MATHEMATICS NOTES

NOTE 36

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Expansion of a Scalar, Vector, or Dyadic Function in Terms of the Spherical Vector Wave Functions

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ABSTRACT

This note describes a technique for determining the coefficients for a vector wave function expansion of the form:

$$\vec{K} = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_{n=o}^{\infty} \sum_{m=-n}^{n} \sum_{\sigma = \substack{e\\o}} A_{nm\sigma}(\gamma)\vec{L}_{nm\sigma}(\vec{r})$$

$$+ B_{nm\sigma}(\gamma)\vec{N}_{nm\sigma}(\vec{r}) + C_{nm\sigma}(\gamma)\vec{M}_{nm\sigma}(\vec{r})$$

The case where $\vec{K}$ is a solution to Maxwell's equations is handled specifically. The difficulty associated with the orthogonality relations for $\vec{L}$, $\vec{M}$, and $\vec{N}$ is circumvented using the orthogonality relations for the vector spherical harmonics.

Both the vector and scalar coefficients are determined for certain types of dyadic and vector expansions. The coefficients for the expansion of the dyadic delta function are also given.

I. INTRODUCTION

The study of System Generated Electromagnetic Pulse (SGEMP) problems frequently requires solutions of Maxwell's equations in regions where there is a distributed current density. Typical solution techniques determine the solutions of the vector wave equation as an eigenfunction expansion in solutions to the homogeneous wave equation. In scattering problems the electric field typically has zero divergence and therefore requires only solenoidal vector functions for its expansion. In static problems there are no radiated fields so only irrotational vector wave functions are required. In many SGEMP problems both types of solutions exist. It may therefore be desirable to be able to expand a general solution to Maxwell's equations in terms of the complete set of solenoidal and irrotational vector wave functions.

The vector wave functions normally used in finding solutions to the inhomogeneous vector wave equation are the $\vec{L}$, $\vec{M}$, and $\vec{N}$ sets of functions (Refs. 2, 5, 6). The difficulty in determining the unknown coefficients of such an expansion is that while the three sets of vector wave functions span a solution space of interest the sets are not completely orthogonal (Ref. 6/418).

It will be shown that this problem of nonorthogonality may be circumvented by expressing the vector wave functions in terms of the vector spherical harmonics and using their orthogonality relations to develop a set of simultaneous linear equations for the expansion coefficients.

In this note the properties of the vector wave functions in spherical coordinates are first discussed. Included in this area are discussions of the vector spherical harmonics and their orthogonality properties and the orthogonality properties of the vector wave functions. In Chapter II the wave functions described above are used to expand an arbitrary vector in a restricted solution space of Maxwell's equations. The coefficients are determined as integral transforms of the radial part of the vector being expanded. The first section of Chapter IV contains a generalization of the vector expansion techniques for use in expanding dyads. Chapter IV is concluded with an expansion of the dyadic delta function as an example. For completeness Chapter V contains a well-known expansion of a scalar function in solutions to the scalar wave equation. The results of the scalar and vector techniques developed in this note are compared by expanding the divergence of an arbitrary vector using both techniques.

II. THE VECTOR WAVE FUNCTIONS

The Vector Spherical Harmonics

Three sets of vector spherical harmonics are useful in expressing the vector wave functions. These functions may be defined in terms of the scalar spherical harmonics in the following way (Ref. 2/Appendix B).

$$\vec{P}_{nm\sigma}(\theta,\phi) = \vec{e}_r Y_{nm\sigma}(\theta,\phi) \tag{1}$$

$$\vec{Q}_{nm\sigma}(\theta,\phi) = \nabla_s Y_{nm\sigma}(\theta,\phi) \tag{2}$$

and

$$\vec{R}_{nm\sigma}(\theta,\phi) = -\vec{e}_r \times \nabla_s Y_{nm\sigma}(\theta,\phi) \tag{3}$$

where

$$\nabla_s F = \vec{e}_\theta \frac{\partial}{\partial\theta} F + \vec{e}_\phi \frac{1}{\sin\theta} \frac{\partial}{\partial\phi} F$$

and

$$Y_{nm\binom{e}{o}}(\theta,\phi) = P_n^m(\cos\theta)\begin{Bmatrix}\cos m\phi \\ \sin m\phi\end{Bmatrix} \qquad m = 0, 1, 2, \cdots, n$$

(∇_s is the gradient operator on the surface of a sphere.)

The vector spherical harmonics defined in equations (1), (2), and (3) have some useful orthogonality properties (Ref. 2/Appendix B). The writing of the orthogonality relations may be simplified considerably by using the following notation, which has been previously established (Ref. 3/Appendix A).

$$\left\langle \vec{A}_{\alpha'} ; \vec{B}_{\alpha} \right\rangle \equiv \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \vec{A}_{\alpha'}(\vec{r}) \cdot \vec{B}_{\alpha}(\vec{r}) \sin\theta \tag{4}$$

where α and α' are two index sets of the form $\alpha = \{n, m, \sigma\}$. The orthogonality relations are then written as:

$$\left\langle \vec{P}_{\alpha} ; \vec{Q}_{\alpha'} \right\rangle = 0 \quad (\alpha \neq \alpha') \tag{5}$$

$$\left\langle \vec{P}_{\alpha} ; \vec{R}_{\alpha'} \right\rangle = 0 \quad (\alpha \neq \alpha') \tag{6}$$

$$\left\langle \vec{Q}_{\alpha} ; \vec{R}_{\alpha'} \right\rangle = 0 \quad (\alpha \neq \alpha') \tag{7}$$

