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The Combined Field: Boundary Conditions, Integral Representation, Dyadic Green's Functions and Some Theorems and Concepts

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Abstract

This note deals with the combined field, current, etc., in the microscopic and macroscopic formulations. The combined field equation, continuity equation, and the Gauss's law for the combined field are derived in macroscopic and microscopic formulations. Boundary conditions between two different media for the combined field are derived. Integral equations in the scalar and dyadic formulations are derived for the combined field. Poynting's theorem and the reciprocity theorem for the combined field are derived along with the matrix representation of the combined field.

CHAPTER 1

INTRODUCTION

In the analysis of electromagnetic radiation and scattering problems, in general, electric and magnetic field quantities are calculated in a dependent manner where either $\tilde{\vec{E}}$ or $\tilde{\vec{H}}$ is first calculated and the remaining quantity is obtained by Maxwell's equations. Because of the inherent difficulties involved in taking the derivative on a computer, a more compact formulation is necessary. The present report which is one in a series of reports to be written will examine the feasibility of the combined field formulation.

The combined field vector in its present formulation was first introduced by Bateman,[1] and was effectively used by Itoh.[2] Tai[3] has derived some useful relationships using the combined field. However, in recent years there appears to be little if any work done on the theory and applications of the combined field. Baum[4] has derived some energy and reciprocity theorems along with some concepts of SEM[5] for the combined fields. It has been shown[6] that the combined field forms a useful tool in obtaining a generalized Babinet's principle. In this report the combined field is studied from the applications point of view. Integral formulations, boundary conditions, Poynting's vector, and reciprocity principle are studied for the combined field. In future reports some applications of the combined field are proposed to be studied with the intention of testing the general utility of the combined field formulation.

In this report, chapter 2 deals with the traditional form of Maxwell's equations, vector and scalar potentials. The combined field, current, charge, vector, and scalar potentials are defined. The combined field equation for the Faraday-Maxwell-Ampere law, combined continuity equation, and Gauss's law for the combined field are derived and are expressed in the microscopic and macroscopic formulations.

In chapter 3, boundary conditions between two different media are obtained for the normal and tangential components of the combined field. These boundary conditions are shown to be much simpler than the traditional boundary conditions. An integral equation for the surface current density is obtained in the combined field formulation. The combined dyadic Green's function is defined and is shown to satisfy both first order and second order differential equations. A combined dyadic operator $\overleftrightarrow{D}_q$ is defined.

In chapter 4, Poynting's theorem and the reciprocity theorem for the combined field are derived. The combined field, current, etc., are also expressed in the matrix notation.

CHAPTER 2

MAXWELL'S EQUATIONS FOR THE COMBINED FIELDS AND SOME INTEGRAL REPRESENTATIONS

2.1 Maxwell's Equations, Vector, Scalar Potentials, and Boundary Conditions

Maxwell's equations for the case where electric and magnetic currents, charges are present are given by

$$\nabla \times \tilde{\vec{E}} = -s\mu\tilde{\vec{H}} - \tilde{\vec{J}}_m \tag{2.1}$$

$$\nabla \times \tilde{\vec{H}} = s\epsilon\tilde{\vec{E}} + \tilde{\vec{J}} \tag{2.2}$$

$$\nabla \cdot \tilde{\vec{B}} = \nabla \cdot (\mu\tilde{\vec{H}}) = \tilde{\rho}_m \tag{2.3}$$

$$\nabla \cdot \tilde{\vec{D}} = \nabla \cdot (\epsilon\tilde{\vec{E}}) = \tilde{\rho} \tag{2.4}$$

where tilde $\sim$ denotes Laplace transformed quantities. If ϵ and μ are assumed to be the material constants, in terms of the scalar and vector potentials, we can write

$$\tilde{\vec{E}} = -\nabla\tilde{\phi} - s\tilde{\vec{A}} - \frac{1}{\epsilon}\nabla \times \tilde{\vec{A}}_m \tag{2.5}$$

$$\tilde{\vec{H}} = \frac{1}{\mu}\nabla \times \tilde{\vec{A}} - \nabla\tilde{\phi}_m - s\tilde{\vec{A}}_m \tag{2.6}$$

with the Lorentz gauge relations

$$\nabla \cdot \tilde{\vec{A}} + \frac{s}{c^2}\tilde{\phi} = 0 \tag{2.7}$$

$$\nabla \cdot \tilde{\vec{A}}_m + \frac{s}{c^2}\tilde{\phi}_m = 0 \tag{2.8}$$

The wave equations for the fields are

$$[\nabla \times \nabla\times + \gamma^2]\tilde{\vec{E}} = -s\mu\tilde{\vec{J}} - \nabla \times \tilde{\vec{J}}_m \tag{2.9}$$

$$[\nabla \times \nabla\times + \gamma^2]\tilde{\vec{H}} = \nabla \times \tilde{\vec{J}} - s\epsilon\tilde{\vec{J}}_m \tag{2.10}$$

where

$$\gamma = \frac{s}{c} \sim \text{propagation constant} \tag{2.11}$$

$$c = \frac{1}{\sqrt{\mu\epsilon}} \sim \text{wave velocity} \tag{2.12}$$

$$Z = \sqrt{\frac{\mu}{\epsilon}} \sim \text{wave impedance} \tag{2.13}$$

Helmholz equations for the potentials are

$$[\nabla^2 - \gamma^2]\tilde{\vec{A}} = -\mu\tilde{\vec{J}} \tag{2.14}$$

$$[\nabla^2 - \gamma^2]\tilde{\vec{A}}_m = -\epsilon\tilde{\vec{J}}_m \tag{2.15}$$

$$[\nabla^2 - \gamma^2]\tilde{\phi} = -\frac{1}{\epsilon}\tilde{\rho} \tag{2.16}$$

$$[\nabla^2 - \gamma^2]\tilde{\phi}_m = -\frac{1}{\mu}\tilde{\rho}_m \tag{2.17}$$

Consider two regions of different constitutive relationships separated by a surface S_o as shown in fig. 1.

