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A METHOD FOR FITTING EMP WAVEFORMS THAT FACILITATES CALCULATION OF THE TIME DERIVATIVE AND FOURIER TRANSFORM

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20. ABSTRACT (Continued)

written in terms of a summation of the discontinuities in the third and fourth derivatives is then derived. Difficulties in evaluating this expression due to round-off error at low frequencies are discussed and sample results are included.

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INTRODUCTION

The output of many EMP phenomenology codes consists of the magnitude of various field components plotted as a function of time. The time scale may typically run over several decades from times of a few nanoseconds to several tens of microseconds. For very early times the waveforms closely resemble a rising exponential while at late times the pulse falls off like a decaying exponential. The exact waveform shape between these two extremes is very important in various vulnerability assessments and it is desirable to have some simple means of describing the waveform in this intermediate time region. It is particularly desirable to have a description that simplifies calculation of the time derivative and Fourier transform of the waveform. It is sometimes difficult to do these calculations directly from waveforms generated by EMP codes because of various small oscillations in the code output due to numerical techniques rather than the actual physics of EMP generation. Thus one desires a method for approximating the output waveform of an EMP code with a relatively simple function which smooths out small numerical variations but is easily differentiated and Fourier transformed. This report describes one method of finding such a function.

The curve fitting technique described here starts with a small number of data points chosen from the output waveform of some EMP generation code. The points should be judiciously chosen to represent a fairly smooth waveform without the numerical oscillations that the code may have generated. In the time interval before the first data point the function is assumed to behave like $e^{\alpha t}$ and after the last data point the curve decays like $e^{-\beta t}$. Between the various data points the waveform is approximated by a polynomial. The polynomial is chosen so that the function passes through the data points, and is continuous. In addition, the first and second derivatives of the function are required to be continuous. Thus we approximate the waveform by an analytic function with coefficients depending on the data points chosen. Similarly, one has an analytic expression for the time derivative of the waveform. It will also be shown that the Fourier transform can be written as a simple summation rather than an integration which must be calculated numerically.

Thus it can be seen that this curve fitting technique gives a simple means of utilizing EMP code output for further calculations without having a complete description of the waveform generated by the code. The choice of a limited number of data points also serves to eliminate spurious numerical oscillations in the code's output waveforms.

SECTION 2

CURVE FITTING TECHNIQUE

FITTING THEORY

Assume we are given some set of data points f_n for a number of times t_n. Let there be a total of N such data points with f_1 being the first point in time. These data points give the general shape of some smooth function of time which we wish to fit for times varying over several decades. We further assume that for $t < t_1$ the function is approximately an exponential which varies as $e^{\alpha t}$; also for $t > t_N$ the assumed dependence varies as $e^{-\beta t}$. Any of a large class of functions which start out and end up very small can be described in this way, with the number of data points required depending on how rapidly the function varies with time.

At any time t in the n^{th} time interval [i.e., $t_n \leq t \leq t_{n+1}$] we fit the curve with the function

$$\begin{aligned} f(t) = {} & [f_{n+1}(t - t_n) + f_n(t_{n+1} - t)]/(t_{n+1} - t_n) \\ & + \frac{1}{2}(B_n + B_{n+1})(t - t_n)(t - t_{n+1}) \\ & + C_n (t - t_n)(t_{n+1} - t)^3 \\ & + D_{n+1} (t_{n+1} - t)(t - t_n)^3 \end{aligned} \tag{1}$$

where f_n and t_n are the given values of f and t respectively at the n^{th} data point and the meaning of the various coefficients will be discussed in the following paragraphs.

The first term in Equation 1 is just a linear fit between the n^{th} and $(n + 1)^{th}$ data points. The second term is the quadratic correction based on an "average" curvature. The curvature, B_n, is given by

$$B_n = \left[\frac{f_{n+1} - f_n}{t_{n+1} - t_n} - \frac{f_n - f_{n-1}}{t_n - t_{n-1}}\right]\left[\frac{1}{t_{n+1} - t_n}\right] \qquad (2)$$

for $2 \le n \le N - 1$.

In the first and last intervals the exponential functions are used to calculate the curvature, giving

$$B_1 = \left[\frac{f_2 - f_1}{t_2 - t_1} - \alpha f_1\right]\left[\frac{1}{t_2 - t_1}\right] \qquad (3)$$

$$B_N = \left[- \beta f_N - \frac{f_n - f_{N-1}}{t_N - t_{N-1}}\right]\left[\frac{1}{t_N - t_{N-1}}\right] \qquad (4)$$

The third and fourth terms in Equation 1 are used to insure that the first and second derivatives of the fitting function are continuous at the data points. The third term has finite first, second, third, and fourth derivatives at t_n but these derivatives are zero when evaluated at t_{n+1}. Thus the coefficient C_n is chosen to match the first and second derivatives of f(t) at the n^{th} data point with the first and second derivatives of f(t) evaluated in the previous interval. Similarly, the coefficient D_{n+1} is used to insure continuous first and second derivatives at the $(n + 1)^{th}$ data point. Matching the first and second derivatives at the n^{th} data point gives two simultaneous equations relating C_n and D_n which can be solved for $2 \le n \le N - 1$ to give

$$C_n = \frac{-\Delta f_n^{[1]} + \Delta f_n^{[2]} (t_n - t_{n-1})/6}{(t_{n+1} - t_n)^2 (t_{n+1} - t_{n-1})} \qquad (5)$$

$$D_n = \frac{-\Delta f_n^{[1]} - \Delta f_n^{[2]}(t_n - t_{n-1})/6}{(t_n - t_{n-1})^2(t_{n+1} - t_{n-1})} \tag{6}$$

where

$$\Delta f_n^{[1]} = \frac{f_{n+1} - f_n}{t_{n+1} - t_n} - \frac{f_n - f_{n-1}}{t_n - t_{n-1}}$$

$$+ \frac{1}{2}(B_n + B_{n+1})(t_{n+1} - t_n) - \frac{1}{2}(B_n + B_{n-1})(t_n - t_{n-1}) \tag{7}$$

and

$$\Delta f_n^{[2]} = B_{n+1} - B_{n-1} \tag{8}$$

Note that these expressions couple the value of f(t) in an interval to the next two data points on either side of the interval; i.e., f(t) in the n^{th} interval depends on the n - 2, n - 1, n + 1, and n + 2 data points.

Due to this coupling one must give special attention as to how to calculate the various coefficients near the first and last data points. B_1 and B_N have already been calculated using the curvature of the exponentials added at both ends of the data interval. With these values one can evaluate all the required coefficients except C_1 and D_N. [Note that the coefficients C_1 and D_N will be chosen so as to help match the first and second derivatives of the function at the first and last data points. To match both of these derivatives one requires an additional correction term to be added to the exponentials. Thus we write:

$$f(t) = A_1e^{\alpha t} + A_3e^{2\alpha t} \quad \text{for} \quad t \le t_1 \tag{9}$$

$$f(t) = A_2e^{-\beta t} + A_4e^{-2\beta t} \quad \text{for} \quad t \ge t_N \; . \tag{10}$$

The first term in each of the above equations gives the basic exponential behavior of the function while the A_3 and A_4 terms are used to match the first and second derivatives at the first and last data points. In general, one requires that the A_3 term be small compared to the A_1 term and the A_4 term be small with respect to the A_2 term so that the function is closely approximated by a single exponential.

To obtain continuous first and second derivatives at t_1 one has two simultaneous equations relating A_3 and C_1. Similarly, at t_N one obtains two equations that can be solved for A_4 and D_N. Thus one finds that

$$A_3 = \frac{k_1 + k_2 (t_2 - t_1)/6}{[\alpha^2 (t_2 - t_1)/2 + \alpha] e^{2\alpha t_1}} \tag{11}$$

$$C_1 = \frac{k_2 + 3\alpha^2 \left[\dfrac{k_1 + k_2 (t_2 - t_1)/6}{\alpha^2 (t_2 - t_1)/2 + \alpha}\right]}{6(t_2 - t_1)^2} \tag{12}$$

where

$$k_1 = \frac{f_2 - f_1}{t_2 - t_1} + \frac{1}{2} (B_1 + B_2)(t_1 - t_2) - \alpha f_1 \tag{13}$$

$$k_2 = B_1 + B_2 - \alpha^2 f_1 \tag{14}$$

and

$$A_4 = \frac{k_3 - k_4 (t_N - t_{N-1})/6}{[\beta + \beta^2 (t_N - t_{N-1})/2] e^{-2\beta t_N}} \tag{15}$$

$$D_N = \frac{k_4 + 3\beta^2 \left[\dfrac{k_3 - k_4 (t_N - t_{N-1})/6}{\beta + \beta^2 (t_N - t_{N-1})/2}\right]}{6(t_N - t_{N-1})^2} \tag{16}$$

where

$$k_3 = -\frac{f_N - f_{N-1}}{t_N - t_{N-1}} - \frac{1}{2} (B_N + B_{N-1})(t_N - t_{N-1}) - \beta f_N \tag{17}$$

$$k_4 = \beta^2 f_N - (B_N + B_{N-1}) \ . \tag{18}$$

Then from Equations 9 and 10 and the requirement that the function pass through the data points, we have

$$A_1 = \left[f_1 - A_3 e^{2\alpha t_1}\right] e^{-\alpha t_1} \tag{19}$$

$$A_3 = \left[f_N - A_4 e^{-2\beta t_N}\right] e^{\beta t_N} \ . \tag{20}$$

Thus we have found the necessary coefficients to evaluate f(t) at any given time t. The function f(t) is continuous, passes through the data points, and has continuous first and second derivatives. The third and fourth derivatives, however, are discontinuous at the data points while the fifth and higher derivatives of f(t) are zero for $t_1 < t < t_N$.