and

$$\left\langle \vec{P}_{nm\sigma} ; \vec{P}_{n'm'\sigma'} \right\rangle = [1 + [\delta_{e\sigma} - \delta_{o\sigma}]\delta_{om}] \cdot \frac{2\pi}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{nn'} \delta_{mm'} \delta_{\sigma\sigma'} \tag{8}$$

where

$$\delta_{nn'} = \begin{cases} 0 & n \neq n' \\ 1 & n = n' \end{cases}$$

$$\left\langle \vec{Q}_{\alpha} ; \vec{Q}_{\alpha'} \right\rangle = \left\langle \vec{R}_{\alpha} ; \vec{R}_{\alpha'} \right\rangle = [1 + [\delta_{e\sigma} - \delta_{o\sigma}]\delta_{om}] \cdot \frac{2\pi n(n+1)}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{\alpha\alpha'} \tag{9}$$

The Radial Functions

A set of radial functions which is useful for expansions of this type is the set of modified spherical Bessel functions denoted by $f_n^{(\ell)}(\gamma r)$. This set of functions can be related to the spherical Bessel functions (Ref. 2/ Appendix B) ($j_n(kr)$ and $h_n^{(2)}(kr)$) in the following way.

$$f_n^{(1)}(\gamma r) = i_n(\gamma r) = i^n j_n(kr) = i^n j_n(-i\gamma r)$$

$$f_n^{(2)}(\gamma r) = k_n(\gamma r) = -i^{-n} h_n^{(2)}(kr) = -i^{-n} h_n^{(2)}(-i\gamma r)$$

where

$$\gamma r = \frac{sr}{c} = ikr$$

The functions $f_n^{(2)}(\gamma r)$ are required for mathematical generality for any series expansion. However, in practice the outgoing wave portion of a solution to Maxwell's equations can frequently be determined from a scattering term. The integrals required for determining coefficients for the outgoing wave portion are difficult and complex. The expansion set will therefore be restricted to functions which can be expressed by using only the $f_n^{(1)}(\gamma r)$ solutions.

The modified spherical Bessel functions can be expressed as a finite series of elementary functions (Ref. 2/Appendix B).

$$i_n(\gamma r) = \frac{e^{\gamma r}}{2\gamma r} \sum_{\beta=o}^{n} \frac{(n+\beta)!}{\beta!(n-\beta)!} (-2\gamma r)^{-\beta}$$

$$+ (-1)^{n+1} \frac{e^{-\gamma r}}{2\gamma r} \sum_{\beta=o}^{n} \frac{(n+\beta)!}{\beta!(n-\beta)!} (2\gamma r)^{-\beta} \quad (10)$$

The Vector Wave Functions

Three sets of vector wave functions may be defined (Ref. 5). These three sets of functions may be formed from solutions to the scalar wave equation

$$[\nabla^2 - \gamma^2]\psi^{(\ell)}_{nm\sigma}(\gamma,\vec{r}) = 0 \tag{11}$$

where

$$\psi^{(\ell)}_{nm{e \atop o}}(\gamma,\vec{r}) = f^{(\ell)}_n(\gamma r)P^m_n(\theta,\phi)\begin{Bmatrix}\cos m\phi \\ \sin m\phi\end{Bmatrix} \qquad m = 0, 1, 2, \cdots, n$$

Since only restricted solutions of Maxwell's equations are being considered, the superscript ℓ will henceforth be restricted to 1 and will be suppressed.

The first of the vector wave functions has vanishing curl but non-vanishing divergence and is used in expressing the longitudinal part of a vector. The curl free vector functions are

$$\vec{L}_{nm\sigma}(\gamma,\vec{r}) = \frac{1}{\gamma}\nabla\psi_{nm\sigma}(\gamma,\vec{r})$$

$$= i'_n(\gamma r)\vec{P}_{nm\sigma}(\theta,\phi) + \frac{i_n(\gamma r)}{\gamma r}\vec{Q}_{nm\sigma}(\theta,\phi) \tag{12}$$

where

$$i'_n(z) \equiv \frac{\partial i_n(z)}{\partial z}$$

The other two vector wave functions have vanishing divergence but non-vanishing curl. These sets of functions may be used to expand the transverse part of a vector. The divergence free vector functions are

$$\vec{M}_{nm\sigma}(\gamma,\vec{r}) = \nabla \times [\vec{r}\psi_{nm\sigma}(\gamma,\vec{r})] = i_n(\gamma r)\vec{R}_{nm\sigma}(\theta,\phi) \tag{13}$$

and

$$\vec{N}_{nm\sigma}(\gamma,\vec{r}) = \frac{1}{\gamma}\nabla \times \vec{M}_{nm\sigma} = \frac{n(n+1)}{\gamma r} i_n(\gamma r)\vec{P}_{nm\sigma}(\theta,\phi) + \frac{[\gamma r\, i_n(\gamma r)]'}{\gamma r}\vec{Q}_{nm\sigma}(\theta,\phi) \tag{14}$$

where the vector spherical harmonics in equations (12), (13), and (14) are defined in equations (1), (2), and (3).

