\substack{e\\o}mn}(K)\, \bar{M}'_{\substack{e\\o}mn}(K) + \right.$$

$$\left. + \frac{K^4}{K^2 - k^2} \bar{N}_{\substack{e\\o}mn}(K)\, \bar{N}'_{\substack{e\\o}mn}(K) \right] \quad . \tag{4}$$

The second integral in Eq. (4) is regular at infinity and it can be evaluated in closed form after the domain of integration is changed from $(0, \infty)$ to $(-\infty, \infty)$. Thus, we obtain

$$\nabla x \bar{\bar{G}}_{mo}(\bar{R}/\bar{R}') = \bar{\bar{s}}_1(\bar{R}/\bar{R}') + k^2 \bar{\bar{S}}(\bar{R}/\bar{R}') \tag{5}$$

where

$$\bar{\bar{S}}(\bar{R}/\bar{R}') = \frac{ik}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{n} (2 - \delta_o) \frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!} \quad .$$

$$\cdot \begin{Bmatrix} \bar{M}^{(1)}_{\substack{e\\o}mn}(k)\, \bar{M}'_{\substack{e\\o}mn}(k) + \bar{N}^{(1)}_{\substack{e\\o}mn}(k)\, \bar{N}'_{\substack{e\\o}mn}(k) \\ \bar{M}_{\substack{e\\o}mn}(k)\, \bar{M}'^{(1)}_{\substack{e\\o}mn}(k) + \bar{N}_{\substack{e\\o}mn}(k)\, \bar{N}'^{(1)}_{\substack{e\\o}mn}(k) \end{Bmatrix} \quad R \gtrless R' \tag{6}$$

Subsequently, we obtain

$$\bar{\bar{G}}_{eo}(\bar{R}/\bar{R}') = \frac{1}{k^2}\left[\nabla \times \bar{\bar{G}}_{mo}(\bar{R}/\bar{R}') - \bar{\bar{I}}\delta(\bar{R}-\bar{R}')\right]$$

$$= \bar{\bar{S}}(\bar{R}/\bar{R}') - \frac{1}{k^2}\left[\bar{\bar{I}}\delta(\bar{R}-\bar{R}') - \bar{\bar{s}}_1(\bar{R}/\bar{R}')\right] \quad . \tag{7}$$

The residue series $\bar{\bar{S}}$ given by Eq. (6) is the same as Eq. (39-18), p. 174, of Reference [1].

In regard to $\bar{\bar{s}}_1(\bar{R}/\bar{R}')$, defined by Eq. (3), the K-integration can be simplified as follows:

$$\bar{M}_{\substack{e \\ o}mn}(K) = j_n(KR)\,\bar{m}_{\substack{e \\ o}mn}$$

where

$$\bar{m}_{\substack{e \\ o}mn} = \left[\mp \frac{m P_n^m(\cos\theta)}{\sin\theta} \begin{matrix}\sin \\ \cos\end{matrix}\, m\phi\,\hat{\theta} - \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \begin{matrix}\cos \\ \sin\end{matrix}\, m\phi\,\hat{\phi}\right]$$

and

$$\int_0^\infty K^2 j_n(KR)\, j_n(KR')\, dK = \frac{\pi\delta(R-R')}{2R^2}$$

hence

$$\bar{\bar{s}}_1(\bar{R}/\bar{R}') = \frac{\pi\delta(R-R')}{2R^2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn}\, \bar{m}_{\substack{e \\ o}mn}\, \bar{m}'_{\substack{e \\ o}mn} \quad . \tag{8}$$

The terms involved in the summation signs of Eq. (8) represent part of a two-dimensional angular singular function. In fact, we have the expansion theorem

$$\frac{\bar{\bar{I}}_t\,\delta(\theta-\theta')\,\delta(\phi-\phi')}{\sin\theta} = \frac{\pi}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} \left[\bar{m}_{\substack{e \\ o}mn}\, \bar{m}'_{\substack{e \\ o}mn} + (\hat{R}\times\bar{m}_{\substack{e \\ o}mn})(\hat{R}'\times\bar{m}'_{\substack{e \\ o}mn})\right] \tag{9}$$