DISCUSSION OF RESULTS

One should note that the exact shape of the function f(t) depends upon both the location and number of data points chosen. It appears that one can fit a typical EMP waveform over four or five decades in time with only twenty or so data points (see Figure 1). The spacing of the points is somewhat arbitrary with the obvious guideline that the density of the data points should be greatest where the function being fitted varies most rapidly. The effect of having too few data points is shown in Figure 2. Note the local minima between data points right after the peak of the waveform. Figure 2 is supposedly a fit of the same function shown in Figure 1. The smooth curve in Figure 1 was achieved by slightly moving several of the points in Figure 2 and adding an additional data point where the slope of the function is large.

The importance of obtaining a smooth curve as shown in Figure 1 is related to the desire of finding the time derivative of the waveform.

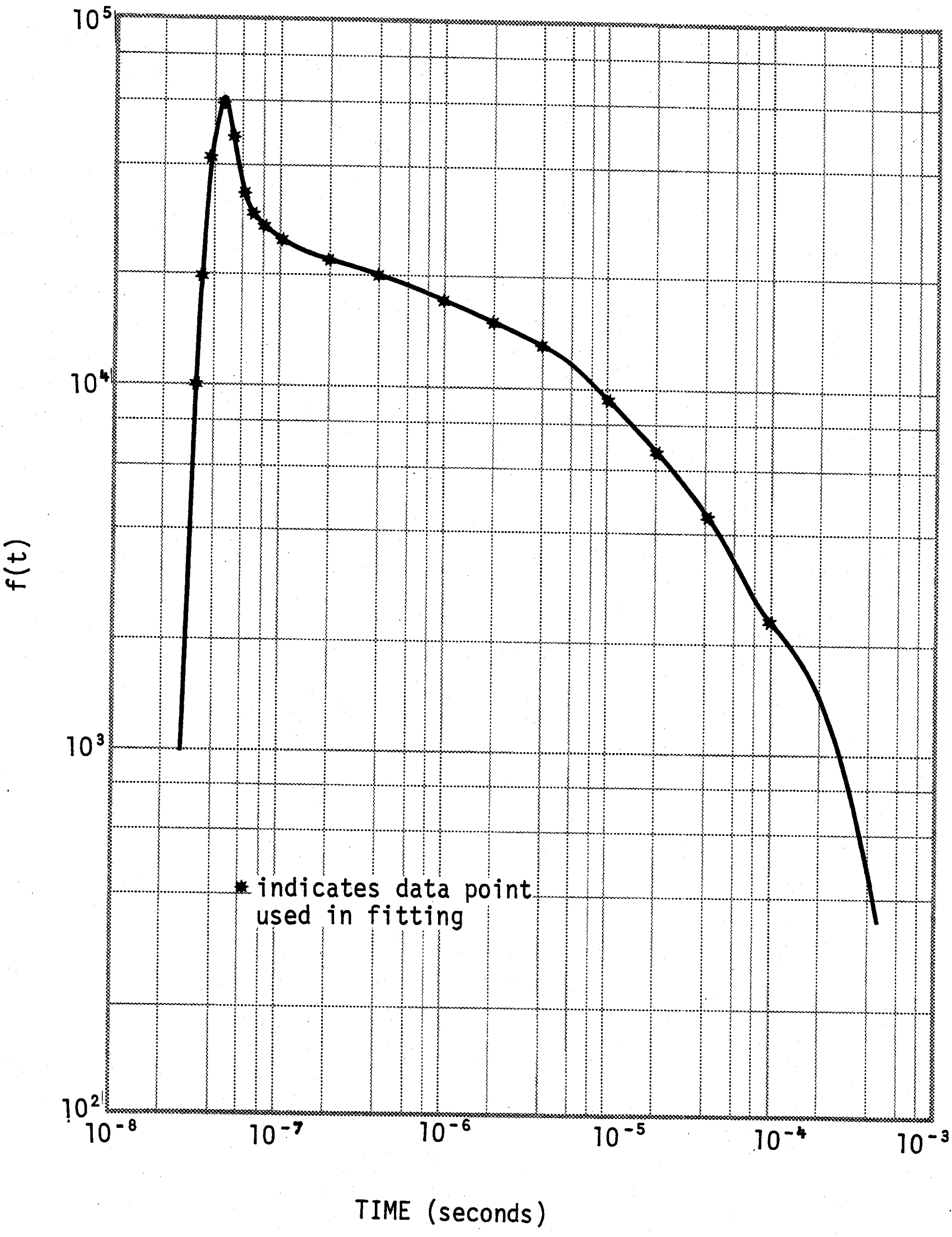

Figure 1. Example of resulting waveform from curve-fitting technique.

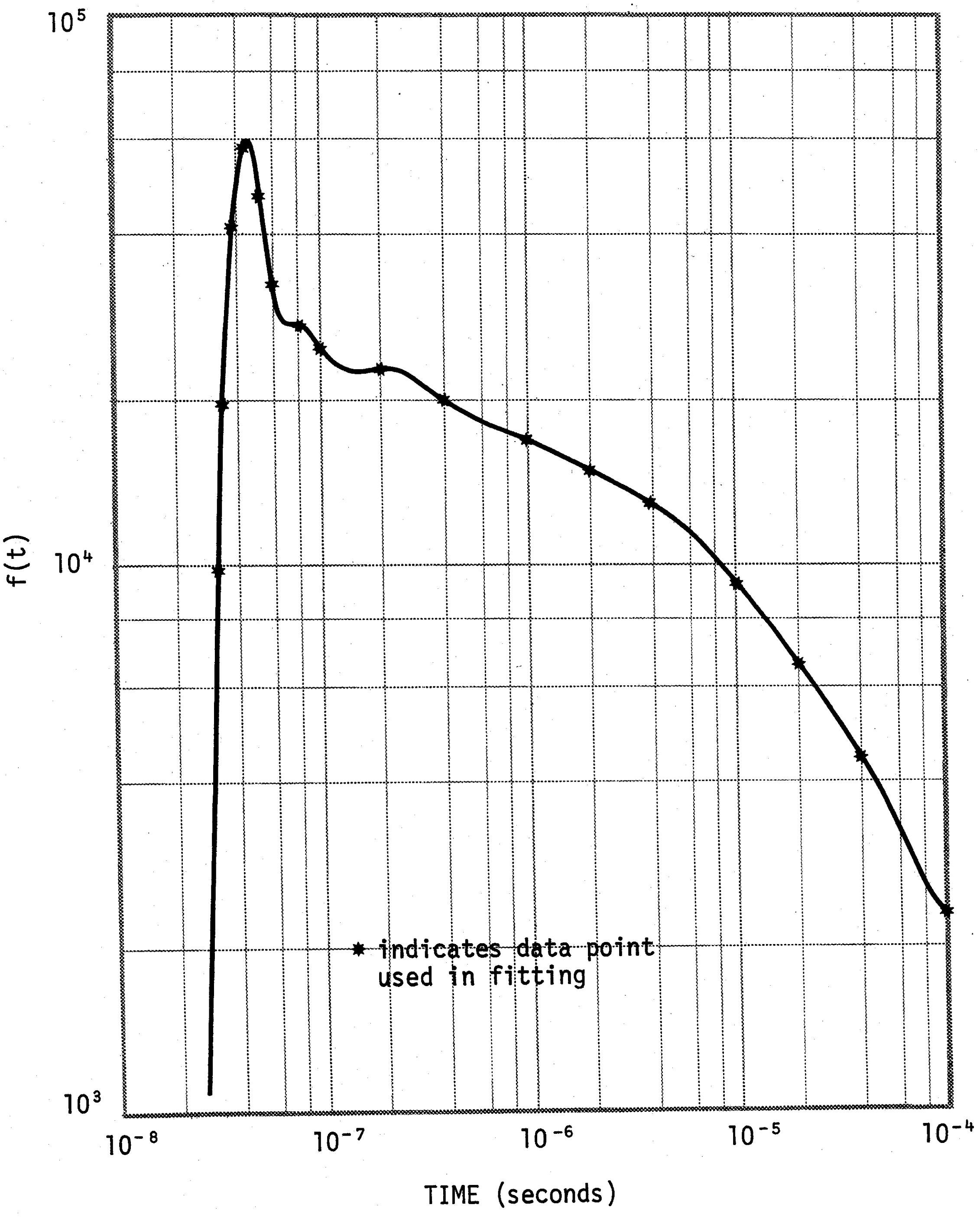

Figure 2. Example of local extrema in the fitting function due to choice of data point location.