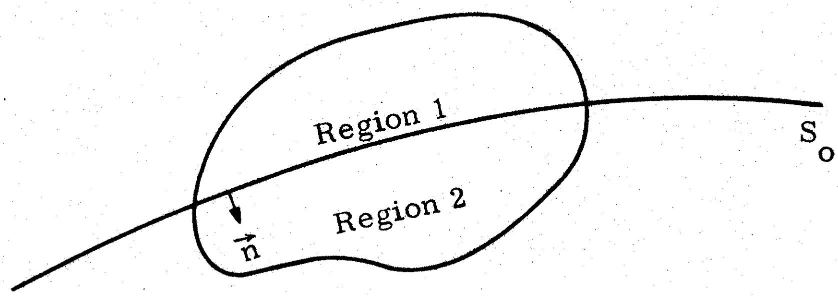

Figure 1. Boundary between two regions

The boundary conditions for the electric and magnetic fields are given by

$$\vec{n} \times [\tilde{\vec{H}}_2 - \tilde{\vec{H}}_1] = \tilde{\vec{J}}_s \tag{2.18}$$

$$\vec{n} \times [\tilde{\vec{E}}_2 - \tilde{\vec{E}}_1] = -\tilde{\vec{J}}_{m_s} \tag{2.19}$$

$$\vec{n} \cdot (\tilde{\vec{D}}_2 - \tilde{\vec{D}}_1) = \tilde{\rho}_s \tag{2.20}$$

$$\vec{n} \cdot (\tilde{\vec{B}}_2 - \tilde{\vec{B}}_1) = \tilde{\rho}_{m_s} \tag{2.21}$$

The radiation condition is given by

$$\lim_{r\to\infty} r\left[\nabla \times \begin{pmatrix} \tilde{\vec{E}} \\ \tilde{\vec{H}} \end{pmatrix} + \gamma \vec{e}_r \times \begin{pmatrix} \tilde{\vec{E}} \\ \tilde{\vec{H}} \end{pmatrix}\right] = 0 \qquad \mathrm{Re}(s) \geq 0 \tag{2.22}$$

The continuity equations are given by

$$\nabla \cdot \tilde{\vec{J}} = -s\tilde{\rho} \tag{2.23}$$

$$\nabla \cdot \tilde{\vec{J}}_m = -s\tilde{\rho}_m \tag{2.24}$$

2.2 Maxwell's Equations for the Combined Fields, Combined Vector, and Scalar Potentials and Polarization

In the literature[3] combined fields, potentials, etc., have always been defined with the free space as the reference. We will make no exception and define them the same way. It is clear that this definition can very simply be extended to uniform material media. We define the combined field, current, etc., as

$$\tilde{\vec{F}}_q \equiv \tilde{\vec{E}} + qiZ_o\tilde{\vec{H}} \tag{2.25}$$

$$\tilde{\vec{K}}_q \equiv \tilde{\vec{J}} + q\frac{i}{Z_o}\tilde{\vec{J}}_m \tag{2.26}$$

$$\tilde{Q}_q = \tilde{\rho} + q\frac{i}{Z_o}\tilde{\rho}_m \tag{2.27}$$

$$\tilde{\vec{C}}_q = \tilde{\vec{A}} + qiZ_o\tilde{\vec{A}}_m \tag{2.28}$$

$$\tilde{\phi}_q = \tilde{\phi} + qiZ_o\tilde{\phi}_m \tag{2.29}$$

where the separation index $q = \pm 1$. The ambiguity sign $\pm$ associated with the separation index is used to construct the fields, currents, etc., from the combined field, current, etc. In (2.25)-(2.29) $\tilde{\vec{F}}_q$, $\tilde{\vec{K}}_q$, $\tilde{Q}_q$, $\tilde{\vec{C}}_q$, $\tilde{\phi}_q$ are defined as combined field, combined current density, combined charge density, combined vector potential, and combined scalar potential, respectively. Z_o is the wave impedance in free space given by $Z_o = \sqrt{\mu_o/\epsilon_o}$ where μ_o and ϵ_o are the permeability and the permittivity, respectively, of free space. Multiplying (2.2) by qiZ_o and adding to (2.1) we obtain

$$[\nabla\times - qi\gamma]\tilde{\vec{F}}_q = qiZ_o\tilde{\vec{K}}_q \tag{2.30}$$

which is the Maxwell's equation for the combined field. Similarly from (2.3) and (2.4),

$$\nabla \cdot \tilde{\vec{F}}_q = \frac{1}{\epsilon_o} \tilde{Q}_q \tag{2.31}$$

which is the divergence equation for the combined field and from (2.23) and (2.24)

$$\nabla \cdot \tilde{\vec{K}}_q = -s\tilde{Q}_q \tag{2.32}$$

From (2.9), (2.10) the combined field is related to the combined potentials by

$$\tilde{\vec{F}}_q = -\nabla\tilde{\phi}_q + [-s + qic\nabla\times]\tilde{\vec{C}}_q \tag{2.33}$$

and the Lorentz gauge is given by

$$\nabla \cdot \tilde{\vec{C}}_q + \frac{s}{c^2}\tilde{\phi}_q = 0 \tag{2.34}$$

The combined field wave equation is given by

$$[\nabla \times \nabla\times + \gamma^2]\tilde{\vec{F}}_q = -[s\mu_o - qiZ_o\nabla\times]\tilde{\vec{K}}_q \tag{2.35}$$

and the Helmholtz equation for the combined potentials is given by

$$[\nabla^2 - \gamma^2]\tilde{\vec{C}}_q = -\mu_o\tilde{\vec{K}}_q \tag{2.36}$$

$$[\nabla^2 - \gamma^2]\tilde{\phi}_q = -\frac{1}{\epsilon_o}\tilde{Q}_q \tag{2.37}$$

The radiation condition for the combined field is given by

$$\lim_{r\to\infty} r[\nabla \times \vec{\tilde{F}}_q + \gamma \vec{e}_r \times \vec{\tilde{F}}_q] = \vec{0} \qquad (\mathrm{Re}(s) \geq 0) \tag{2.38}$$

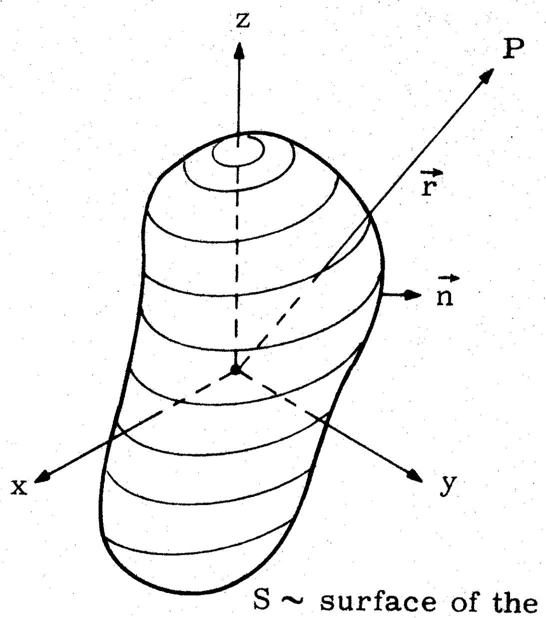

Figure 2. Coordinate picture for the scatterer

If the region of interest is a simple material medium, we can define

$$s\vec{\tilde{B}} = s(\mu - \mu_o)\vec{\tilde{H}} + s\mu_o\vec{\tilde{H}} \equiv \vec{\tilde{J}}_m + s\mu_o\vec{\tilde{H}} \tag{2.39}$$

$$s\vec{\tilde{D}} = s(\epsilon - \epsilon_o)\vec{\tilde{E}} + s\epsilon_o\vec{\tilde{E}} \equiv \vec{\tilde{J}} + s\epsilon_o\vec{\tilde{E}} \tag{2.40}$$