where $\bar{\bar{I}}_t = (\hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi})$. This theorem can be proved as we recognize that the two angular functions $\bar{m}_{\substack{e\\o}mn}$ and $\hat{R} \times \bar{m}_{\substack{e\\o}mn}$ represent two orthogonal sets of functions. This theorem is the dyadic counterpart of the scalar expansion theorem involving the spherical surface harmonics discussed by Stratton [9]. By means of this theorem, any vector angular function $\bar{f}(\theta, \phi) = f_\theta \hat{\theta} + f_\phi \hat{\phi}$ can be expressed in terms of $\bar{m}_{\substack{e\\o}mn}$ and $\hat{R} \times \bar{m}_{\substack{e\\o}mn}$. Thus, by taking the anterior scalar product of $\bar{f}(\theta, \phi)$ and Eq. (9) and integrating with respect to $\sin\theta\, d\theta\, d\phi$ we obtain

$$\bar{f}(\theta', \phi') = \frac{\pi}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} \left[a_{\substack{e\\o}mn} \bar{m}'_{\substack{e\\o}mn} + b_{\substack{e\\o}mn} (\hat{R}' \times \bar{m}'_{\substack{e\\o}mn}) \right] \tag{10}$$

where

$$a_{\substack{e\\o}mn} = \int_0^\pi \int_{-\pi}^{\pi} \bar{f}(\theta, \phi) \cdot \bar{m}_{\substack{e\\o}mn} \sin\theta d\theta d\phi$$

$$b_{\substack{e\\o}mn} = \int_0^\pi \int_{-\pi}^{\pi} \bar{f}(\theta, \phi) \cdot (\hat{R} \times \bar{m}_{\substack{e\\o}mn}) \sin\theta d\theta d\phi \quad .$$

Returning now to our discussion of $\bar{\bar{s}}_1(\bar{R}/\bar{R}')$, if we denote

$$\bar{\bar{s}}_2(\bar{R}/\bar{R}') = \frac{\pi \delta(R - R')}{2R^2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} (\hat{R} \times \bar{m}_{\substack{e\\o}mn}) (\hat{R}' \times \bar{m}'_{\substack{e\\o}mn})$$

then

$$\frac{\bar{\bar{I}}_t \delta(\bar{R} - \bar{R}') \delta(\theta - \theta') \delta(\phi - \phi')}{R^2 \sin\theta} = \bar{\bar{I}}_t \delta(\bar{R} - \bar{R}') = \bar{\bar{s}}_1(\bar{R}/\bar{R}') + \bar{\bar{s}}_2(\bar{R}/\bar{R}') \quad . \tag{11}$$

This shows $\overline{\overline{s}}_1(\overline{R}/\overline{R}')$ is only part of the singular angular dyadic function $\overline{\overline{I}}_t\,\delta(\overline{R}-\overline{R}')$. As a result of Eq. (11) an alternative expression for $\overline{\overline{G}}_{eo}(\overline{R}/\overline{R}')$, as given by Eq. (7), is

$$\overline{\overline{G}}_{eo}(\overline{R}/\overline{R}') = \overline{\overline{S}}(\overline{R}/\overline{R}') - \frac{1}{k^2}\left[\hat{R}\hat{R}\,\delta(\overline{R}-\overline{R}') + \overline{\overline{s}}_2(\overline{R}/\overline{R}')\right] .$$

For $\overline{R} \neq \overline{R}'$ only $\overline{\overline{S}}(\overline{R}/\overline{R}')$ remains in $\overline{\overline{G}}_{eo}(\overline{R}/\overline{R}')$.

IX. $\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}')$ Pertaining to a Perfectly Conducting Cone

The synthesis of $\overline{\overline{G}}_{e1}$ pertaining to a perfectly conducting cone is very similar to the one for a perfectly conducting wedge. As far as the structure of the functions is concerned the conical vector wave functions are similar to the spherical vector wave functions. We outlined below the key equations omitting the detailed derivations.