Local extrema such as seen in Figure 2 will result in large changes in the derivative of the function unrelated to the real physics of the problem.

One should also keep in mind the source of the data points used. This fitting method was developed to smooth the numerical oscillations in the output waveforms calculated by various EMP codes. The data points used in fitting these waveforms are chosen by simply drawing a smooth curve through a numerically generated graph and picking as few points as pos$ible to describe the general shape of the waveform. Thus the locations of the data points are not highly accurate in any case and slightly moving the points to obtain a smooth fit probably gives far less error than the initial reading of the points did.

SECTION 3

CALCULATING THE TIME DERIVATIVE

THEORY

From Equations 1, 9, and 10 it is simple to write down an analytic expression for the first derivative of f(t).

$$\frac{df(t)}{dt} = \frac{f(t)}{\alpha} + \frac{A_3 e^{2\alpha t}}{\alpha} \quad \text{for} \quad t \leq t_1 \tag{21a}$$

$$= \frac{f_{n+1} - f_n}{t_{n+1} - t_n} + \frac{1}{2}(B_n + B_{n+1})[(t - t_{n+1}) + (t - t_n)]$$

$$+ C_n [(t_{n+1} - t)^3 - 3(t - t_n)(t_{n+1} - t)^2]$$

$$+ D_{n+1} [-(t - t_n)^3 + 3(t_{n+1} - t)(t - t_n)^2]$$

$$\text{for } t_n \leq t \leq t_{n+1} \tag{21b}$$

$$= -\frac{f(t)}{\beta} - \frac{A_4 e^{-2\beta t}}{\beta} \quad \text{for} \quad t \geq t_N \, . \tag{21c}$$

From the nature of f(t) it is obvious that df/dt is continuous and has a continuous first derivative.

DISCUSSION OF RESULTS

As an example, the time derivative of the function shown in Figure 1 is plotted in Figure 3. One should note that the details of the derivative waveform are probably much more sensitive to data point location and spacing than the fit function itself is. For instance, the derivative of the function shown in Figure 2 varies considerably from that of the function in Figure 1 especially near the local extrema.

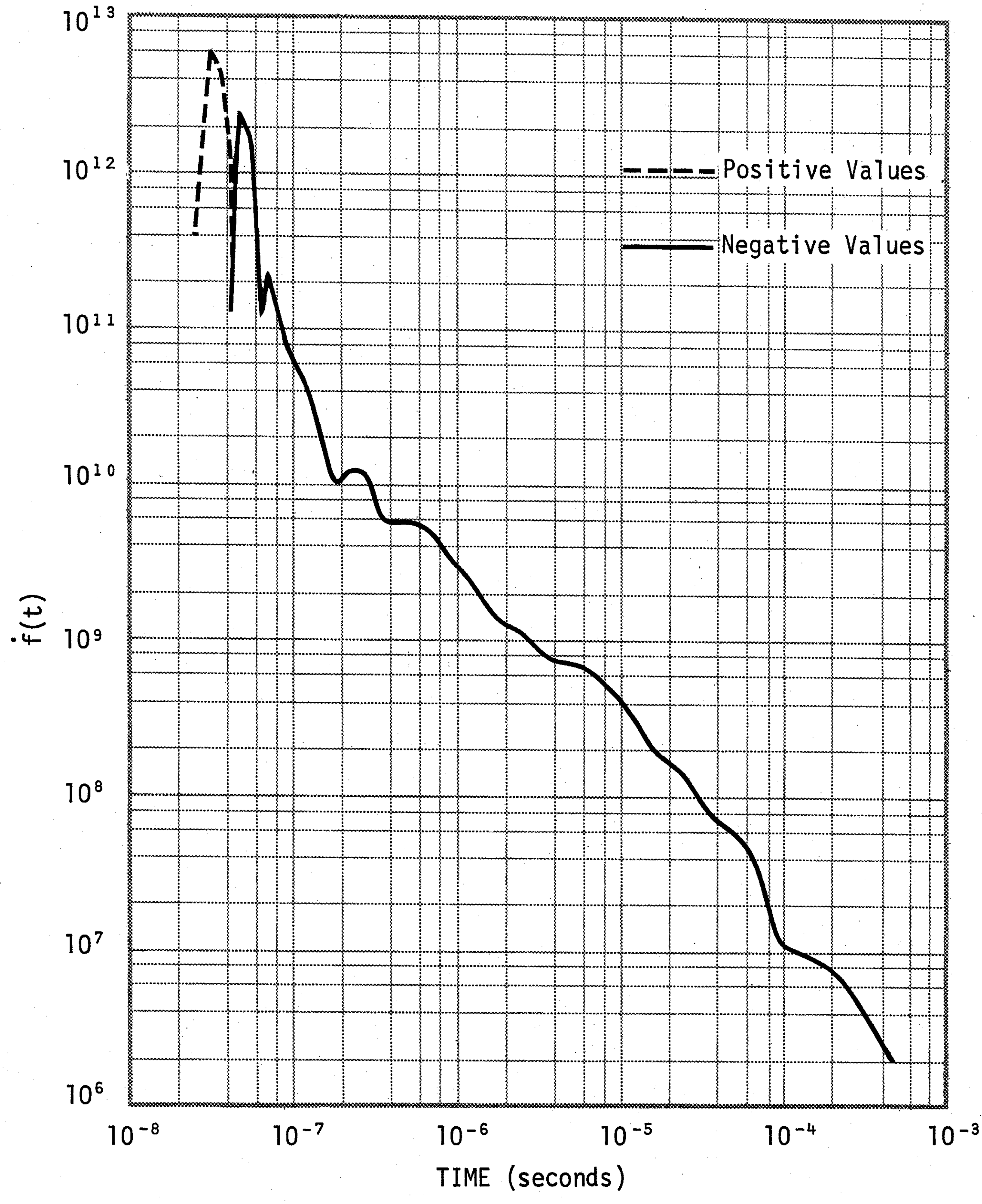

Figure 3. Time derivative of the waveform shown in Figure 1.

SECTION 4

THE FOURIER TRANSFORM

THEORY

Once we have the function f(t) it is very useful to have a simple means of calculating the function's Fourier transform, F(ω). Knowing the Fourier transform of an EMP waveform greatly simplifies many vulnerability assessment calculations. If the transfer function of a system can be calculated, one merely has to multiply the Fourier transform of the incident EMP by the transfer function to obtain the resulting voltage or current waveform at some critical circuit element. In addition, the response of certain systems to an incident electromagnetic field may be highly frequency dependent, making an understanding of the frequency content of the incident pulse very important.

By definition, the Fourier transform of f(t) is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt \; . \qquad (22)$$

It is relatively easy to evaluate this integral over the exponential parts of f(t) with the result that

$$F(\omega) = \frac{A_1 e^{(\alpha+i\omega)t_1}}{\alpha+i\omega} + \frac{A_3 e^{(2\alpha+i\omega)t_1}}{2\alpha+i\omega} - \frac{A_2 e^{(-\beta+i\omega)t_N}}{-\beta+i\omega} - \frac{A_4 e^{(-2\beta+i\omega)t_N}}{-2\beta+i\omega} + \int_{t_1}^{t_N} f(t) e^{i\omega t} dt \; . \qquad (23)$$

For the present, let us consider only the integral

$$I_1 \equiv \int_{t_1}^{t_N} f(t) e^{i\omega t} dt \; . \tag{24}$$

Integrating by parts several times, one obtains

$$I_1 = \frac{1}{i\omega}\left(e^{i\omega t_N} f_N - e^{i\omega t_1} f_1\right) + \frac{1}{\omega^2}\left(e^{i\omega t_N} f_N^{[1]} - e^{i\omega t_1} f_1^{[1]}\right)$$

$$+ \frac{i}{\omega^3}\left(e^{i\omega t_N} f_N^{[2]} - e^{i\omega t} \; f_1^{[2]}\right)$$

$$+ \left(\frac{i}{\omega}\right)^3 \int_{t_1}^{t_N} f^{[3]}(t) \; e^{i\omega t} dt \tag{25}$$

where $f^{[i]}(t)$ indicates the i^{th} time derivative of $f(t)$.

The reason for repeatedly integrating by parts is related to the fact that $f^{[5]}(t)$ and higher derivatives are zero over the range of integration. One must be careful, however, to remember that the third and fourth derivatives are not continuous. Consider only the integral in the last term of the previous equation,

$$I_2 \equiv \int_{t_1}^{t_N} f^{[3]}(t) \; e^{i\omega t} dt \; . \tag{26}$$

This integral can be written as

$$I_2 = \sum_{n=1}^{N-1} \left[\int_{t_n}^{t_{n+1}} f^{[3]}(t) \; e^{i\omega t} dt \right] \tag{27}$$

where now the integrand of each integral in the sum is continuous over the range of integration. One can integrate by parts over each interval, giving

$$I_2 = \sum_{n=1}^{N-1} \left[\frac{e^{i\omega t}}{i\omega} f^{[3]}(t) \Big|_{t_n}^{t_{n+1}} + \frac{e^{i\omega t}}{\omega^2} f^{[4]}(t) \Big|_{t_n}^{t_{n+1}} + \left(\frac{i}{\omega}\right)^2 \int_{t_n}^{t_{n+1}} f^{[5]}(t) e^{i\omega t} dt \right] \quad (28)$$

However, $f^{[5]}(t) = 0$ over each interval in the summation. Thus the integral in the above equation can be dropped and the Fourier transform integral has become just a summation.