The vector wave functions are all solutions of:

$$[\nabla^2 - \gamma^2]\begin{Bmatrix}\vec{L}\\ \vec{M}\\ \vec{N}\end{Bmatrix} = \vec{0} \tag{15}$$

Due to the identity

$$\nabla \times \nabla \times = \nabla\nabla \cdot - \nabla^2 \tag{16}$$

they also obey the following useful relations:

$$\nabla\nabla \cdot \begin{Bmatrix}\vec{M}\\ \vec{N}\end{Bmatrix} = \vec{0} \tag{17}$$

$$\nabla\nabla \cdot \vec{L} = \gamma^2\vec{L} \tag{18}$$

Using these relations it may be shown that

$$[\nabla \times \nabla \times + \gamma^2] \left\{ \begin{matrix} \vec{M} \\ \vec{N} \end{matrix} \right\} = \vec{0} \tag{19}$$

Orthogonality Relations

The vector wave functions have some interesting orthogonality properties. The writing of these equations can be greatly simplified using the notation defined in equation (4).

Two orthogonality relations are given by

$$\left\langle \vec{L}_{\alpha'} ; \vec{M}_{\alpha} \right\rangle \equiv 0 \tag{20}$$

$$\left\langle \vec{M}_{\alpha'} ; \vec{N}_{\alpha} \right\rangle \equiv 0 \tag{21}$$

The most interesting relation and the one which makes expansions difficult in these wave functions is:

$$\left\langle \vec{L}_{\alpha'} ; \vec{N}_{\alpha} \right\rangle = i_n(\gamma r) \frac{n(n+1)}{\gamma r} i_n'(\gamma r) \left\langle \vec{P}_{\alpha'} ; \vec{P}_{\alpha} \right\rangle + \frac{i_n(\gamma r)}{\gamma r} \frac{[\gamma r\, i_n(\gamma r)]'}{\gamma r} \left\langle \vec{Q}_{\alpha'} ; \vec{Q}_{\alpha} \right\rangle \tag{22}$$

Equation (22) prevents the set $\vec{L}$, $\vec{M}$, and $\vec{N}$ from being mutually orthogonal but does not prevent its use as a set used for expansion of some vectors.

Additional orthogonality relations exist for $\vec{L}$, $\vec{M}$, and $\vec{N}$, but they will not be used in this note.

III. EXPANSION OF A VECTOR IN THE VECTOR WAVE FUNCTIONS

Consider a vector function $\vec{K}(\vec{r})$. This vector may be expanded in the vector wave functions with certain restrictions to assure existence of the expansion coefficients. $\vec{K}(\vec{r})$ will be a function of r, θ, ϕ. If r is held constant, the function remaining may be expanded in the vector spherical harmonics if the θ, ϕ function is L^2. The expansion is complete in these functions. The remaining coefficients in r can be expanded only if the integral $\int_o^\infty f(r) i_n(\gamma r) r^2 dr$ exists. γ is restricted to be to the right of a contour from $\gamma_o - i\infty$ to $\gamma_o + i\infty$ where γ_o is an arbitrary real constant. The radius variable r is real and positive. Therefore for some γ_o this integral will converge if f(r) is Fourier transformable. Actually the integral will converge somewhat faster than the Fourier transform because of the $1/r^n$ behavior of the $i_n(\gamma r)$ functions. With these restrictions $\vec{K}(\vec{r})$ may be expanded as

$$\vec{K} = \frac{1}{2\pi i} \int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_{n=o}^{\infty} \sum_{m=-n}^{n} \sum_{\sigma=e,o} [A_{nm\sigma}(\gamma) \vec{L}_{nm\sigma}(\gamma, \vec{r})$$

$$+ B_{nm\sigma}(\gamma) \vec{N}_{nm\sigma}(\gamma, \vec{r}) + C_{nm\sigma}(\gamma) \vec{M}_{nm\sigma}(\gamma, \vec{r})] \tag{23}$$

Equation (23) may be expressed in terms of $\vec{P}$, $\vec{Q}$, and $\vec{R}$ using equations (12), (13), and (14) as

$$\vec{K} = \frac{1}{2\pi i} \int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_{\alpha} A_\alpha \; i_n'(\gamma r) \vec{P}_\alpha(\theta, \phi) + \frac{i_n(\gamma r)}{\gamma r} \vec{Q}_\alpha(\theta, \phi)$$

$$+ B_\alpha \left[\frac{n(n+1)}{\gamma r} i_n(\gamma r) \vec{P}_\alpha(\theta, \phi) + \left(\frac{i_n(\gamma r)}{\gamma r} + i_n'(\gamma r) \right) \vec{Q}(\theta, \phi) \right]$$

$$+ C_\alpha i_n(\gamma r) \vec{R}_\alpha(\theta, \phi) \tag{24}$$

where α is an index set consisting of m, n, and σ.

Scalar multiplying equation (24) by $\vec{P}_{\alpha'}$, $\vec{Q}_{\alpha'}$, and $\vec{R}_{\alpha'}$, respectively, and integrating over θ and ϕ provides the following relations.

$$\rho_\alpha(r) \equiv \frac{\langle \vec{P}_\alpha ; \vec{K}_\alpha \rangle}{\langle \vec{P}_\alpha ; \vec{P}_\alpha \rangle} = \frac{1}{2\pi i} \int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \left[A_\alpha i'_n(\gamma r) + B_\alpha \frac{n(n+1)}{\gamma r} i_n(\gamma r) \right] \tag{25}$$

$$\sigma_\alpha(r) \equiv \frac{\langle \vec{Q}_\alpha ; \vec{K}_\alpha \rangle}{\langle \vec{Q}_\alpha ; \vec{Q}_\alpha \rangle} = \frac{1}{2\pi i} \int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \left[A_\alpha \frac{i_n(\gamma r)}{\gamma r} + B_\alpha \left[\frac{i_n(\gamma r)}{\gamma r} + i'_n(\gamma r) \right] \right] \tag{26}$$

$$\tau_\alpha(r) \equiv \frac{\langle \vec{R}_\alpha ; \vec{K}_\alpha \rangle}{\langle \vec{R}_\alpha ; \vec{R}_\alpha \rangle} = \frac{1}{2\pi i} \int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma i_n(\gamma r) C_\alpha(\gamma) \tag{27}$$