$$\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \int_0^\infty dK \sum_m \frac{(2-\delta_o)K^3}{\pi^2(K^2-k^2)} \cdot \left[\cdot \sum_\mu \frac{1}{\mu(\mu+1)I_\mu} \overline{M}_{\substack{e\\o}m\mu}(K)\,\overline{N}'_{\substack{e\\o}m\mu}(K) + \sum_\lambda \frac{1}{\lambda(\lambda+1)I_\lambda} \overline{N}_{\substack{e\\o}m\lambda}(K)\,\overline{M}'_{\substack{e\\o}m\lambda}(K)\right] \quad (1)$$

where

$$\overline{M}_{\substack{e\\o}m\lambda}(K) = \nabla \times \left[j_\lambda(KR)\,P_\lambda^m(\cos\theta)\,{\cos \atop \sin}\,m\phi\,\overline{R}\right]$$

$$\overline{N}_{\substack{e\\o}m\lambda}(K) = \frac{1}{K}\nabla \times \overline{M}_{\substack{e\\o}m\lambda}(K)$$

$$\overline{M}_{\substack{e\\o}m\mu}(K) = \nabla \times \left[j_\mu(KR)\,P_\mu^m(\cos\theta)\,{\cos \atop \sin}\,m\phi\,\overline{R}\right]$$

$$\overline{N}_{\substack{e\\o}m\mu}(K) = \frac{1}{K}\nabla \times \overline{M}_{\substack{e\\o}m\mu}(K)$$

$$I_\mu = \int_{\theta_o}^{\pi} \left[P_\mu^m(\cos\theta)\right]^2 \sin\theta\, d\theta$$

$$I_\lambda = \int_{\theta_o}^{\pi} \left[P_\lambda^m(\cos\theta)\right]^2 \sin\theta\, d\theta$$

$$P_\mu^m(\cos\theta_o) = 0, \quad \text{characteristic equation for } \mu$$

$$\frac{\partial P_\lambda^m(\cos\theta_o)}{\partial\theta_o} = 0, \quad \text{characteristic equation for } \lambda$$

$$\delta_o = \begin{Bmatrix} 1, & m = 0 \\ 0, & m \neq 0 \end{Bmatrix}$$

$$\nabla \times \overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \int_0^\infty dK \sum_m \frac{(2-\delta_o)K^4}{\pi^2(K^2-k^2)} \cdot$$

$$\left[\cdot \sum_\mu \frac{1}{\mu(\mu+1)I_\mu} \overline{N}_{\substack{e\\o}m\mu}(K)\overline{N}'_{\substack{e\\o}m\mu}(K) + \sum_\lambda \frac{1}{\lambda(\lambda+1)I_\lambda} \overline{M}_{\substack{e\\o}m\lambda}(K)\overline{M}'_{\substack{e\\o}m\lambda}(K)\right] =$$

$$= k^2 \overline{\overline{S}}(\overline{R}/\overline{R}') + \overline{\overline{s}}_1(\overline{R}/\overline{R}') \tag{2}$$

where

$$\bar{\bar{S}}(\bar{R}/\bar{R}') = \frac{ik}{2\pi}\sum_m (2-\delta_o)\cdot$$

$$\left[\cdot\sum_\lambda \frac{1}{\lambda(\lambda+1)I_\lambda}\begin{Bmatrix}\bar{M}^{(1)}_{\substack{e\\o}m\lambda}(k)\,\bar{M}'_{\substack{e\\o}m\lambda}(k)\\ \bar{M}_{\substack{e\\o}m\lambda}(k)\,\bar{M}'^{(1)}_{\substack{e\\o}m\lambda}(k)\end{Bmatrix} + \right.$$

$$\left. + \sum_\mu \frac{1}{\mu(\mu+1)I_\mu}\begin{Bmatrix}\bar{N}^{(1)}_{\substack{e\\o}m\mu}(k)\,\bar{N}'_{\substack{e\\o}m\mu}(k)\\ \bar{N}_{\substack{e\\o}m\mu}(k)\,\bar{N}'^{(1)}_{\substack{e\\o}m\mu}(k)\end{Bmatrix}\right], \quad R \gtrless R' \tag{3}$$

$$\bar{\bar{s}}_1(\bar{R}/\bar{R}') = \frac{\delta(R-R')}{2\pi R^2}\sum_m\sum_\lambda \frac{2-\delta_o}{\lambda(\lambda+1)I_\lambda}\bar{m}_{\substack{e\\o}m\lambda}\bar{m}'_{\substack{e\\o}m\lambda} \tag{4}$$

$$\bar{\bar{G}}_{e1}(\bar{R}/\bar{R}') = \frac{1}{k^2}\left[\nabla\times\bar{\bar{G}}_{m2}(\bar{R}/\bar{R}') - \bar{\bar{I}}\delta(\bar{R}-\bar{R}')\right]$$