Note that the previous expression for I_2 may be somewhat confusing to evaluate since $f^{[3]}(t)$ and $f^{[4]}(t)$ have different limits just to the left of a data point from those just to the right. It is therefore more convenient to express the summation in terms of the discontinuity in $f^{[3]}(t)$ and $f^{[4]}(t)$ at each data point. One obtains

$$I_2 = \sum_{n=1}^{N} \left[\frac{e^{i\omega t_n}}{i\omega} \Delta f_n^{[3]} + \frac{e^{i\omega t_n}}{\omega^2} \Delta f_n^{[4]} \right] \quad (29)$$

where

$$\Delta f_1^{[3]} = -[18C_1 + 6D_2](t_2 - t_1) \quad (30)$$

$$\Delta f_1^{[4]} = 24[C_1 + D_2] \quad (31)$$

$$\Delta f_N^{[3]} = -[6C_{N-1} + 18D_N](t_N - t_{N-1}) \quad (32)$$

$$\Delta f_N^{[4]} = -24[C_{N-1} + D_N] \quad (33)$$

$$\Delta f_n^{[3]} = -[18C_n + 6D_{n+1}](t_{n+1} - t_n)$$

$$-[6C_{n-1} + 18D_n](t_n - t_{n+1}) \quad \text{for} \quad 2 \le n \le N-1 \quad (34)$$

$$\Delta f_n^{[4]} = 24[C_n + D_{n+1}] - 24[C_{n-1} + D_n] \quad \text{for } 2 \le n \le N-1 \, . \quad (35)$$

Putting all these results together

$$F(\omega) = \frac{A_1 e^{(\alpha+i\omega)t_1}}{\alpha + i\omega} + \frac{A_3 e^{(2\alpha+i\omega)t_1}}{2\alpha + i\omega}$$

$$- \frac{A_2 e^{(-\beta+i\omega)t_N}}{-\beta + i\omega} - \frac{A_4 e^{(-2\beta+i\omega)t_N}}{-2\beta + i\omega}$$

$$- \frac{i}{\omega}\left(e^{i\omega t_N} f_N - e^{i\omega t_1} f_1\right) + \frac{1}{\omega^2}\left(e^{i\omega t_N} f_N^{[1]} - e^{i\omega t_1} f_1^{[1]}\right)$$

$$+ \frac{i}{\omega^3}\left(e^{i\omega t_N} f_N^{[2]} - e^{i\omega t_1} f_1^{[2]}\right) - \frac{1}{\omega^4}\left(\sum_{n=1}^{N} e^{i\omega t_n} \Delta f_n^{[3]}\right)$$

$$- \frac{i}{\omega^5}\left(\sum_{n=1}^{N} e^{i\omega t_n} \Delta f_n^{[4]}\right). \qquad (36)$$

Although this equation is correct, it is somewhat clumsy due to its length. Note that in deriving the above equation, the exponential parts of the function were transformed directly while the rest was integrated by parts several times. One can also integrate the exponential terms by parts several times with the result that $F(\omega)$ can be written as

$$F(\omega) = \frac{i}{\omega^5}\left(\frac{A_1 \alpha^5}{\alpha + i\omega}\right) e^{(\alpha+i\omega)t_1} + \frac{i}{\omega^5}\left(\frac{A_3 (2\alpha)^5}{2\alpha + i\omega}\right) e^{(2\alpha + i\omega)t_1}$$

$$+ \frac{i}{\omega^5}\left(\frac{A_2 \beta^5}{-\beta + i\omega}\right) e^{(-\beta+i\omega)t_N} + \frac{i}{\omega^5}\left(\frac{A_4 (2\beta)^5}{-2\beta + i\omega}\right) e^{(-2\beta+i\omega)t_N}$$

$$- \frac{1}{\omega^4}\sum_{n=1}^{N}\left(e^{i\omega t_n} \delta f_n^{[3]}\right) - \frac{i}{\omega^5}\sum_{n=1}^{N}\left(e^{i\omega t_n} \delta f_n^{[4]}\right) \qquad (37)$$

where

$$\delta f_1^{[3]} = \Delta f_1^{[3]} + A_1 \alpha^3 e^{\alpha t_1} + A_3 (2\alpha)^3 e^{2\alpha t_1} \qquad (38)$$

$$\delta f_1^{[4]} = \Delta f_1^{[4]} + A_1 \alpha^4 e^{\alpha t_1} + A_3 (2\alpha)^4 e^{2\alpha t_1} \qquad (39)$$

$$\delta f_N^{[3]} = \Delta f_N^{[3]} + A_2 \beta^3 e^{-\beta t_N} + A_4 (2\beta)^3 e^{-2\beta t_N} \qquad (40)$$

$$\delta f_N^{[4]} = \Delta f_N^{[4]} - A_2\beta^4 e^{-\beta t_N} - A_4(2\beta)^4 e^{-2\beta t_N} \quad (41)$$

$$\delta f_n^{[3]} = \Delta f_n^{[3]} \text{ for } 2 \le n \le N - 1 \quad (42)$$

$$\delta f_n^{[4]} = \Delta f_n^{[4]} \text{ for } 2 \le n \le N - 1 \; . \quad (43)$$

Thus we have an expression for the Fourier transform in terms of a simple summation whose coefficients can be calculated and tabulated from the original data points used to describe the time waveform. Considerably fewer data points are required with this technique than with a strictly numerical transform method. Also, the above expression is the exact transforms of f(t) and is thus accurate to the degree that f(t) matches the actual waveform. Note, however, that f(t) will never exactly match a real waveform due to the discontinuity in its third and fourth derivatives.

POWER SERIES EXPANSION

The expressions for the Fourier transform of f(t) shown in Equations 36 and 37 are exact but one runs into some trouble numerically evaluating these expressions at low frequencies. The difficulty arises do to ω appearing in the denominator of the equations. Such terms get very large as ω becomes small unless the numerator is also zero. We will show that such terms are indeed zero, but numerical evaluation of the expressions will not have proper behavior as ω becomes small because round-off error will give non-zero numerators. As a result of this problem, we were forced to explicitly subtract out from the expressions for $F(\omega)$ those terms in the power series expansion where ω appears in the denominator. The coefficients of the first two non-negative powers of ω in the series were also evaluated.

To obtain a simple power series in ω from Equation 36 it is necessary to expand the exponentials; i.e., write

$$e^{i\omega t} = 1 + \frac{i\omega t}{1!} + \frac{(i\omega t)^2}{2!} + \ldots + \frac{(i\omega t)^n}{n!} + \ldots \tag{44}$$

One must also expand terms such as

$$\frac{1}{\alpha + i\omega} = \frac{1}{\alpha}\left[1 - \frac{i\omega}{\alpha} + \left(\frac{i\omega}{\alpha}\right)^2 - \left(\frac{i\omega}{\alpha}\right)^3 + \ldots\right] . \tag{45}$$

For the moment consider only the part of $F(\omega)$ that involves summation over the $f_n^{[3]}$ and $f_n^{[4]}$ terms; i.e., let

$$S_1 = \sum_{n=1}^{N}\left[- \frac{1}{\omega^4} e^{i\omega t_n}\Delta f_n^{[3]} - \frac{i}{\omega^5} e^{i\omega t_n}\Delta f_n^{[4]}\right] .$$

Upon expanding $e^{i\omega t_n}$ and collecting coefficients of equal powers of ω

$$S_1 = \sum_{n=1}^{N}\left[- \frac{1}{\omega^5}\Delta f_n^{[4]} + \frac{1}{\omega^4}\left(t_n\Delta f_n^{[4]} - \Delta f_n^{[3]}\right)\right.$$
$$+ \frac{i}{\omega^3}\left(\frac{t_n^2}{2}\Delta f_n^{[4]} - t_n\Delta f_n^{[3]}\right) + \frac{1}{\omega^2}\left(-\frac{t_n^3}{6}\Delta f_n^{[4]} + \frac{t_n^2}{2}\Delta f_n^{[3]}\right)$$
$$+ \frac{i}{\omega}\left(-\frac{t_n^4}{24}\Delta f_n^{[4]} + \frac{t_n^3}{6}\Delta f_n^{[3]}\right) + \left(\frac{t_n^5}{120} f_n^{[4]} - \frac{t_n^4}{24} f_n^{[3]}\right)$$
$$\left. + O(\omega)\right] . \tag{46}$$