The first two equations above form a set of linear simultaneous equations for $A_\alpha(\gamma)$ and $B_\alpha(\gamma)$. Equation (27) is a single equation in $C_\alpha(\gamma)$ and may be solved immediately by integration. We would like to maintain the analogy that r represents the radial coordinate and as such is real and has meaning only for $r \geq 0$. Carrying out the required multiplication by $i_n(\gamma' r) r^2$ and integration over r results in the following set of equations.

$$\int_0^\infty dr i_n(\gamma' r) \frac{\langle \vec{R}_\alpha ; \vec{K}_\alpha \rangle}{\langle \vec{R}_\alpha ; \vec{R}_\alpha \rangle} r^2$$

$$= \frac{1}{2\pi i} \int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \int_0^\infty dr i_n(\gamma r) i_n(\gamma' r) C_\alpha(\gamma) r^2$$

$$= \frac{(-1)^{n+1}}{4} \int_{\gamma_o - i\infty}^{\gamma_o + i\infty} \delta(\gamma - \gamma') C_\alpha(\gamma) \tag{28}$$

Performing the required integration in equation (28) and interchanging γ' and γ yields

$$C_\alpha(\gamma) = 4(-1)^{n+1} \int_o^\infty dr \frac{\left\langle \vec{R}_\alpha ; \vec{K} \right\rangle}{\left\langle \vec{R}_\alpha ; \vec{R}_\alpha \right\rangle} i_n(\gamma r) r^2 \tag{29}$$

or $C_\alpha(\gamma)$ may be written as:

$$C_\alpha(\gamma) = 4(-1)^{n+1} \int_o^\infty \tau_\alpha(r) i_n(\gamma r) r^2$$

Note that the function coefficient $C_\alpha(\gamma)$ is an integral transform of the radial portion of $\vec{K}$, i.e., $\tau_\alpha(r)$.

Solution of equations (25) and (26) for $A_\alpha(\gamma)$ and $B_\alpha(\gamma)$ is a bit more complicated since the r dependence must be eliminated before the linear equations are solved. Equations (25) and (26) may be written as:

$$\rho_\alpha(r) = \frac{1}{2\pi i} \int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma A_\alpha(\gamma) i'_n(\gamma r) + B_\alpha(\gamma) \frac{n(n+1)}{\gamma r} i_n(\gamma r) \tag{30}$$

and

$$\sigma_\alpha(r) = \frac{1}{2\pi i} \int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma A_\alpha(\gamma) \frac{i_n(\gamma r)}{\gamma r} + B_\alpha(\gamma) \left[\frac{i_n(\gamma r)}{\gamma r} - i'_n(\gamma r) \right] \tag{31}$$

The modified spherical Bessel functions in equations (30) and (31) may be expanded using the following recursion relations (Ref. 1, p. 444)

$$\frac{i_n(\gamma r)}{\gamma r} = \frac{1}{2n + 1}[i_{n-1}(\gamma r) - i_{n+1}(\gamma r)] \tag{32}$$

and

$$i'_n(\gamma r) = \frac{1}{2n + 1}[ni_{n-1}(\gamma r) + (n + 1)i_{n+1}(\gamma r)] \tag{33}$$

Substitution of equations (32) and (33) into equations (30) and (31) yields

$$(2n + 1)\rho_\alpha(r) = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma n A_\alpha(\gamma) i_{n-1}(\gamma r)$$

$$+ (n + 1)\frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma A_\alpha i_{n+1}(\gamma r)$$

$$+ n(n + 1)\frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma B_\alpha(\gamma) i_{n-1}(\gamma r)$$

$$- n(n + 1)\frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma B_\alpha(\gamma) i_{n+1}(\gamma r) \tag{34}$$

$$(2n + 1)\sigma_\alpha(r) = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma A_\alpha(\gamma) i_{n-1}(\gamma r)$$

$$- \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma A_\alpha(\gamma) i_{n+1}(\gamma r)$$

$$+ \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma B_\alpha(\gamma) i_{n-1}(\gamma r)$$

$$+ \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma n B_\alpha(\gamma) i_{n+1}(\gamma r) \qquad (35)$$

Rearranging terms

$$(2n + 1)\rho_\alpha(r) = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma [nA_\alpha(\gamma) + n(n + 1)B_\alpha(\gamma)] i_{n-1}(\gamma r)$$

$$+ \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma [(n + 1)A_\alpha(\gamma) - n(n + 1)B_\alpha(\gamma)] i_{n+1}(\gamma r) \qquad (36)$$

$$(2n + 1)\sigma_\alpha(r) = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma [A_\alpha(\gamma) + (n + 1)B_\alpha(\gamma)] i_{n-1}(\gamma r)$$

$$+ \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma [-A_\alpha(\gamma) + nB_\alpha(\gamma)] i_{n+1}(\gamma r) \qquad (37)$$