$$= \bar{\bar{S}}(\bar{R}/\bar{R}') - \frac{1}{k^2}\left[\bar{\bar{I}}\delta(\bar{R}-\bar{R}') - \bar{\bar{s}}_1(\bar{R}/\bar{R}')\right]$$

$$= \bar{\bar{S}}(\bar{R}/\bar{R}') - \frac{1}{k^2}\left[\hat{R}\hat{R}\delta(\bar{R}-\bar{R}') + \bar{\bar{s}}_2(\bar{R}/\bar{R}')\right] \tag{5}$$

where

$$\bar{\bar{s}}_2(\bar{R}/\bar{R}') = \frac{\delta(R-R')}{2\pi R^2}\sum_m\sum_\mu \frac{2-\delta_o}{\mu(\mu+1)I_\mu}(\hat{R}\times\bar{m}_{\substack{e\\o}m\mu})(\hat{R}'\times\bar{m}'_{\substack{e\\o}m\mu}) \quad . \tag{6}$$

The residue series $\bar{\bar{S}}(\bar{R}/\bar{R}')$ is the same as the one given by Eq. (22), p. 191, of Reference [1].

X. Rectangular Waveguide with a Moving Isotropic Medium

The basic equations which govern the field in a waveguide with moving isotropic medium are:

$$\nabla \times \left[\overline{\overline{b}} \cdot \overline{E}^{(b)} \right] = i\omega\mu \overline{H}^{(b)} \tag{1}$$

$$\nabla \times \left[\overline{\overline{b}} \cdot \overline{H}^{(b)} \right] = \overline{J} e^{i\omega\Omega z} - i\omega\epsilon \overline{E}^{(b)} \; . \tag{2}$$

The actual fields are related to the auxiliary fields $\overline{E}^{(b)}$ and $\overline{H}^{(b)}$ by

$$\overline{E} = e^{-i\omega\Omega z} \overline{\overline{b}} \cdot \overline{E}^{(b)} \tag{3}$$

$$\overline{H} = e^{-i\omega\Omega z} \overline{\overline{b}} \cdot \overline{H}^{(b)} \tag{4}$$

where

$$\overline{\overline{b}} = \frac{1}{a} (\hat{x}\hat{x} + \hat{y}\hat{y}) + \hat{z}\hat{z}$$

$$a = \frac{1-\beta^2}{1-n^2\beta^2}$$

$$n = \left(\frac{\mu\epsilon}{\mu_o \epsilon_o} \right)^{1/2} , \qquad \beta = \frac{v}{c}$$

$\overline{v} = v\hat{z} =$ velocity of the moving medium

$c =$ velocity of light

$$\Omega = \frac{(n^2-1)\, v}{(1-n^2\beta^2)\, c^2} \; .$$

To integrate Eqs. (1) and (2) we introduce the dyadic Green's functions $\overline{\overline{G}}_{m2}$ and $\overline{\overline{G}}_{e1}$ defined by

$$\nabla \times \left[\overline{\overline{b}} \cdot \overline{\overline{G}}_{e1} \right] = \overline{\overline{G}}_{m2} \tag{5}$$

$$\nabla \times \left[\bar{\bar{b}} \cdot \bar{\bar{G}}_{m2} \right] = \bar{\bar{I}} \delta(\bar{R} - \bar{R}') + k^2 \bar{\bar{G}}_{e1} \quad . \tag{6}$$

By eliminating $\bar{\bar{G}}_{e1}$ between Eqs. (5) and (6) we find that the function $\bar{\bar{G}}_{m2}$ satisfies the differential equation

$$\nabla \times \left[\bar{\bar{b}} \cdot \nabla \times (\bar{\bar{b}} \cdot \bar{\bar{G}}_{m2}) \right] - k^2 \bar{\bar{G}}_{m2} = \nabla \times \left[\bar{\bar{b}} \delta(\bar{R} - \bar{R}') \right] . \tag{7}$$