Now consider looking for some means of simplifying the various terms in the above summation so that we can show that the Fourier transform does not diverge as $\omega \to 0$. First of all, it is easily seen that

$$f^{[4]}(t_N^-) - f^{[4]}(t_1^+) = \int_{t_1^+}^{t_N^-} f^{[5]}(t)dt \tag{47}$$

where t_N^- denotes the limit as we approach t_N from the left (from $t < t_N$) and t_1^+ denotes the limit as we approach t_1 from the right ($t > t_1$). The above equation can also be written

$$\sum_{n=1}^{N}\left\{f^{[4]}(t_{n+1}^{-}) - f^{[4]}(t_n^{+})\right\} = \sum_{n=1}^{N}\int_{t_n^{+}}^{t_{n+1}^{-}} f^{[5]}(t)dt \; . \qquad (48)$$

But $f^{[5]}(t) = 0$ for each interval $t_n < t < t_{n+1}$ and thus Equation 48 becomes

$$\sum_{n=1}^{N}\left\{f^{[4]}(t_{n+1}^{-}) - f^{[4]}(t_n^{+})\right\} = \sum_{n=1}^{N}\Delta f_n^{[4]} = 0 \; . \qquad (49)$$

By integrating by parts several times one can use the same technique to show that

$$\sum_{n=1}^{N}\Delta f_n^{[3]} = \sum_{n=1}^{N} t_n \Delta f_n^{[4]} \qquad (50)$$

$$\sum_{n=1}^{N}\left(t_n\Delta f_n^{[3]} - \frac{t_n^2}{2}\Delta f_n^{[4]}\right) = f^{[2]}\Big|_{t_1}^{t_N} = f^{[2]}(t_N) - f^{[2]}(t_1) \qquad (51)$$

$$\sum_{n=1}^{N}\left(-\frac{t_n^2}{2}\Delta f_n^{[3]} + \frac{t_n^3}{6}\Delta f_n^{[4]}\right) = f^{[1]}\Big|_{t_1}^{t_N} - tf^{[2]}\Big|_{t_1}^{t_N} \qquad (52)$$

$$\sum_{n=1}^{N}\left(\frac{t_n^3}{6}\Delta f_n^{[3]} - \frac{t_n^4}{24}\Delta f_n^{[4]}\right) = f\Big|_{t_1}^{t_N} - tf^{[1]}\Big|_{t_1}^{t_N} + \frac{t^2}{2} f^{[2]}\Big|_{t_1}^{t_N} \; . \qquad (53)$$

From these expressions the ω^{-5} and ω^{-4} terms in S_1 have coefficients identically equal to zero. Also the ω^{-3}, ω^{-2}, and ω^{-1} terms can now be written in terms of f, $f^{[1]}$, and $f^{[2]}$evaluated at t_N and t_1 only, rather than at each data point.

Now, if we go back to Equation 36 and expand the other exponentials, it is easily seen that the ω^{-3}, ω^{-2}, and ω^{-1} terms all have coefficients that add to zero. The power series for $F(\omega)$ then begins with a constant term [the $\omega = 0$ value of $F(\omega)$] which must be equal to the complete time integral of f(t). By keeping track of the various coefficients the complete series can be written; i.e., first write the power series

$$F(\omega) = W_0 + W_1\omega + W_2\omega^2 + \ldots \tag{54}$$

where it can be shown that

$$W_0 = \sum_{n=1}^{N} \left[\frac{t_n^5}{5!}\Delta f_n^{[4]} - \frac{t_n^4}{4!}\Delta f_n^{[3]}\right] + \left[t_N f_N - t_1 f_1\right]$$
$$+ \left[\frac{(it_N)^2}{2!} f_N^{[1]} - \frac{(it_1)^2}{2!} f_1^{[1]}\right] + i\left[\frac{(it_N)^3}{3!} f_N^{[2]} - \frac{(it_1)^3}{3!} f_1^{[2]}\right]$$
$$+ \frac{A_1 e^{\alpha t_1}}{\alpha} + \frac{A_3 e^{2\alpha t_1}}{2\alpha} + \frac{A_2 e^{-\beta t_N}}{\beta} + \frac{A\ e^{-2\beta t_N}}{2\beta} \tag{55}$$

and

$$W_1 = \sum_{n=1}^{N} i\left[\frac{t_n^6}{6!}\Delta f_n^{[4]} - \frac{t_n^5}{5!}\Delta f_n^{[3]}\right]$$
$$+ i\left(\frac{t_N}{2!} f_N - \frac{t_1^2}{2!} f_1\right) + \left(\frac{(it_N)^3}{3!} f_N^{[1]} - \frac{(it_1)^3}{3!} f_1^{[1]}\right)$$
$$+ i\left(\frac{(it_N)^4}{4!} f_N^{[2]} - \frac{(it_1)^4}{4!} f_1^{[2]}\right) + iA_1 e^{\alpha t_1}\left[\frac{t_1}{\alpha} - \frac{1}{\alpha^2}\right]$$
$$+ iA_3 e^{2\alpha t}\left[\frac{t_1}{2\alpha} - \frac{1}{4\alpha^2}\right] + 1A_2 e^{-\beta t_N}\left[\frac{t_N}{-\beta} - \frac{1}{\beta^2}\right]$$
$$+ iA_4 e^{-2\beta t_N}\left[\frac{t_N}{-2\beta} - \frac{1}{4\beta^2}\right]\ . \tag{56}$$

With sufficient effort, the coefficients of higher powers of ω can also be evaluated.

NUMERICAL PROBLEMS

Now let us consider some problems encountered in numerically evaluating the expressions for the Fourier transform. It turned out to be easier to work with Equation 36 than Equation 37. The exponentials $e^{i\omega t_n}$ were written as $(\cos\omega t_n + i \sin\omega t_n)$ and $F(\omega)$ was separated into real and imaginary parts. The round-off error problem can be seen by considering just the $\cos\omega t_n$ and $\sin\omega t_n$ terms. First, write the cosine in terms of its series expansion

$$\cos\omega t_n = 1 - \frac{(\omega t_n)^2}{2!} + \frac{(\omega t_n)^4}{4!} - \frac{(\omega t_n)^6}{6!} + \cdots \tag{57}$$

Now, in the last term of Equation 36 the cosine is divided by a ω^5 term which means that the first three terms in the above series contribute to the ω^{-5}, ω^{-3}, and ω^{-1} terms of a power series expansion for $F(\omega)$. But we have previously seen that the coefficients of these terms sum to zero so that they give no contribution to the Fourier transform. Thus only the $(\omega t_n)^6$ and higher order terms in the above series are important in calculating $F(\omega)$. However, if we consider times of the order of 10^{-8} seconds and ω about 10^4 radians/second, $\omega t_n \simeq 10^{-4}$. For these values the $(\omega t_n)^6$ term is 24 decades below the first term of the series. And since the CDC 7600 only has 14 digit accuracy (single precision), information in the sixth and higher order terms will be lost when the cosine is evaluated and it is just these terms that are required to find $F(\omega)$. It is easily seen that a similar problem occurs in evaluating the $\sin\omega t_n$ terms.

To get around this round-off error problem it was necessary to explicitly remove the first few terms from the $\sin\omega t_n$ and $\cos\omega t_n$ expressions. The number of terms removed depends on the power of ω by which the $\sin\omega t_n$ or $\cos\omega t_n$ term is being divided. For example, the last term of Equation 36 has an ω^5 term in the denominator so that the first three terms of the cosine series and the first two terms of the sine series are removed. Since we have seen that the coefficients of negative powers of ω are identically zero, subtracting each of these terms from the sine and cosine is equivalent to just subtracting zero from the equation for $F(\omega)$.

Thus the last term in Equation 36 can be written as

$$- \frac{i}{\omega^5}\left(\sum_{n=1}^{N} e^{i\omega t_n}\Delta f_n^{[4]}\right) = - \frac{i}{\omega^5}\sum_{n=1}^{N}\left\{\Delta f_n^{4}\left[\left(\cos\omega t_n - 1 + \frac{(\omega t_n)^2}{2!} - \frac{(\omega t_n)^4}{4!}\right)\right.\right.$$

$$\left.\left. + i\left(\sin\omega t_n - \omega t_n + \frac{(\omega t_n)^3}{3!}\right)\right]\right\} \qquad (58)$$

Note that for numerical reasons one cannot evaluate the above expressions just as written. For example, if one evaluates $\cos\omega t_n$ first and then subtracts the first few terms of the cosine series, the round-off problem will not be helped since much of the useful information in the $\cos\omega t_n$ term will already have been lost. Instead, a routine for calculating and summing the terms remaining in the sine or cosine series after the first few are subtracted was developed and used in evaluating $F(\omega)$. The magnitude and phase of the Fourier transform of the function shown in Figure 1 have been calculated and are shown in Figures 4 and 5.

CHECK ON VALIDITY AND DISCUSSION OF RESULTS

To insure the validity of the Fourier transform expressions developed here and to test the accuracy of the computer code evaluating these expressions, a function that has an exactly calcuable Fourier transform was used. The function used is given by the equation

$$f_1(t) = Ak \frac{e^{\alpha(t - t_0)}}{1 + e^{\beta(t - t_0)}} \qquad (59)$$

where

$$k = \frac{\beta}{\alpha}\left(\frac{\alpha}{\gamma}\right)^{\gamma/\beta} \qquad (60)$$

and A is the peak value of the function. At early times, $f_1(t)$ varies like $e^{\alpha t}$ while at late times $f_1(t)$ goes as $e^{-\gamma t}$ where $\beta \equiv \alpha + \gamma$. The variable t_0 is used to arbitrarily shift the function along the time scale.