Multiplying equation (37) by n and subtracting equation (37) from equation (36) yields:

$$\frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma[(2n+1)A_\alpha(\gamma) - n(2n+1)B_\alpha(\gamma)]i_{n+1}(\gamma r)$$

$$= (2n+1)\rho_\alpha(r) - n(2n+1)\sigma_\alpha(r) \tag{38}$$

Multiplying equation (37) by (n + 1), adding equations (36) and (37), and dividing by (2n + 1) yields:

$$\frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma[A_\alpha(\gamma) + (n+1)B_\alpha(\gamma)]i_{n-1}(\gamma r)$$

$$= \rho_\alpha(r) + (n+1)\sigma_\alpha(r) \tag{39}$$

Multiplying equation (38) by $i_{n+1}(\gamma' r)r^2$, equation (39) by $i_{n-1}(\gamma' r)r^2$, integrating them both over r from 0 to ∞, carrying out the required γ integration, etc., yields:

$$A_\alpha(\gamma) - nB_\alpha(\gamma) = 4(-1)^{n+2}\int_0^\infty dr[\rho_\alpha(r) - n\sigma_\alpha(r)]i_{n+1}(\gamma r)r^2$$

$$\equiv \alpha(\gamma) \tag{40}$$

and

$$A_\alpha(\gamma) + (n+1)B_\alpha(\gamma) = 4(-1)^{n}\int_0^\infty dr\rho_\alpha(r) + (n+1)\sigma_\alpha(r)i_{n-1}(\gamma r)r^2$$

$$\equiv \beta(\gamma) \tag{41}$$

Since the elements of equations (40) and (41) are now functions only of γ, this set of equations may be solved for $A_\alpha(\gamma)$ and $B_\alpha(\gamma)$.

$$A_\alpha(\gamma) - nB_\alpha(\gamma) = \alpha(\gamma) \tag{42}$$

and

$$A_\alpha(\gamma) + (n + 1)B_\alpha(\gamma) = \beta(\gamma) \tag{43}$$

The solutions for this set of equations are:

$$A_\alpha(\gamma) = \frac{1}{2n + 1}[(n + 1)\alpha + n\beta] \tag{44}$$

$$B_\alpha(\gamma) = \frac{1}{2n + 1}[\beta - \alpha] \tag{45}$$

Expanding equations (44) and (45) using the definitions of $\alpha(\gamma)$ and $\beta(\gamma)$ yields:

$$A_\alpha(\gamma) = \frac{(-1)^n 4(n + 1)}{(2n + 1)} \int_0^\infty dr[\rho_\alpha(r) - n\sigma_\alpha(r)]i_{n+1}(\gamma r)r^2$$

$$+ \frac{4n(-1)^n}{2n + 1} \int_0^\infty dr[\rho_\alpha(r) + (n + 1)\sigma_\alpha(r)]i_{n-1}(\gamma r)r^2$$

$$B_\alpha(\gamma) = \frac{(-1)^n 4}{2n + 1} \int_0^\infty dr[\rho_\alpha(r) + (n + 1)\sigma_\alpha(r)]i_{n-1}(\gamma r)r^2$$

$$- \frac{4(-1)^n}{(2n + 1)} \int_0^\infty dr[\rho_\alpha(r) - n\sigma_\alpha(r)]i_{n+1}(\gamma r)r^2$$

These equations may be greatly simplified using the recursion relations in equations (32) and (33) and the definitions in equations (25), (26), and (27)

$$A_\alpha(\gamma) = 4(-1)^n \int_0^\infty dr i_n'(\gamma r) \frac{\langle \vec{P}_\alpha ; \vec{K} \rangle}{\langle \vec{P}_\alpha ; \vec{P}_\alpha \rangle}$$

$$+ \frac{n(n+1)}{\gamma r} \frac{\langle \vec{Q}_\alpha ; \vec{K} \rangle}{\langle \vec{Q}_\alpha ; \vec{Q}_\alpha \rangle} i_n(\gamma r) r^2 \qquad (46)$$

$$B_\alpha(\gamma) = 4(-1)^n \int_0^\infty dr \frac{i_n(\gamma r)}{\gamma r} \frac{\langle \vec{P}_\alpha ; \vec{K} \rangle}{\langle \vec{P}_\alpha ; \vec{P}_\alpha \rangle}$$

$$+ \frac{[\gamma r i_n(\gamma r)]}{\gamma r} \frac{\langle \vec{Q}_\alpha ; \vec{K} \rangle}{\langle \vec{Q}_\alpha ; \vec{Q}_\alpha \rangle} r^2 \qquad (47)$$

$$C_\alpha(\gamma) = 4(-1)^{n+1} \int_0^\infty dr \frac{\langle \vec{R}_\alpha ; \vec{K} \rangle}{\langle \vec{R}_\alpha ; \vec{R}_\alpha \rangle} i_n(\gamma r) r^2 \qquad (29)$$

Where equation (29) has been included for convenience. The three sets of coefficients given above contain all the unknown information needed in the expansion for $\vec{K}(\vec{r})$ given in equation (23). There is, however, some addition symmetry information contained in the above three equations.