Since the polarization vectors are associated with the properties of the matter, from (2.1) and (2.40) we obtain

$$\nabla \times \vec{\tilde{E}} = -s\mu_o\vec{H} - \left(\vec{\tilde{J}}_{m_p} + \vec{\tilde{J}}_{m_{imp}}\right) \tag{2.41a}$$

or alternately

$$\nabla \times \vec{\tilde{E}} = -s\mu_o \vec{\tilde{H}} - \vec{\tilde{J}}_m \tag{2.41b}$$

where the term associated with the magnetic polarization combined with the magnetic impressed current $\vec{\tilde{J}}_{m_{imp}}$ can be considered as the effective or total magnetic current. Similarly from (2.2) and (2.39)

$$\nabla \times \vec{\tilde{H}} = s\epsilon_o \vec{\tilde{E}} + (\vec{\tilde{J}}_p + \vec{\tilde{J}}_{imp}) \tag{2.42}$$

where $(\vec{\tilde{J}}_p + \vec{\tilde{J}}_{imp})$ is taken as the total electric current. This formulation is different from that followed by Stratton[7] and Van Bladel.[8] However, for our purposes this is more suitable than the conventional formulation. It must be noted that in the present formulation there exists a possibility for magnetic materials to be associated with magnetic charge density $\tilde{\rho}_m$. This, however, should come as no surprise to us because of our familiarity in handling magnetic currents and charges. The problem of obtaining the boundary conditions for the combined field will be delegated to another section.

2.3 Some Integral Representation of Maxwell's Combined Field Equation

In the preceding discussions we have been concerned with the macroscopic equation. In terms of the problems associated with obtaining the boundary conditions, microscopic equations are more convenient. For convenience we will rewrite (2.30), (2.31), and (2.32) here.

$$[\nabla\times - qi\gamma]\vec{\tilde{F}}_q = qiZ_o\vec{\tilde{K}}_q \tag{2.43}$$

This can be called the Faraday-Maxwell-Ampere law for the combined field. The equation of continuity for the combined current is given by

$$\nabla \cdot \vec{\tilde{K}}_q = -s\tilde{Q}_q \tag{2.44}$$

These two equations constitute the independent equations of the combined field. These encompass the traditional four equations generally used as the independent equations given by (2.1), (2.2), (2.23), and (2.24). The Gauss' law for the combined field is given by

$$\nabla \cdot \tilde{\vec{F}}_q = \frac{1}{\epsilon_o}\tilde{Q}_q \tag{2.45}$$

This, however, is a dependent equation and can easily be derived from the independent equations (2.43) and (2.44).

Consider a regular region of volume V bounded by a surface S as shown in fig. 2. Since the region is assumed to be regular, we can draw a unit outward pointing normal $\vec{n}$ to the closed surface S at every point on S. This restriction is not an essential consideration in our development but does make the development simpler. Upon integration of (2.43), (2.44), and (2.45) throughout the volume V and expressing as a volume integral over V,

$$\iiint_V \nabla \times \tilde{\vec{F}}_q dV = qi\gamma \iiint_V \tilde{\vec{F}}_q dV + qiZ_o \iiint_V \tilde{\vec{K}}_q dV \tag{2.46}$$

$$\iiint_V \nabla \cdot \tilde{\vec{K}}_q dV = -s \iiint_V \tilde{Q}_q dV \tag{2.47}$$

$$\iiint_V \nabla \cdot \tilde{\vec{F}}_q dV = \frac{1}{\epsilon_o} \iiint_V \tilde{Q}_q dV \tag{2.48}$$

Applying the vector Stokes theorem

$$\iiint_V (\nabla \times \tilde{\vec{G}}) dV = \oiint_S (\vec{n} \times \tilde{\vec{G}}) dS \tag{2.49}$$

to (2.46) we obtain

$$\oiint_S (\vec{n} \times \tilde{\vec{F}}_q) dS = qi\gamma \iiint_V \tilde{\vec{F}}_q dV + qiZ_o \iiint_V \tilde{\vec{K}}_q dV \tag{2.50}$$

Similarly applying Gauss's theorem

$$\iiint_V \nabla \cdot \tilde{\vec{G}} dV = \oiint_S \tilde{\vec{G}} \cdot \vec{n} dS \tag{2.51}$$

to (2.47) and (2.48) gives

$$\oiint_S \tilde{\vec{K}}_q \cdot \vec{n} dS = -s \iiint_V \tilde{Q}_q dV \tag{2.52}$$

$$\oiint_S \tilde{\vec{F}}_q \cdot \vec{n} dS = \frac{1}{\epsilon_o} \iiint_V \tilde{Q}_q dV \tag{2.53}$$

From the Stokes theorem, (2.46) can be written as

$$\oint_{C_s} \tilde{\vec{F}}_q \cdot d\vec{\ell} = qi\gamma \oiint_S \tilde{\vec{F}}_q \cdot \vec{n} dS + qiZ_o \oiint_S \tilde{\vec{K}}_q \cdot \vec{n} dS \tag{2.54}$$

where C_s is a closed contour and S is the surface bounded by C_s as shown in fig. 3.

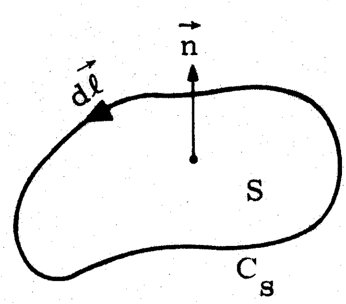

Figure 3. Surface with contour boundary for Stokes' theorem

Different authors have different views on which of (2.50), (2.54) is easier to work with in obtaining the boundary conditions. No preference will be indicated here because of the equally simple formulations of (2.50) and (2.54).