The Fourier transform of $f_1(t)$ can be evaluated exactly as

$$F_1(\omega) = \int_{-\infty}^{\infty} f_1(t) e^{i\omega t} dt$$

$$= Ak\left(-\frac{2\pi i}{\beta}\right) \frac{e^{-\pi\frac{\omega}{\beta}} e^{\pi i \frac{\alpha}{\beta}}}{1 - e^{-2\pi\frac{\omega}{\beta}} e^{2\pi i \frac{\alpha}{\beta}}} e^{i\omega t_0} \qquad (61)$$

where the last term is merely a phase shift factor which results from translating the function along the t-axis.

The results of fitting this function by the method described in Section 2 are shown in Figure 6. The input data points indicated on the graph were accurately calculated from Equation 59. The resulting curve is within a few percent of the analytic function at all points calculated.

Figures 7 and 8 plot the magnitude of the Fourier transform as a function of the radian frequency, ω. Figure 7 shows the low-frequency end of the spectrum. In the frequency range shown the exact transform and that of the fitted function agree within the accuracy that can be plotted on the graph. Figure 8 shows the higher frequency end of the spectrum where the transform of the fitting function begins to diverge from the exact transform. The transforms agree fairly well up to $\omega \simeq 2 \times 10^8$ radians/second. For higher frequencies the transform of the fitting function tends to fall off approximately as ω^{-4}, which is to be expected of a function with discontinuous third derivatives. The point at which the transform of the fitting function tends to deviate considerably from the analytic transform probably depends on the location and spacing chosen for the data points used in fitting the analytic function but the exact details of the variation are not well understood.. However, unless one is particularly interested in very high frequencies, the transform technique described here should be extremely useful, especially considering the limited number of data points required for the fit.

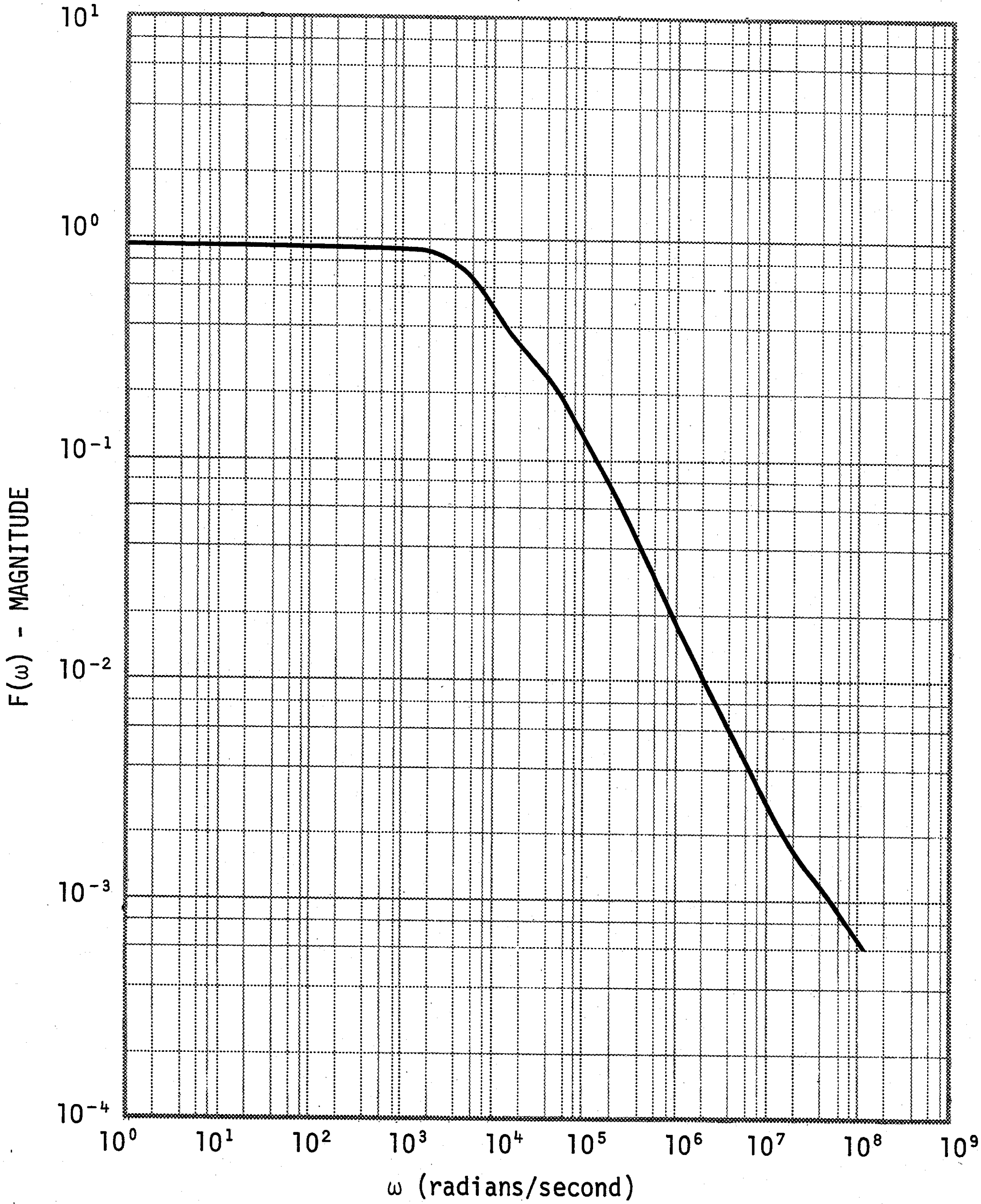

Figure 4. Magnitude of the Fourier transform of the time waveform shown in Figure 1.

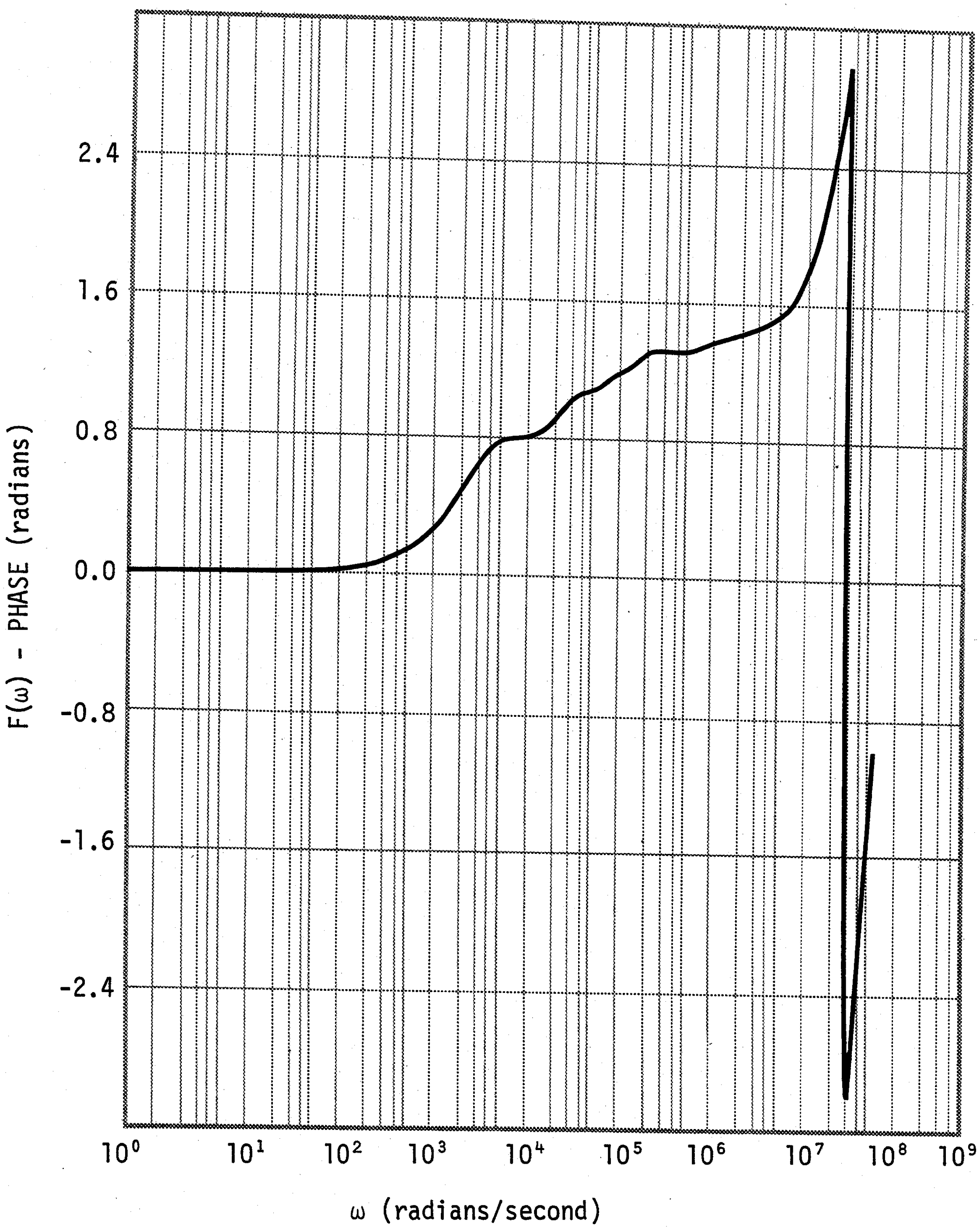

Figure 5. Phase of the Fourier transform of the time waveform shown in Figure 1.