The solenoidal vector wave functions to be used for expanding $\bar{\bar{G}}_{m2}$ are:

$$\bar{M}_{\substack{e \\ o}mn}(h) = \nabla \times \left[\psi_{\substack{e \\ o}mn}(h) \hat{z} \right] \tag{8}$$

$$\begin{aligned} \bar{N}_{\substack{e \\ o}mn}(h) &= \frac{1}{K} \nabla \times \left[\bar{\bar{b}} \cdot \bar{M}_{\substack{e \\ o}mn}(h) \right] \\ &= \frac{1}{Ka} \nabla \times \nabla \times \left[\psi_{\substack{e \\ o}mn}(h) \hat{z} \right] \end{aligned} \tag{9}$$

where

$$\psi_{\substack{e \\ o}mn}(h) = \begin{Bmatrix} \cos \dfrac{m\pi}{x_o} x \cos \dfrac{n\pi}{y_o} y \\ \\ \sin \dfrac{m\pi}{x_o} x \sin \dfrac{n\pi}{y_o} y \end{Bmatrix} e^{ihz}$$

$$K^2 a^2 = h^2 + a k_c^2$$

$$k_c^2 = \left(\frac{m\pi}{x_o} \right)^2 + \left(\frac{n\pi}{y_o} \right)^2$$

(x_o, y_o) denote the lateral dimensions of the rectangular waveguide. The scalar functions $\psi_{\substack{e \\ o}mn}$ are solutions to the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{a} \frac{\partial^2}{\partial z^2} + K^2 a \right) \psi = 0$$

and the vector wave functions are solutions to the equation

$$\nabla \times \left[\bar{\bar{b}} \cdot \nabla \times (\bar{\bar{b}} \cdot \bar{F}) \right] - K^2 \bar{F} = 0 \quad .$$

The vector wave functions have the orthogonal property that

$$\iiint \bar{M}_{\substack{e\\o}mn}(h) \cdot \bar{N}_{\substack{e\\o}m'n'}(-h')\, dv = 0$$

$$\iiint \bar{M}_{\substack{e\\o}mn}(h) \cdot \bar{M}_{\substack{e\\o}m'n'}(-h')\, dv = \begin{cases} 0, & m \neq m',\ n \neq n', \\ \frac{1}{2}(1+\delta_o)\pi k_c^2 x_o y_o \delta(h-h'), & m = m',\ n = n'. \end{cases}$$

$$\iiint \bar{N}_{\substack{e\\o}mn}(h) \cdot \bar{N}_{\substack{e\\o}m'n'}(-h')\, dv = \begin{cases} 0, & m \neq m',\ n \neq n', \\ \frac{1}{2}(1+\delta_o)\pi k_c^2 x_o y_o \dfrac{h^2 + k_c^2}{h^2 + a k_c^2}\, \delta(h-h'), & \\ & m = m',\ n = n'. \end{cases}$$

They also share the symmetrical property

$$\nabla \times \left[\bar{\bar{b}} \cdot \bar{M}_{\substack{e\\o}mn} \right] = K \bar{N}_{\substack{e\\o}mn}$$

$$\nabla \times \left[\bar{\bar{b}} \cdot \bar{N}_{\substack{e\\o}mn} \right] = K \bar{M}_{\substack{e\\o}mn} \quad .$$

These properties have been discussed in detail previously in Sec. 48 of Reference [1]. We repeat here for completeness mainly because the eigen-function expansion of $\bar{\bar{G}}_{e1}$ discussed in that reference was not properly done and the result was wrong even for $\bar{R} \neq \bar{R}'$. A more thorough reformulation is therefore necessary.

To find the expansion of $\bar{\bar{G}}_{m2}$ we start with Eq. (7) and let

$$\nabla \times \left[\bar{\bar{b}}\, \delta(\bar{R}-\bar{R}') \right] = \int_{-\infty}^{\infty} dh \sum_{m,n} \left[\bar{M}_{omn}(h)\, \bar{A}_{omn}(h) + \bar{N}_{emn}(h)\, \bar{B}_{emn}(h) \right] \qquad (10)$$

where $\bar{A}_{omn}$ and $\bar{B}_{emn}$ are two sets of unknown coefficients to be determined. By taking the scalar product of Eq. (10) with $\bar{M}_{om'n'}(-h')$ and integrating through the entire volume of the guide we find, as a result of the orthogonal property of the vector functions,

$$\frac{1}{2}(1+\delta_o)\pi k_c'^2 x_o y_o \bar{A}_{om'n'}(h') = \iiint \bar{M}_{om'n'}(-h') \cdot \nabla \times \left[\bar{\bar{b}}\ \delta(\bar{R}-\bar{R}') \right] dv$$