CHAPTER 3

BOUNDARY CONDITIONS AND INTEGRAL EQUATION FOR THE COMBINED FIELD

3.1 Boundary Conditions

It is clear that in electromagnetic problems uniqueness of solutions is solely based on defining the boundary conditions in a consistent manner. This uniqueness problem obviously applies to the combined field as well simply because the combined field is a unique linear combination of the electric and magnetic fields. As a consequence, a unique solution of the combined field wave equation (2.35) is completely dependent upon defining the boundary conditions on $\tilde{\vec{F}}_q$ in a self-consistent manner. The boundary condition at infinity for the combined field is already given by (2.38).

Consider a regular region bounded by a closed surface S as shown in fig. 1. The surface S_o separates the region into two sub regions as shown in fig. 1. It is assumed that region 1 differs from 2 in the constitutive relationships. Hence from (2.50) we obtain

$$n \times \left[\tilde{\vec{F}}_{2_q} - \tilde{\vec{F}}_{1_q}\right] = qiZ_o\tilde{\vec{K}}_{s_q} \tag{3.1}$$

where $\tilde{\vec{K}}_{s_q}$ is the combined surface current. This relationship expresses the boundary condition for the tangential components of the combined field. Similarly for the normal component from (2.53)

$$\vec{n} \cdot \left[\tilde{\vec{F}}_{2_q} - \tilde{\vec{F}}_{1_q}\right] = \frac{1}{\epsilon_o}\tilde{Q}_{s_q} \tag{3.2}$$

where $\tilde{Q}_{s_q}$ is the surface charge density. If the first region is perfectly conducting and region 2 is free space

$$\vec{n} \times [\tilde{\vec{F}}_q + \tilde{\vec{F}}_{-q}] = \vec{0} \tag{3.3}$$

$$\vec{n} \times [\tilde{\vec{F}}_q - \tilde{\vec{F}}_{-q}] = 2qiZ_o\tilde{\vec{J}}_s \tag{3.4}$$

$$\vec{n} \cdot [\tilde{\vec{F}}_q - \tilde{\vec{F}}_{-q}] = 0 \tag{3.5}$$

$$\vec{n} \cdot [\tilde{\vec{F}}_q + \tilde{\vec{F}}_{-q}] = \frac{2}{\epsilon_o}\tilde{\rho}_s \tag{3.6}$$

3.2 Integral Equation for the Combined Field

If we consider a volume V bounded by a closed surface S which contains scatterers with closed surfaces $S_1, \cdots, S_n$ and sources $\tilde{\vec{J}}$, $\tilde{\vec{J}}_m$, $\tilde{\rho}$, $\tilde{\rho}_m$ (not contained within the S_j) we can write the electric and magnetic fields at a point $P \notin S_i$ as[9]

$$\tilde{\vec{E}}_P = -\int_V \left[s\mu_o\tilde{\vec{J}}\tilde{G}_o + \tilde{\vec{J}}_m \times \nabla'\tilde{G}_o - \frac{\tilde{\rho}}{\epsilon_o}\nabla'\tilde{G}_o \right] dV$$

$$+ \int_{S_1+\cdots+S_n} \left[-s\mu_o\tilde{G}_o(\vec{n}\times\tilde{\vec{H}}) + (\vec{n}\times\tilde{\vec{E}})\times\nabla'\tilde{G}_o + (\vec{n}\cdot\tilde{\vec{E}})\nabla'\tilde{G}_o \right] dS' \tag{3.7}$$

where ∇' operates on the primed coordinates, and on S_j $\vec{n}$ is a unit outward pointing normal (outward with respect to the S_j (not as a boundary of V).

$$\tilde{\vec{H}}_P = -\int_V \left[s\epsilon_o\tilde{G}_o\tilde{\vec{J}}_m - \tilde{\vec{J}} \times \nabla'\tilde{G}_o - \frac{\tilde{\rho}_m}{\mu_o}\nabla'\tilde{G}_o \right] dV$$

$$+ \int_{S_1+\cdots+S_n} \left[s\epsilon_o(\vec{n}\times\tilde{\vec{E}})\tilde{G}_o + (\vec{n}\times\tilde{\vec{H}})\times\nabla'\tilde{G}_o + (\vec{n}\cdot\tilde{\vec{H}})\nabla'\tilde{G}_o \right] dS \tag{3.8}$$

$$\tilde{G}_o(\vec{r},\vec{r}') = \frac{1}{4\pi}\,\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \tag{3.9}$$

where $\vec{r}$ is the field point while $\vec{r}'$ represents the source point. It is clear that (3.8) may be obtained from (3.7) by using the duality of Maxwell's equations. A similar equation can be obtained for the combined field $\vec{F}_q$ by starting from (2.35) and following the same procedure as for $\vec{E}_P$ and $\vec{H}_P$. However, a simpler method is to use (3.7), (3.8) in conjunction with (2.25)-(2.27). Combining (3.7) with (3.8) we obtain

$$\begin{aligned}\tilde{\vec{F}}_{P_q} = &-\int_V \left[s\tilde{G}_o\left(\mu_o\tilde{\vec{J}} + qiZ_o\epsilon_o\tilde{\vec{J}}_m\right) + \left(\tilde{\vec{J}}_m - qiZ_o\tilde{\vec{J}}\right)\times\nabla'\tilde{G}_o - \left(\frac{\tilde{\rho}}{\epsilon_o} + qiZ_o\frac{\tilde{\rho}_m}{\mu_o}\right)\nabla'\tilde{G}_o\right] dV \\ &+ \int_{S_1+\cdots+S_n} \left[-s\tilde{G}_o\vec{n}\times\left(\mu_o\tilde{\vec{H}} - qiZ_o\epsilon_o\tilde{\vec{E}}\right) + \left\{\vec{n}\times\left(\tilde{\vec{E}} + qiZ_o\tilde{\vec{H}}\right)\right\}\times\nabla'\tilde{G}_o \right. \\ &\left. + \vec{n}\cdot\left(\tilde{\vec{E}} + qiZ_o\tilde{\vec{H}}\right)\nabla'\tilde{G}_o\right] dS'\end{aligned} \tag{3.10}$$