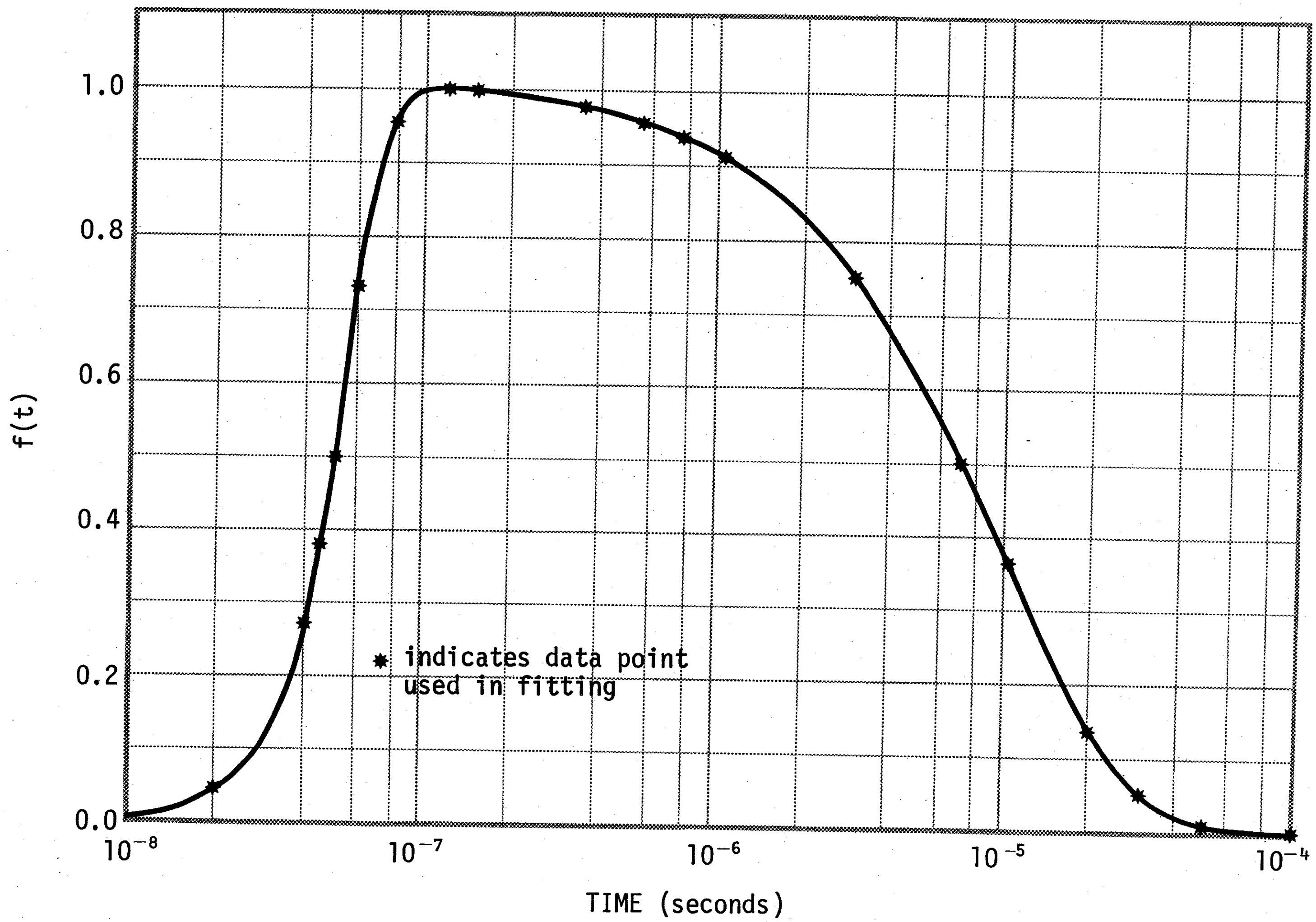

Figure 6. Time domain waveform fitted to the function $Ak\ e^{\alpha t}/(1+e^{\beta t})$ where $\alpha = 10^8$, $\beta = 1.001 \times 10^8$.

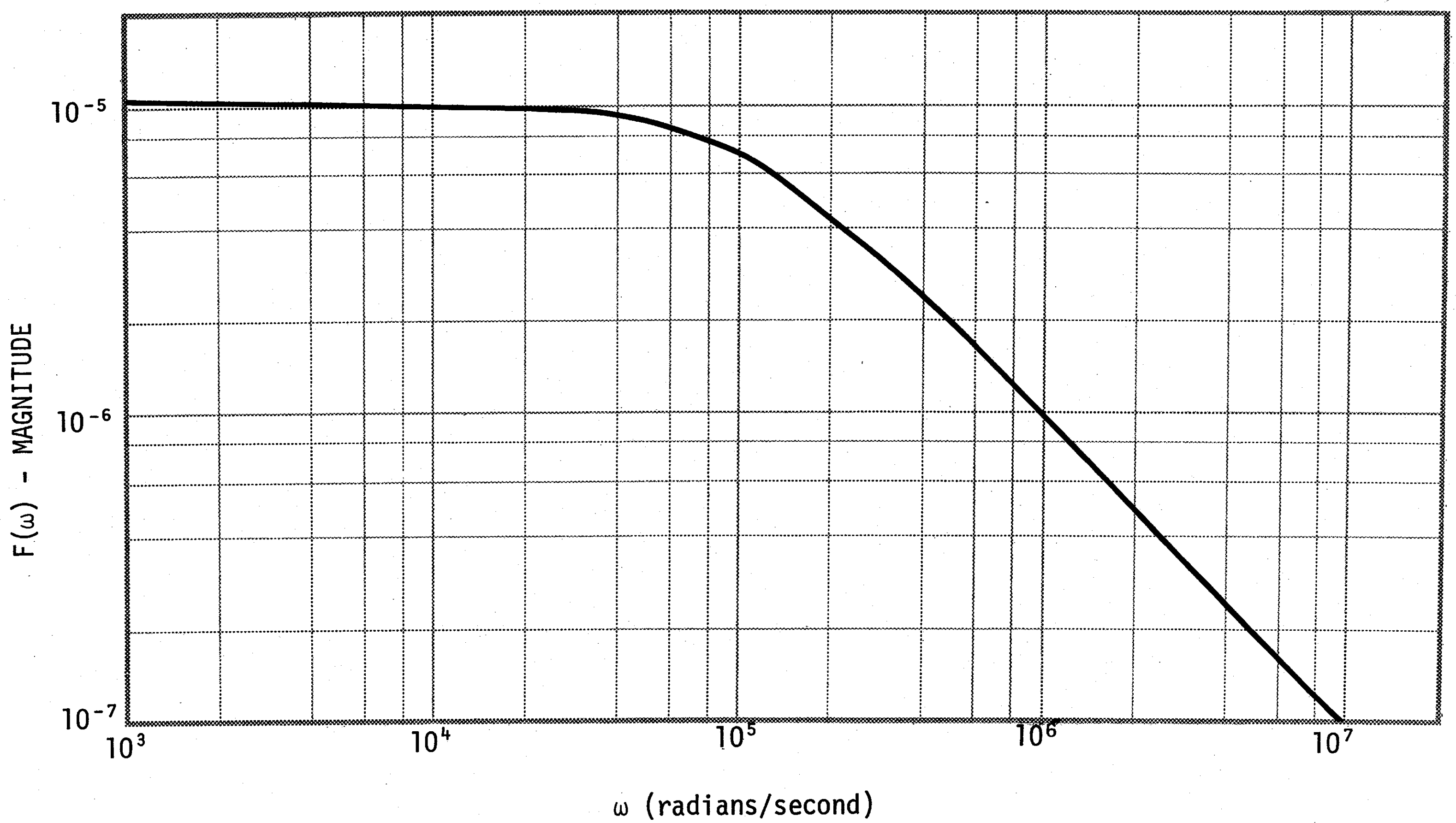

Figure 7. Magnitude of the Fourier transform of the time waveform shown in Figure 6.