Equations (29), (46), and (47) may be written in matrix form. The first column vector is made up of the expansion coefficients. The next vector is a row vector which is made up of a set of integral operators. The matrix which transforms the integral operators into the sets of expansion coefficients consists of the kernals of the various r transforms required to derive the expansion coefficients.

$$\begin{pmatrix} A_\alpha \\ B_\alpha \\ C_\alpha \end{pmatrix} = 4(-1)^n \left(\int_0^\infty dr\, r^2 \frac{\langle \vec{P}_\alpha ; \vec{K} \rangle}{\langle \vec{P}_\alpha ; \vec{P}_\alpha \rangle}, \right.$$

$$\left. \int_0^\infty dr\, r^2 \frac{\langle \vec{Q}_\alpha ; \vec{K} \rangle}{\langle \vec{Q}_\alpha ; \vec{Q}_\alpha \rangle}, -\int_0^\infty dr\, r^2 \frac{\langle \vec{R}_\alpha ; \vec{K} \rangle}{\langle \vec{R}_\alpha ; \vec{R}_\alpha \rangle} \right)$$

$$\cdot \begin{pmatrix} i'_n(\gamma r) & \dfrac{i_n(\gamma r)}{\gamma r} & 0 \\ \dfrac{n(n+1) i_n(\gamma r)}{\gamma r} & \dfrac{[\gamma r i_n(\gamma r)]'}{\gamma r} & 0 \\ 0 & 0 & i_n(\gamma r) \end{pmatrix} \qquad (47a)$$

The parallel matrix formulation of the $\vec{L}$, $\vec{M}$, and $\vec{N}$ functions in the $\vec{P}$, $\vec{Q}$, and $\vec{R}$ spherical harmonics has similar transformation matrix. In this case the spherical harmonics themselves form the constant matrix, with no integration implied. Post multiplication is used and the transformation matrix is the transpose of the transformation matrix given above. The matrix formulation of the $\vec{L}$, $\vec{M}$, and $\vec{N}$ functions is given below.

The methodology for evaluating the above coefficients for a specific vector is straightforward, but for clarity, an expansion of the dyadic delta function is given in Chapter IV.

$$\begin{pmatrix} \vec{L}_\alpha \\ \vec{N}_\alpha \\ \vec{M}_\alpha \end{pmatrix} = \begin{pmatrix} i'_n(\gamma r) & \dfrac{n(n+1)}{\gamma r} i_n(\gamma r) & 0 \\ \dfrac{i_n(\gamma r)}{\gamma r} & \dfrac{[\gamma r i_n(\gamma r)]'}{\gamma r} & 0 \\ 0 & 0 & i_n(\gamma r) \end{pmatrix} \begin{pmatrix} \vec{P}_\alpha \\ \vec{Q}_\alpha \\ \vec{R}_\alpha \end{pmatrix} \qquad (47b)$$

IV. EXPANSION OF THE DYADIC DELTA FUNCTION

Extension of the Wave Function Expansion to Dyads

Equation (23) and the derivation steps leading to the coefficients given in equations (29), (46), and (47) may be easily generalized to include the expansion of dyads. Consider the expansion of an arbitrary dyadic function in three dimensions. The dyadic function has the same general restrictions as the vector expanded above. In spherical coordinates the expansion can be given as:

$$\vec{\vec{K}}(\vec{r}) = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_{n=o}^{\infty} \sum_{m=-n}^{n} \sum_{\sigma = {e \atop o}} \vec{L}_{nm\sigma}(\gamma, \vec{r})\vec{A}_{nm\sigma}(\gamma)$$

$$+ \vec{N}_{nm\sigma}(\gamma, \vec{r})\vec{B}_{nm\sigma}(\gamma) + \vec{M}_{nm\sigma}(\gamma, \vec{r})\vec{C}_{nm\sigma}(\gamma) \qquad (48)$$

The vector products above are outer or dyadic products. The dyadic expansion coefficients analogous to the vector expansion coefficients are given in equations (49), (50), and (51). The products forming the coefficients in the example are anterior dyadic products. An analogous case may be developed using a posterior outer product. For this case the vector products in equation (48) must be commuted. Also, the bracket products of the form

$$\left\langle \vec{Q}_{\alpha} \, ; \, \vec{\vec{K}} \right\rangle$$

must be commuted. The coefficients for the anterior outer product case follow.

$$\vec{A}_\alpha(\gamma) = 4(-1)^n \int_0^\infty dri_n'(\gamma r) \frac{\left\langle \vec{P}_\alpha ; \vec{\vec{K}} \right\rangle}{\left\langle \vec{P}_\alpha ; \vec{P}_\alpha \right\rangle}$$

$$+ \frac{n(n+1)}{\gamma r} \frac{\left\langle \vec{Q}_\alpha ; \vec{\vec{K}} \right\rangle}{\left\langle \vec{Q}_\alpha ; \vec{Q} \right\rangle} i_n(\gamma r) r^2 \tag{49}$$

$$\vec{B}_\alpha(\gamma) = 4(-1)^n \int_0^\infty dr \frac{i_n(\gamma r)}{\gamma r} \frac{\left\langle \vec{P}_\alpha ; \vec{\vec{K}} \right\rangle}{\vec{P}_\alpha ; \vec{P}_\alpha}$$

$$+ \frac{[\gamma r i_n(\gamma r)]'}{\gamma r} \frac{\left\langle \vec{Q}_\alpha ; \vec{\vec{K}} \right\rangle}{\left\langle \vec{Q}_\alpha ; \vec{Q} \right\rangle} \tag{50}$$

$$\vec{C}_\alpha(\gamma) = 4(-1)^{n+1} \int_0^\infty dr \frac{\left\langle \vec{R}_\alpha ; \vec{\vec{K}} \right\rangle}{\left\langle \vec{R}_\alpha ; \vec{R}_\alpha \right\rangle} i_n(\gamma r) r^2 \tag{51}$$

These coefficients complete the expansion for a dyad in the function space of interest. They may now be applied specifically to the dyadic delta function.