Using (2.25)-(2.27) this can be rewritten as

$$\begin{aligned}\tilde{\vec{F}}_{P_q} = &-\int_V \left[s\mu_o\tilde{G}_o\tilde{\vec{K}}_q - qiZ_o\tilde{\vec{K}}_q\times\nabla'\tilde{G}_o - \frac{1}{\epsilon_o}\tilde{Q}_q\nabla'\tilde{G}_o\right] dV \\ &+ \int_{S_1+\cdots+S_n}\left[qi\gamma\tilde{G}_o\left(\vec{n}\times\tilde{\vec{F}}_q\right) + \left(\vec{n}\times\tilde{\vec{F}}_q\right)\times\nabla'\tilde{G}_o + \left(\vec{n}\cdot\tilde{\vec{F}}_q\right)\nabla'\tilde{G}_o\right] dS'\end{aligned} \tag{3.11}$$

The surfaces $S_1, \ldots, S_n$ represent the surfaces of the scatterers. If we consider a source free region containing a single scattering surface S as in figure 2 with the incident field $\vec{F}_{inc_q}$,

$$\tilde{\vec{F}}_{P_q} = \tilde{\vec{F}}_{inc_q} + \int_S \left[qi\gamma\tilde{G}_o\left(\vec{n}\times\tilde{\vec{F}}_q\right) + \left(\vec{n}\times\tilde{\vec{F}}_q\right)\times\nabla'\tilde{G}_o + \left(\vec{n}\cdot\tilde{\vec{F}}_q\right)\nabla'\tilde{G}_o\right] dS' \qquad P\notin S \tag{3.12}$$

Using Maue's integral[8, 9] we can write (3.12) as

$$\tilde{\vec{F}}_q(\vec{r}) = T\tilde{\vec{F}}_{inc_q}(\vec{r}) + T\int_S \left[qi\gamma\tilde{G}_o\left(\vec{n}\times\tilde{\vec{F}}_q\right) + \left(\vec{n}\times\tilde{\vec{F}}_q\right)\times\nabla'\tilde{G}_o + \left(\vec{n}\cdot\tilde{\vec{F}}_q\right)\nabla'\tilde{G}_o\right] dS' \tag{3.13}$$

where

$$T = \begin{cases} 1 & \vec{r} \notin S \\ 2 & \vec{r} \in S \text{ (S is regular)} \end{cases} \tag{3.14}$$

If S is not regular, for $\vec{r} \in S$

$$T = \left(1 - \frac{\Omega}{4\pi}\right)^{-1} \tag{3.15}$$

where Ω is the exterior solid angle subtended at $\vec{r}$. If $\vec{r} \notin S$, $\Omega = 0$; $\vec{r} \in S$ for S regular or on the smooth portion of S, $\Omega = 2\pi$ and is some other value for a non regular region given by

$$d\Omega = \frac{\vec{n}\cdot\vec{e}_r}{r^2}\, dS \tag{3.16}$$

where $\vec{n}$ is a unit outward pointing normal to S, $\vec{e}_r$ is a unit vector in the $\vec{r}$ direction. To calculate the current on a scatterer, one would impose appropriate boundary conditions for the scatterer in (3.13). Taking the cross product of (3.13) with $\vec{n}$,

$$\vec{n}\times\tilde{\vec{F}}_q = T\vec{n}\times\tilde{\vec{F}}_{inc_q}$$

$$+ T\vec{n}\times\int_S \left[qi\gamma\tilde{G}_o\left(\vec{n}\times\tilde{\vec{F}}_q\right) + \left(\vec{n}\times\tilde{\vec{F}}_q\right)\times\nabla'\tilde{G}_o + \left(\vec{n}\cdot\tilde{\vec{F}}_q\right)\nabla'\tilde{G}_o\right] dS' \tag{3.17}$$

If the scatterer is assumed to be perfectly conducting, we can rewrite (3.1) and (3.2) as

$$\vec{n} \times \tilde{\vec{F}}_q = qiZ_o \tilde{\vec{J}}_s \tag{3.18}$$

$$\vec{n} \cdot \tilde{\vec{F}}_q = \frac{1}{\epsilon_o} \tilde{\rho}_s = -\frac{1}{s\epsilon_o} \nabla \cdot \tilde{\vec{J}}_s \tag{3.19}$$

and

$$\vec{n} \times \tilde{\vec{F}}_{q_{inc}} = qiZ_o \tilde{\vec{K}}_{s_{q_{inc}}} \tag{3.20}$$

where

$$qiZ_o \tilde{\vec{K}}_{s_{q_{inc}}} \equiv qiZ_o \left[\tilde{\vec{J}}_{inc} + q \frac{i}{Z_o} \tilde{\vec{J}}_{m_{inc}} \right] \equiv qiZ_o \left(\vec{n} \times \tilde{\vec{H}}_{inc} \right) + \left(\vec{n} \times \tilde{\vec{E}}_{inc} \right) \tag{3.21}$$

Substituting (3.18)-(3.20) into (3.17) we obtain

$$qiZ_o \tilde{\vec{J}}_s = 2qiZ_o \tilde{\vec{K}}_{s_{q_{inc}}}$$

$$+ 2 \int_S \left[-\gamma Z_o \tilde{G}_o \left(\vec{n} \times \tilde{\vec{J}}_s \right) + qiZ_o \left(\vec{n} \times \tilde{\vec{J}}_s \right) \times \nabla' \tilde{G}_o + \frac{\tilde{\rho}_s}{\epsilon_o} \left(\vec{n} \times \nabla' \tilde{G}_o \right) \right] dS' \tag{3.22a}$$

or

$$qiZ_o \tilde{\vec{J}}_s = 2qiZ_o \tilde{\vec{K}}_{s_{q_{inc}}}$$

$$+ 2 \int_S \left[-\gamma Z_o \tilde{G}_o \left(\vec{n} \times \tilde{\vec{J}}_s \right) + qiZ_o \left(\vec{n} \times \tilde{\vec{J}}_s \right) \times \nabla' \tilde{G}_o - \frac{\nabla' \cdot \tilde{\vec{J}}_s}{s\epsilon_o} \left(\vec{n} \times \nabla' \tilde{G}_o \right) \right] dS' \tag{3.22b}$$

which can be called the combined field integral equation (CFIE). Using the definition (3.21), it is easy to obtain the conventional electric and magnetic field integral equations from (3.22a) or (3.22b).