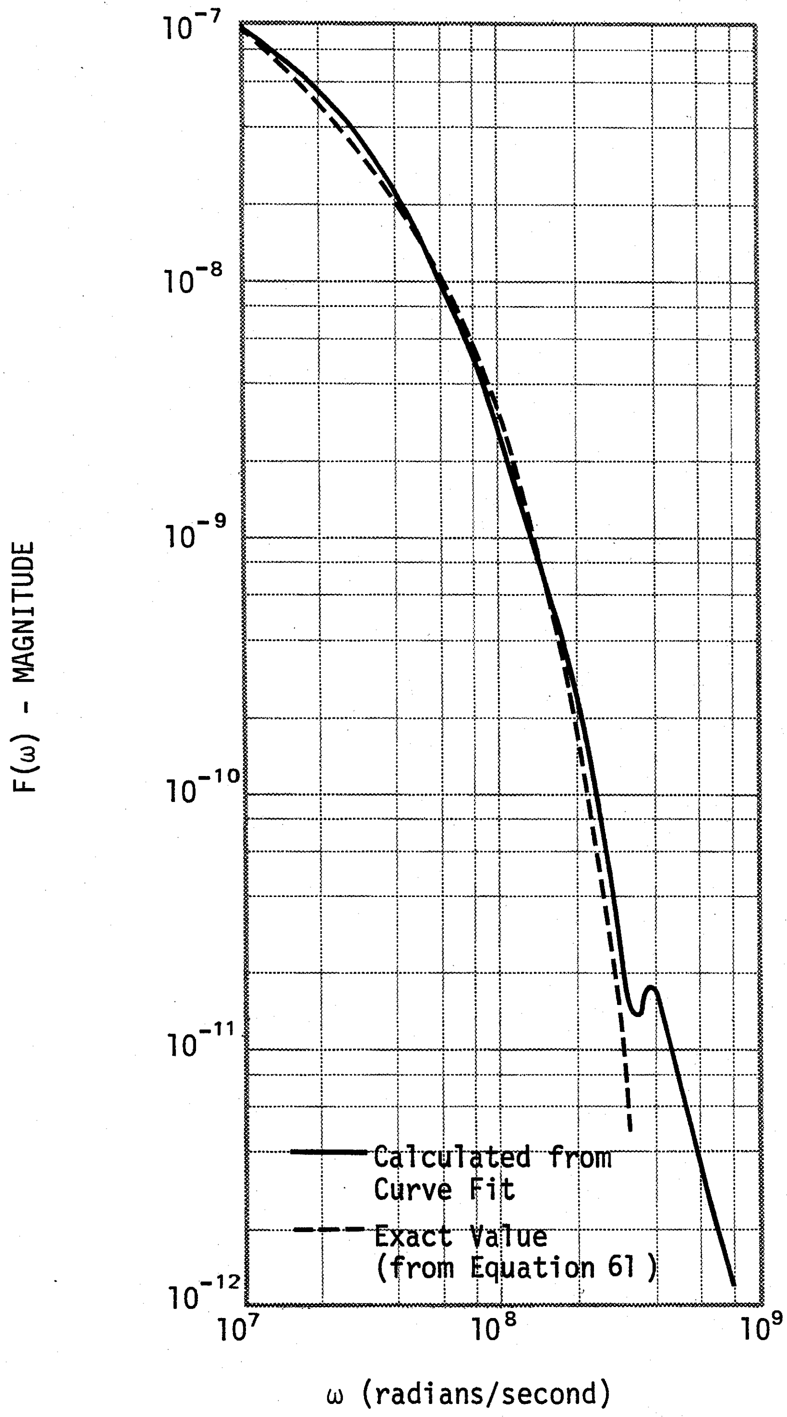

Figure 8. Comparison of high frequency Fourier transform magnitude with exact values.

The phase of the Fourier transform of the fitting function is shown in Figure 9. As with the magnitude, this phase plot is almost identical to analytic values calculated from Equation 61, at least up to $\omega \simeq 10^8$ radians/second.

FOURIER TRANSFORM OF THE TIME DERIVATIVE

One should note that if this scheme is used to calculate the Fourier transform of the fitting function, f(t), then the Fourier transform of the time derivative, $\dot{f}(t)$, is readily calculated. If $F(\omega)$ is the Fourier transform of f(t), then

$$\int_{-\infty}^{\infty} \dot{f}(t)e^{i\omega t}dt = -i\omega F(\omega) \tag{62}$$

where $F(\omega)$ is given by Equation 36 or 37. Thus, in any computer code for evaluating $F(\omega)$ the Fourier transform of $\dot{f}(t)$ is obtained by shifting the phase of $F(\omega)$ by $\pi/2$ and multiplying the magnitude of $F(\omega)$ by ω.

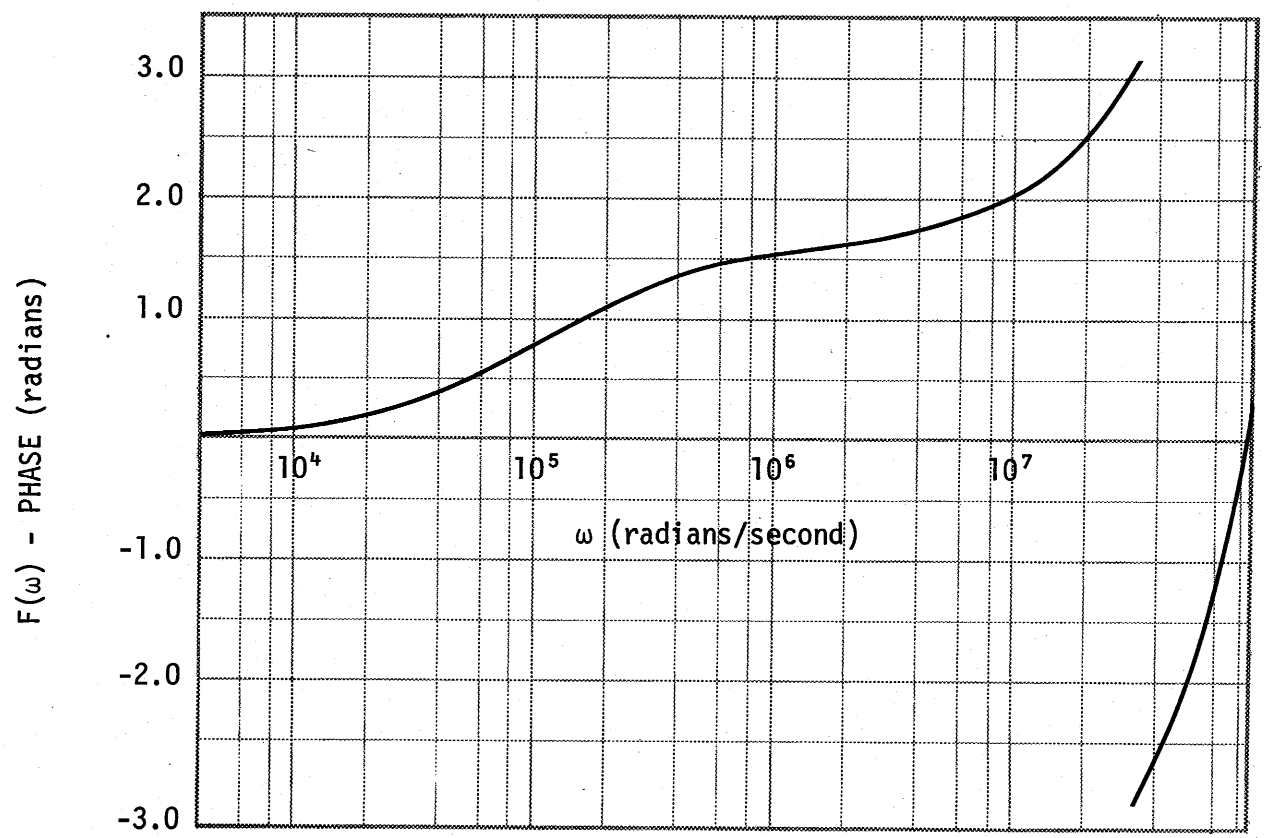

Figure 9. Phase of the Fourier transform of the time waveform shown in Figure 6.

SECTION 5

SUMMARY

In this paper we have developed a method of using a relatively small number of data points to generate a fairly smooth function for approximating an EMP waveform. This function is chosen so that it passes through the input data points and has continuous first and second derivatives. The third and fourth derivatives are discontinuous at the data points however.

This fitting function is particularly useful in that the numerically generated oscillations in the output of an EMP code can be smoothed out. This smoothing process is especially important if one desires to calculate the time derivative of the waveform.

Once the various coefficients required by the fitting function are calculated from the input data points, one can easily derive expressions for both the time derivative and the Fourier transform. The shape of the time derivative curve will be somewhat dependent on the choice of data points and one may want to go back and change or add data points to help "smooth" the fitting function and its time derivative.

By integrating by parts several times, the Fourier transform of the fit can be written as a summation of the discontinuities in the third and fourth derivatives at the data points. There was some difficulty in numerically evaluating these expressions due to round-off error but these problems were solved and numerical results were presented comparing this Fourier transform technique with the transform that can be calculated analytically. It should be noted that no attempt is made to claim that this transform technique should be used to replace more conventional numerical transform schemes. However, it would appear that the technique described here has great utility if only a small number of data points describing the time domain waveform are known.