Expansion of the Dyadic Delta Function

The dyadic delta function $\vec{\vec{I}}\,\delta(\vec{r} - \vec{r}\,')$ has the following form

$$\vec{\vec{I}}\,\delta(\vec{r} - \vec{r}\,') = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \delta(\vec{r} - \vec{r}\,') \tag{52}$$

This function is useful as a driving function for use in obtaining a dyadic Green's function for various vector differential equations. The mose useful form of the dyadic delta function is as an expansion of eigenfunctions of the appropriate homogeneous differential equation. An expansion of equation (52) in eigenfunctions of

$$(\nabla^2 - \gamma^2)\vec{\Lambda}_{nm\sigma} = \vec{0} \tag{53}$$

is given in the form of equation (48) in equation (53). Note that the $\vec{L}_n(\gamma\vec{r})$ functions are not solutions of $[\nabla \times \nabla \times + \gamma^2]\vec{\Lambda} = \vec{0}$. These two differential equations are not equivalent in the source region. Note that the anterior and posterior forms of equation (53) lead to the same expansion because of the symmetry of the dyadic delta function.

$$\vec{\vec{I}}\,\delta(\vec{r} - \vec{r}\,') = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_{n=o}^{\infty}\sum_{m=-n}^{n}\sum_{\sigma = {e \atop o}} \vec{L}_{nm\sigma}(\gamma, \vec{r})\vec{A}_{nm\sigma}(\gamma, \vec{r}\,')$$

$$+ \vec{N}_{nm\sigma}(\gamma, r)\vec{B}_{nm\sigma}(\gamma, \vec{r}\,') + \vec{M}_{nm\sigma}(\gamma, \vec{r})\vec{C}_{nm\sigma}(\gamma, r')$$

$$= \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_{\alpha} \vec{L}_{\alpha}(\gamma, \vec{r}\,')\vec{A}_{\alpha}(\gamma, \vec{r}) + \vec{M}_{\alpha}(\gamma, \vec{r}\,')\vec{B}_{\alpha}(\gamma, \vec{r})$$

$$+ \vec{N}_{\alpha}(\gamma, \vec{r}\,')\vec{C}_{\alpha}(\gamma, \vec{r}) \tag{54}$$

The coefficients are then derived from equations (49), (50), and (51)

$$\vec{A}_{\alpha}(\gamma) = 4(-1)^n i_n'(\gamma r') \frac{\vec{P}_{\alpha}(\theta', \phi')}{\langle \vec{P}_{\alpha} ; \vec{P}_{\alpha} \rangle}$$

$$+ \frac{4n(n+1)(-1)^n}{\gamma r'} \frac{\vec{Q}_{\alpha}(\theta', \phi')}{\langle \vec{Q}_{\alpha} ; \vec{Q}_{\alpha} \rangle} i_n(\gamma r') \tag{55}$$

$$\vec{B}_\alpha(\gamma) = 4(-1)^n \frac{i_n(\gamma r')}{\gamma r'} \frac{\vec{P}_\alpha(\theta', \phi')}{\left\langle \vec{P}_\alpha ; \vec{P}_\alpha \right\rangle}$$

$$+ \frac{4[\gamma r' i_n(\gamma r')]'(-1)^n}{\gamma r'} \frac{\vec{Q}_\alpha(\theta', \phi')}{\left\langle \vec{Q}_\alpha ; \vec{Q}_\alpha \right\rangle} \tag{56}$$

$$\vec{C}_\alpha(\gamma) = 4(-1)^{n+1} \frac{\vec{R}_\alpha(\theta', \phi')}{\left\langle \vec{R}_\alpha ; \vec{R}_\alpha \right\rangle} i_n(\gamma r') \tag{57}$$

The normalization products are defined specifically in equations (8) and (9). Knowledge of the coefficients given in equations (55), (56), and (57) completes the expansion of the dyadic delta function.

This expansion is also an example of how the technique developed in this note may be used to calculate the coefficients of any vector or dyad in the space of solutions to the vector Helmholtz equation or vector wave equation.

V. SCALAR FUNCTIONS

Introduction

Completeness of this discussion of the expansion of vectors and dyads in terms of the vector wave functions requires a discussion of the parallel expansion of scalar functions in terms of the scalar wave functions. If the scalar function that is being used in the expansion is represented as the divergence of a vector then the expansion may be used to check the coefficients of the vector expansions developed earlier in this note.

Expansion of a Scalar Function

A scalar function $K_L(r, \theta, \phi)$ may be expanded in an orthogonality expansion using solutions of the scalar Helmholtz equation as the expansion set. Let

$$K_L(\vec{r}) = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_\alpha D_\alpha \psi_\alpha(\gamma, \vec{r}) \tag{58}$$

where the $\psi_\alpha(\gamma\vec{r})$ are solutions of

$$(\nabla^2 - \gamma^2)\psi_\alpha(\gamma\vec{r}) = 0$$

and may be written as

$$\psi_\alpha(\gamma\vec{r}) = i_n(\gamma r)Y_\alpha(\theta, \phi)$$

Equation (58) may be written as

$$K_L(\vec{r}) = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_\alpha D_\alpha i_n(\gamma r)Y_\alpha(\theta, \phi)$$

$$K_L(\vec{r}) = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_\alpha D_\alpha i_n(\gamma r) Y_\alpha(\theta, \phi)$$

The unknown coefficients, D_α, may be determined in a straightforward manner by using the orthogonality relations of the $Y_\alpha(\theta, \phi)$.

$$\frac{\langle Y_\alpha, K_L \rangle}{\langle Y_\alpha, Y_\alpha \rangle} = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma D_\alpha i_n(\gamma r)$$

Multiplying both sides by $i_n(\gamma' r)r^2$, integrating over r, and carrying out the required γ integration yields

$$D_\alpha = 4(-1)^{n+1}\int_0^\infty dr \frac{\langle Y_\alpha, K_L \rangle}{\langle Y_\alpha, Y_\alpha \rangle} i_n(\gamma r) r^2 \tag{59}$$