3.3 Dyadic Green's Function Formulation

The scalar Green's function $\tilde{G}_o(\vec{r},\vec{r}\,';s)$ is defined as the solution to

$$[\nabla^2 - \gamma^2]\tilde{G}_o(\vec{r},\vec{r}\,';s) = -\delta(\vec{r} - \vec{r}\,') \tag{3.23}$$

with the radiation condition

$$\lim_{r\to\infty} r\left[\frac{\partial}{\partial r} + \gamma\right]\tilde{G}_o(\vec{r},\vec{r}\,';s) = 0 \qquad \mathrm{Re}(s) \geq 0 \tag{3.24}$$

The Green's function is given by

$$\tilde{G}_o(\vec{r},\vec{r}\,';s) = \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r} - \vec{r}\,'|} \tag{3.25}$$

The dyadic Green's function $\tilde{\overleftrightarrow{G}}_o(\vec{r},\vec{r}\,';s)$ is the solution of

$$[\nabla\times\nabla\times + \gamma^2]\tilde{\overleftrightarrow{G}}_o(\vec{r},\vec{r}\,';s) = \overleftrightarrow{I}\delta(\vec{r} - \vec{r}\,') \tag{3.26}$$

with the radiation condition

$$\lim_{r\to\infty} r[\nabla\times + \gamma\vec{e}_r\times]\tilde{\overleftrightarrow{G}}_o(\vec{r},\vec{r}\,';s) = \vec{0} \qquad (\mathrm{Re}(s) \geq 0) \tag{3.27}$$

given by

$$\tilde{\vec{\vec{G}}}_o(\vec{r}, \vec{r}\,'; s) = \left[\vec{\vec{I}} - \frac{1}{\gamma^2}\nabla\nabla\right]\tilde{G}_o(\vec{r}, \vec{r}\,'; s) \tag{3.28}$$

The component form of (3.28) is given in SSN 179. Following the notation used in SSN 179, we can write the electric and magnetic fields as

$$\tilde{\vec{E}}(\vec{r}) = -s\mu_o \left\langle \tilde{\vec{\vec{G}}}_o \, ; \, \tilde{\vec{J}} \right\rangle - \left\langle \nabla\tilde{G}_o \overset{\times}{,} \tilde{\vec{J}}_m \right\rangle \tag{3.29}$$

and

$$\tilde{\vec{H}}(\vec{r}) = -s\epsilon_o \left\langle \tilde{\vec{\vec{G}}}_o \, ; \, \tilde{\vec{J}}_m \right\rangle + \left\langle \nabla\tilde{G}_o \overset{\times}{,} \tilde{\vec{J}} \right\rangle \tag{3.30}$$

Combining (3.29) and (3.30) we can write

$$\tilde{\vec{F}}_q = -s\mu_o \left[\left\langle \tilde{\vec{\vec{G}}}_o \, ; \, \tilde{\vec{K}}_q \right\rangle - \frac{qi}{\gamma} \left\langle \nabla\tilde{G}_o \overset{\times}{,} \tilde{\vec{K}}_q \right\rangle \right] \tag{3.31}$$

or using the relationship

$$\nabla\tilde{G}_o \times = \nabla \times \tilde{\vec{\vec{G}}}_o \, . \tag{3.32}$$

we can rewrite (3.31) as

$$\tilde{\vec{F}}_q = -s\mu_o \left\langle \left[\tilde{\vec{\vec{G}}}_o - \frac{qi}{\gamma} \nabla \times \tilde{\vec{\vec{G}}}_o \right] ; \, \tilde{\vec{K}}_q \right\rangle \tag{3.33}$$

(3.33) can also be rewritten as

$$\tilde{\vec{F}}_q = -qiZ_o \left\langle \left[qi\gamma\tilde{\vec{\vec{G}}}_o + \nabla \times \tilde{\vec{\vec{G}}}_o \right] ; \, \tilde{\vec{K}}_q \right\rangle \tag{3.34}$$

We now define combined dyadic Green's functions of first and second kinds represented by $\tilde{\vec{\vec{G}}}^{(1)}_{o_q}$ and $\tilde{\vec{\vec{G}}}^{(2)}_{o_q}$ as

$$\tilde{\vec{\vec{G}}}_{o_q}^{(1)} \equiv qi\gamma\tilde{\vec{\vec{G}}}_o + \nabla \times \tilde{\vec{\vec{G}}}_o \tag{3.35}$$

and

$$\tilde{\vec{\vec{G}}}_{o_q}^{(2)} \equiv \tilde{\vec{\vec{G}}}_o - \frac{qi}{\gamma}\nabla \times \tilde{\vec{\vec{G}}}_o \tag{3.36}$$

Hence we can write

$$\tilde{\vec{F}}_q = -qiZ_o \left\langle \tilde{\vec{\vec{G}}}_{o_q}^{(1)} ; \tilde{\vec{K}}_q \right\rangle \tag{3.37}$$

or

$$\tilde{\vec{F}}_q = -s\mu_o \left\langle \tilde{\vec{\vec{G}}}_{o_q}^{(2)} ; \tilde{\vec{K}}_q \right\rangle \tag{3.38}$$

and

$$\tilde{\vec{\vec{G}}}_{o_q}^{(2)} = -\frac{qi}{\gamma}\tilde{\vec{\vec{G}}}_{o_q}^{(1)} = \frac{1}{qi\gamma}\tilde{\vec{\vec{G}}}_{o_q}^{(1)} \tag{3.39}$$

From (3.35), (3.36), (3.37), and (3.38) we note that $\vec{\vec{G}}_{o_q}^{(2)}$ is a generalization of $\vec{\vec{G}}_e$ and $\vec{\vec{G}}_{o_q}^{(1)}$ is a generalization of $\vec{\vec{G}}_m$ as defined by C. T. Tai.[10] It is easy to show that $\vec{\vec{G}}_{o_q}^{(1)}$ is a solution of

$$[\nabla\times - qi\gamma]\tilde{\vec{\vec{G}}}_{o_q}^{(1)} = -\vec{\vec{I}}\delta(\vec{r} - \vec{r}\,') \tag{3.40}$$

with the boundary condition

$$\lim_{r\to\infty} r\left[\nabla \times \tilde{\vec{\vec{G}}}_{o_q}^{(1)} + \gamma\vec{e}_\gamma \times \tilde{\vec{\vec{G}}}_{o_q}^{(1)}\right] = \vec{\vec{0}} \quad (\mathrm{Re}(s) \geq 0) \tag{3.41}$$

The combined dyadic Green's function of the second kind also satisfies (3.40) with the definition (3.39). Unlike the dyadic Green's function for a vector wave equation, the combined dyadic Green's function satisfies a first order differential equation. Taking the curl of both sides of (3.40), we obtain

$$[\nabla \times \nabla\times + \gamma^2]\tilde{\vec{\vec{G}}}_{o_q}^{(1)} = -[qi\gamma\delta(\vec{r} - \vec{r}') + \nabla\delta(\vec{r} - \vec{r}')\times]\vec{\vec{I}} \tag{3.42}$$

or

$$[\nabla \times \nabla\times + \gamma^2]\tilde{\vec{\vec{G}}}_{o_q}^{(2)} = -\left[\delta(\vec{r} - \vec{r}') - \frac{qi}{\gamma}\nabla\delta(\vec{r} - \vec{r}')\times\right]\vec{\vec{I}} \tag{3.43}$$