$$= \iiint \nabla \cdot \left[\bar{\bar{b}}\, \delta(\bar{R}-\bar{R}') \times \bar{M}_{om'n'}(-h') \right] dv$$

$$+ \iiint \bar{\bar{b}}\, \delta(\bar{R}-\bar{R}') \cdot \nabla \times \bar{M}_{om'n'}(-h')\, dv$$

$$= \bar{\bar{b}} \cdot \nabla' \times M'_{om'n'}(-h')$$

hence

$$\bar{A}_{omn}(h) = \frac{(2-\delta_o)}{\pi k_c^2 x_o y_o}\, \bar{\bar{b}} \cdot \nabla' \times \bar{M}'_{omn}(-h)$$

$$= \frac{(2-\delta_o) K a}{\pi k_c^2 x_o y_o}\, \bar{\bar{b}} \cdot \bar{N}'_{omn}(-h) \quad . \qquad (11)$$

By taking the scalar product of Eq. (10) with $\bar{N}_{em'n'}(-h')$ and integrating through the entire volume of the guide we obtain

$$\frac{1}{2}(1+\delta_o)\,\pi k'^2_c x_o y_o \frac{h'^2+k_c^2}{h'^2+ak_c^2}\overline{B}_{em'n'}(h') =$$

$$= \iiint \overline{N}_{em'n'}(-h') \cdot \nabla x \left[\overline{\overline{b}}\,\delta(\overline{R}-\overline{R}')\right] dv$$

$$= \overline{\overline{b}} \cdot \nabla' x \overline{N}'_{em'n'}(-h')$$

thus,

$$\overline{B}_{emn}(h) = \frac{(2-\delta_o)}{\pi k_c^2 x_o y_o} \frac{h^2+ak_c^2}{h^2+k_c^2} \overline{\overline{b}} \cdot \nabla' x \overline{N}'_{emn}(-h) \qquad (12)$$

now

$$\nabla' x \overline{N}'_{emn}(-h) = \frac{h^2+k_c^2}{Ka} \overline{M}'_{emn}(-h)$$

and

$$h^2 + ak_c^2 = K^2 a^2$$

hence

$$\overline{B}_{emn}(h) = \frac{(2-\delta_o)\,Ka}{\pi k_c^2 x_o y_o} \overline{\overline{b}} \cdot \overline{M}'_{emn}(-h) \quad . \qquad (13)$$

The expansion of $\nabla x \left[\overline{\overline{b}}\,\delta(\overline{R}-\overline{R}')\right]$ is therefore found to be

$$\nabla x \left[\overline{\overline{b}}\,\delta(\overline{R}-\overline{R}')\right] = \int_{-\infty}^{\infty} dh \sum_{m,n} C_{mn} Ka \left[\overline{M}_{omn}(h)\overline{\overline{b}} \cdot \overline{N}'_{omn}(-h) + \right.$$

$$\left. + \overline{N}_{emn}(h)\,\overline{\overline{b}} \cdot \overline{M}'_{emn}(-h)\right] \qquad (14)$$

where

$$C_{mn} = \frac{2-\delta_o}{\pi k_c^2 x_o y_o} \quad .$$

In view of Eq. (7), $\overline{\overline{G}}_{m2}$ has the following representation

$$\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} dh \sum_{m,n} \frac{C_{mn} Ka}{K^2 - k^2} \left[\overline{M}_{omn}(h) \overline{\overline{b}} \cdot \overline{N}'_{omn}(-h) + \right.$$

$$\left. + \overline{N}_{emn}(h) \overline{\overline{b}} \cdot \overline{M}'_{emn}(-h) \right] \tag{15}$$

hence

$$\nabla x \left[\overline{\overline{b}} \cdot \overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') \right] = \int_{-\infty}^{\infty} dh \sum_{m,n} \frac{C_{mn} K^2 a}{K^2 - k^2} \left[\overline{N}_{omn}(h) \overline{\overline{b}} \cdot \overline{N}'_{omn}(-h) + \right.$$

$$\left. + \overline{M}_{emn}(h) \overline{\overline{b}} \cdot \overline{M}'_{emn}(-h) \right] . \tag{16}$$