Let the scalar function $K_L(\vec{r})$ represent the divergence of a vector $\vec{K}(\vec{r})$. Substituting this definition into equation (59) yields

$$D_\alpha = 4(-1)^{n+1}\int_0^\infty \frac{\langle Y_\alpha, \nabla \cdot \vec{K} \rangle}{\langle Y_\alpha, Y_\alpha \rangle} i_n(\gamma r) r^2 \tag{60}$$

for the expansion

$$\nabla \cdot \vec{K}(\vec{r}) = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_\alpha D_\alpha \psi_\alpha \tag{61}$$

The Equivalent Vector Expansion

The expansion described in equations (60) and (61) may also be found by expanding the vector $\vec{K}(\vec{r})$ as in Chapter III and taking the divergence of the expansion.

$$\vec{K}(\vec{r}) = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_\alpha A_\alpha \vec{L}_\alpha(\gamma\vec{r}) + B_\alpha \vec{N}_\alpha(\gamma\vec{r}) + C_\alpha \vec{M}_\alpha(\gamma\vec{r}) \qquad (23)$$

The appropriate coefficients are given in equations (29), (46), and (47). Taking the divergence of equation (23) yields

$$\nabla \cdot \vec{K}(\vec{r}) = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_\alpha A_\alpha \nabla \cdot \vec{L}_\alpha(\gamma\vec{r}) \qquad (62)$$

since $\nabla \cdot \vec{M}_\alpha \equiv \nabla \cdot \vec{N}_\alpha \equiv 0$. The divergence operator may be commuted with the integral and sum operators since γ is independent of the coordinates and the series is convergent by assumption.

Equation (62) may be reduced using the relations:

$$\nabla \cdot \vec{L}_\alpha(\vec{r}) \equiv \frac{1}{\gamma}\nabla \cdot \nabla\psi_\alpha(\vec{r}) \equiv \gamma\psi_\alpha(\vec{r})$$

Therefore:

$$\nabla \cdot \vec{K}(\vec{r}) = \frac{1}{2\pi i}\int_{\gamma_o - i\infty}^{\gamma_o + i\infty} d\gamma \sum_\alpha \gamma A_\alpha \psi_\alpha(\gamma\vec{r}) \qquad (63)$$

Comparing equations (61) and (63) demonstrates that the two are expansions of the same function in terms of the same set of orthogonal functions. The coefficients must be the same. Therefore we would like to show

$$D_\alpha = \gamma A_\alpha \qquad (64)$$

Equation (64) may be proved by substituting equations (46) and (60) into equation (64) and then reducing the resulting expressions to an identity. This may be accomplished if the normal component of the product $\psi\vec{K}(\gamma\vec{r})$ is restricted to vanish at infinity. This is a less stringent restriction on

$\vec{K}(\gamma\vec{r})$ than the restrictions required to assure convergence of the transforms in equations (29), (46), and (47). If these coefficients do not exist then the comparison is meaningless.

Comparison of the Techniques

Substituting equations (46) and (60) into equation (64) yields

$$4(-1)^{n+1}\int_o^\infty dr \frac{\left\langle Y_\alpha, \nabla \cdot \vec{K}(\vec{r})\right\rangle}{\left\langle Y_\alpha, Y_\alpha\right\rangle} i_n(\gamma r) r^2$$

$$= 4(-1)^n \int_o^\infty dr \left\{ i'_n(\gamma r) \frac{\left\langle \vec{P}_\alpha ; \vec{K}\right\rangle}{\left\langle \vec{P}_\alpha ; \vec{P}_\alpha\right\rangle} \right.$$

$$\left. + \; n(n+1) \frac{\left\langle \vec{Q}_\alpha ; \vec{K}\right\rangle}{\left\langle \vec{Q}_\alpha ; \vec{Q}_\alpha\right\rangle} \frac{i_n(\gamma r)}{\gamma r} \right\} \gamma r^2 \tag{65}$$

Dividing through by the various normalization constants yields

$$\int_o^\infty dr \left\langle Y_\alpha, \nabla \cdot \vec{K}\right\rangle i_n(\gamma r) r^2 = -\gamma \int_o^\infty dr \left\langle \vec{P}_\alpha ; \vec{K}\right\rangle i_n(\gamma r)$$

$$+ \left\langle \vec{Q} ; \vec{K}\right\rangle \frac{i_n(\gamma r)}{\gamma r} r^2 \tag{66}$$

Observing that the parts of the right side of equation (66) look suspicously like $\vec{L}_n(\gamma\vec{r})$ functions and combining the two integration notations allows equation (66) to be written as

$$\int_V \psi_\alpha \nabla \cdot \vec{K} dV = -\int_V \nabla\psi_\alpha \cdot \vec{K} dV \tag{67}$$

The two sides of this equation now differ only by a term of the form

$$\int_V \nabla \cdot (\psi \vec{K}) dV = \oint_S \psi \vec{K}(\gamma r) \cdot \hat{n}\, dS \tag{68}$$

The integral over the boundary described on the right side of equation (68) has been restricted to vanish as S is extended to infinity. Therefore the equivalence of the coefficients in equation (64) is established.

VI. RECOMMENDATIONS

The expansion of the dyadic delta function and the expansion technique itself provide useful information for formulating a general dyadic Green's function in spherical coordinates. Additional work is needed to determine the completeness and uniqueness properties of expansion techniques for dyadic Green's functions near a distributed source density.

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