Hence the combined dyadic Green's function of the second kind $\tilde{\vec{\vec{G}}}_{o_q}^{(2)}$ is a solution of a second order differential equation (3.43) with the radiation condition

$$\lim_{r\to\infty} r\left[\nabla \times \tilde{\vec{\vec{G}}}_{o_q}^{(2)} + \gamma\vec{e}_r \times \tilde{\vec{\vec{G}}}_{o_q}^{(2)}\right] = \vec{\vec{0}} \quad (\mathrm{Re}(s) \geq 0) \tag{3.44}$$

Since the combined dyadic Green's functions of the first and second kinds differ by only a constant multiple factor, each of them is a solution of a first order differential equation along with a second order differential equation with the radiation condition imposed. Although the dyadic delta function used on the right-hand side of (3.40) is the traditional dyadic delta function, the driving function on the right-hand side of (3.43) involves the derivative of a delta function. There is a distinct similarity between the combined dyadic Green's functions as defined by (3.35) and (3.36) and the electric and magnetic dyadic Green's functions defined by C. T. Tai.[10] A more detailed version of this will be dealt with in future reports.

Returning to the first order differential equation (3.40), the operator $\nabla\times$ can be denoted by $\overset{\rightrightarrows}{D}\cdot$, where $\overset{\rightrightarrows}{D}$ is given in rectangular coordinates by

$$\overset{\rightrightarrows}{D} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \tag{3.45}$$

It is clear that $\overset{\rightrightarrows}{D}$ is an antisymmetric dyadic operator. Hence we can write

$$[\nabla \times -qi\gamma] = [\overset{\rightrightarrows}{D} - qi\gamma \overset{\rightrightarrows}{I}]\cdot \equiv \tilde{\overset{\rightrightarrows}{D}}_q \cdot \tag{3.46}$$

where $\tilde{\overset{\rightrightarrows}{D}}_q$ is defined as the combined operator. Using the combined dyadic operator $\tilde{\overset{\rightrightarrows}{D}}_q$, we can write (3.40) as

$$\tilde{\overset{\rightrightarrows}{D}}_q \cdot \tilde{\overset{\rightrightarrows}{G}}_{o_q}(\vec{r},\vec{r}\,';s) = -\tilde{\overset{\rightrightarrows}{I}}\delta(\vec{r}-\vec{r}\,') \tag{3.47}$$

and

$$\tilde{\overset{\rightrightarrows}{D}}_{-q} \cdot \left[\tilde{\overset{\rightrightarrows}{D}}_q \cdot \tilde{\overset{\rightrightarrows}{G}}_{o_q}(\vec{r},\vec{r}\,';s)\right] = -\tilde{\overset{\rightrightarrows}{D}}_{-q} \cdot [\tilde{\overset{\rightrightarrows}{I}}\delta(\vec{r}-\vec{r}\,')] \tag{3.48}$$

Using the operator notation, we can also write (3.42) as

$$[\nabla \times \nabla \times + \gamma^2]\tilde{\vec{\vec{G}}}_{o_q}^{(1)} = -\tilde{\vec{\vec{D}}}_{-q} \cdot [\vec{\vec{I}}\delta(\vec{r} - \vec{r}')] \tag{3.49}$$

where $\tilde{\vec{\vec{D}}}_q$ is given by

$$\tilde{\vec{\vec{D}}}_q \equiv \begin{bmatrix} -qi\gamma & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & -qi\gamma & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & -qi\gamma \end{bmatrix}, \qquad q = \pm 1 \tag{3.50}$$

and

$$\vec{\vec{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.51}$$

This work should be considered as preliminary for the theory and applications of the combined dyadic Green's function formulation. In future reports, this work is proposed to be extended.

CHAPTER 4

SOME THEOREMS AND CONCEPTS CONCERNING THE COMBINED FIELD

In this chapter we shall develop the combined Poynting's theorem and the combined reciprocity theorem.

4.1 Poynting's Theorem for the Combined Field

Consider a regular region bounded by a closed surface S as shown below.

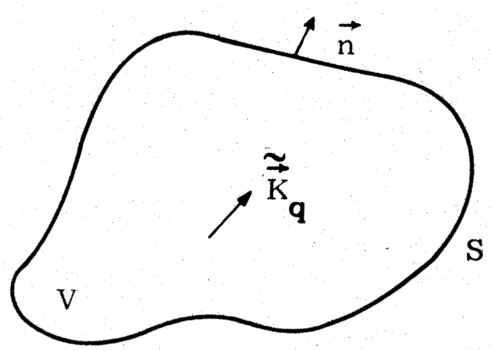

Figure 4. Source volume

The volume V is assumed to contain a combined source $\tilde{\vec{K}}_q$. We define the combined Poynting's vector $\tilde{\vec{S}}'_q$ as

$$\tilde{\vec{S}}'_q(s) = \tilde{\vec{F}}_q(s) \times \tilde{\vec{F}}_q(-s) = \tilde{\vec{S}}'_{-q}(-s) \tag{4.1}$$

The combined Poynting's vector as defined above contains terms which are generally not included in the traditional definition of the Poynting's vector. Taking the divergence of both sides of (4.1) and using a well known vector identity we obtain

$$\nabla \cdot \tilde{\vec{S}}'_q = \tilde{\vec{F}}_q(-s) \cdot \nabla \times \tilde{\vec{F}}_q(s) - \tilde{\vec{F}}_q(s) \cdot \nabla \times \tilde{\vec{F}}_q(-s) \tag{4.2}$$

Substituting (2.43) into (4.2)

$$\nabla \cdot \tilde{\vec{S}}'_q = 2qi\gamma\tilde{\vec{F}}_q(s) \cdot \tilde{\vec{F}}_q(-s) - qiZ_o[\tilde{\vec{F}}_q(s) \cdot \tilde{\vec{K}}_q(-s) - \tilde{\vec{F}}_q(-s) \cdot \tilde{\vec{K}}_q(s)] \tag{4.3}$$

Integrating both sides over a volume V and using the divergence theorem we obtain

$$\oint\!\!\oint_S \tilde{\vec{S}}'_q \cdot \vec{n}ds = 2qi\gamma \iiint_V [\tilde{\vec{F}}_q(s) \cdot \tilde{\vec{F}}_q(-s)]dV - qiZ_o \iiint_V [\tilde{\vec{F}}_q(s) \cdot \tilde{\vec{K}}_q(-s) - \tilde{\vec{F}}_q(-s) \cdot \tilde{\vec{K}}_q(s)]dV \tag{4.4}$$

where we interpret the first term on the right side (4.4) as representing the total rate of change of combined energy while the second term as the total combined energy input. These are clearly new definitions which may have to be refined in later work.