In the integral on the right side of Eq. (16), there is a singular term which must be removed before we can apply the method of contour integration in simplifying Eq. (16). It can be shown that using $\overline{M}_e(h)$ and $\overline{N}_{ot}(h)$ as two orthogonal sets of functions we have the following expansion theorem

$$\overline{\overline{I}}_t \delta(\overline{R} - \overline{R}') = \int_{-\infty}^{\infty} dh \sum_{m,n} C_{mn} a \left[\frac{K^2 a^2}{h^2} \overline{N}_{ot} \overline{\overline{b}} \cdot \overline{N}'_{ot} + \overline{M}_e \overline{\overline{b}} \cdot \overline{M}'_e \right] \tag{17}$$

where $\overline{N}_{ot}$ denotes the transversal part of $\overline{N}_{omn}(h)$ and $\overline{M}_e$ is an abbreviation for $\overline{M}_{emn}(h)$. The integral in Eq. (16) can now be split into two parts

$$\nabla x \left[\overline{\overline{b}} \cdot \overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') \right] = \int_{-\infty}^{\infty} dh \sum_{m,n} C_{mn} a \left[\frac{K^2 a^2}{h^2} \overline{N}_{ot} \overline{\overline{b}} \cdot \overline{N}'_{ot} + \overline{M}_e \overline{\overline{b}} \cdot \overline{M}'_e \right] +$$

$$+ \int_{-\infty}^{\infty} dh \sum_{m,n} C_{mn} a \left[\frac{K^2 a (k^2 a - k_c^2)}{h^2 (K^2 - k^2)} \right] \overline{N}_{ot} \overline{\overline{b}} \cdot \overline{N}'_{ot} +$$

$$+\frac{K^2}{K^2-k^2}(\overline{N}_{oz}\overline{\overline{b}}\cdot\overline{N}'_{ot}+\overline{N}_{ot}\overline{\overline{b}}\cdot\overline{N}'_{oz}+\overline{N}_{oz}\overline{\overline{b}}\cdot\overline{N}'_{oz})+\frac{k^2}{K^2-k^2}\overline{M}_e\overline{\overline{b}}\cdot\overline{M}'_e\Bigg] \quad . \tag{18}$$

The first integral contained in Eq. (18) represents $\overline{\overline{I}}_t\delta(\overline{R}-\overline{R}')$ and the second integral can be evaluated in closed form by means of the residue theorem which yields

$$\nabla x\left[\overline{\overline{b}}\cdot\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}')\right]=\overline{\overline{I}}_t\delta(\overline{R}-\overline{R}')+k^2\overline{\overline{S}}(\overline{R}/\overline{R}') \tag{19}$$

where

$$\overline{\overline{S}}(\overline{R}/\overline{R}')=\sum_{m,n}\frac{i\pi a^3}{k_g}C_{mn}\Big[\overline{N}_{omn}(\pm k_g)\overline{\overline{b}}\cdot\overline{N}'_{omn}(\mp k_g)+$$

$$+\overline{M}_{emn}(\pm k_g)\overline{\overline{b}}\cdot\overline{M}'_{emn}(\mp k_g)\Big] \qquad z\gtrless z' \tag{20}$$

with

$$k_g=(k^2a^2-k_c^2a)^{1/2}\quad,\quad C_{mn}=\frac{2-\delta_o}{\pi k_c^2 x_o y_o}\quad,$$

$$\delta_o=\begin{cases}1, & m \text{ or } n=0\\ 0, & m\neq 0,\ n\neq 0.\end{cases}$$

In view of Eq. (6) we obtain

$$\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}')=\overline{\overline{S}}(\overline{R}/\overline{R}')-\frac{1}{k^2}\hat{z}\hat{z}\delta(\overline{R}-\overline{R}') \quad . \tag{21}$$

It should be pointed out that the residue series given by Eq. (14), p. 219, of Reference [1] is not the same as Eq. (20). The error committed therein affects not only the missing singular term $-\hat{z}\hat{z}\delta(\overline{R}-\overline{R}')/k^2$ but also the residue series.

XI. Conclusion and Acknowledgment

We have amended in this note a serious mistake committed in Reference [1]. The complete expressions for various dyadic Green's functions have now been derived based on the method of $\bar{\bar{G}}_m$ which bypassed the use of nonsolenoidal vector wave functions.

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