4.2 Combined Reciprocity Theorem

Here we will only be concerned with developing a simple reciprocity theorem. The reader is referred to a note by C. E. Baum[4] for a more detailed version. Consider two combined sources $\tilde{\vec{K}}_q^{(a)}$ and $\tilde{\vec{K}}_q^{(b)}$ which produce combined fields $\tilde{\vec{F}}_q^{(a)}$ and $\tilde{\vec{F}}_q^{(b)}$ in a linear matter. The combined Maxwell's equations for these cases can be written as

$$[\nabla\times - qi\gamma]\tilde{\vec{F}}_q^{(a)} = qiZ_o\tilde{\vec{K}}_q^{(a)} \tag{4.5}$$

and

$$[\nabla\times - qi\gamma]\tilde{\vec{F}}_q^{(b)} = qiZ_o\tilde{\vec{K}}_q^{(b)} \tag{4.6}$$

Using these two equations, we can write

$$\nabla \cdot \left[\tilde{\vec{F}}_q^{(a)} \times \tilde{\vec{F}}_q^{(b)}\right] = qiZ_o\left[\tilde{\vec{F}}_q^{(b)} \cdot \tilde{\vec{K}}_q^{(a)} - \tilde{\vec{F}}_q^{(a)} \cdot \tilde{\vec{K}}_q^{(b)}\right] \tag{4.7}$$

Integrating (4.7) over all space and using the divergence theorem and the radiation condition we obtain

$$\iiint_V \tilde{\vec{F}}_q^{(b)} \cdot \tilde{\vec{K}}_q^{(a)} dV = \iiint_V \tilde{\vec{F}}_q^{(a)} \cdot \tilde{\vec{K}}_q^{(b)} dV \tag{4.8}$$

This is the reciprocity principle for the combined field. The reciprocity principle as defined above is much more general than the conventional reciprocity principle. From (4.7) for the case of a source free medium, we obtain

$$\oiint_S \left[\tilde{\vec{F}}_q^{(a)} \times \tilde{\vec{F}}_q^{(b)}\right] \cdot \vec{n} dS = 0 \tag{4.9}$$

where the integration is over a closed surface whose interior or the boundary does not contain any sources. This is the combined reciprocity principle for the source free medium. This again is a much more general reciprocity compared to the Lorentz reciprocity principle.

Noting that

$$\left(\tilde{\vec{F}}_q^{(a)} \times \tilde{\vec{F}}_q^{(b)}\right) = \left(\tilde{\vec{E}}^{(a)} \times \tilde{\vec{E}}^{(b)}\right) + qiZ_o\left(\tilde{\vec{E}}^{(a)} \times \tilde{\vec{H}}^{(b)}\right)$$

$$- qiZ_o\left(\tilde{\vec{H}}^{(a)} \times \tilde{\vec{E}}^{(b)}\right) - Z_o^2\left(\tilde{\vec{H}}^{(a)} \times \tilde{\vec{H}}^{(b)}\right) \tag{4.10}$$

we can write

$$\nabla \cdot \left[\left(\tilde{\vec{E}}^{(a)} \times \tilde{\vec{H}}^{(b)}\right) - \left(\tilde{\vec{E}}^{(b)} \times \tilde{\vec{H}}^{(a)}\right)\right]$$

$$= \frac{1}{2qiZ_o} \nabla \cdot \left[\left(\tilde{\vec{F}}_q^{(a)} \times \tilde{\vec{F}}_q^{(b)}\right) - \left(\tilde{\vec{F}}_{-q}^{(a)} \times \tilde{\vec{F}}_{-q}^{(b)}\right)\right] \tag{4.11}$$

Using (4.5), (4.6), and (4.7) and integrating (4.11) over a closed space we have

$$\oiint_S \left[\left(\tilde{\vec{F}}^{(a)}_q \times \tilde{\vec{F}}^{(b)}_q\right) - \left(\tilde{\vec{F}}^{(a)}_{-q} \times \tilde{\vec{F}}^{(b)}_{-q}\right)\right] \cdot \vec{n}dS$$

$$= qiZ_o \iiint_V \left[\left(\tilde{\vec{F}}^{(b)}_q \cdot \tilde{\vec{K}}^{(a)}_q + \tilde{\vec{F}}^{(b)}_{-q} \cdot \tilde{\vec{K}}^{(a)}_{-q}\right) - \left(\tilde{\vec{F}}^{(a)}_q \cdot \tilde{\vec{K}}^{(b)}_q + \tilde{\vec{F}}^{(a)}_{-q} \cdot \tilde{\vec{K}}^{(b)}_{-q}\right)\right] dV \tag{4.12}$$

If we consider a source free medium, we can rewrite (4.12) as

$$\oiint_S \left[\left(\tilde{\vec{F}}^{(a)}_q \times \tilde{\vec{F}}^{(b)}_q\right) - \left(\tilde{\vec{F}}^{(a)}_{-q} \times \tilde{\vec{F}}^{(b)}_{-q}\right)\right] \cdot \vec{n}dS = 0 \tag{4.13}$$

This is simply the Lorentz reciprocity principle expressed in terms of the combined field. Letting the surface S recede to infinity and imposing the radiation condition,

$$\iiint_V \left(\tilde{\vec{F}}^{(b)}_q \cdot \tilde{\vec{K}}^{(a)}_q + \tilde{\vec{F}}^{(b)}_{-q} \cdot \tilde{\vec{K}}^{(a)}_{-q}\right) \cdot \vec{n}dS$$

$$= \iiint_V \left(\tilde{\vec{F}}^{(a)}_q \cdot \tilde{\vec{K}}^{(b)}_q + \tilde{\vec{F}}^{(a)}_{-q} \cdot \tilde{\vec{K}}^{(b)}_{-q}\right) \cdot \vec{n}dS \tag{4.14}$$

This is the reaction concept of Rumsey in terms of the combined field. There are several other reaction concepts, etc., which can be derived using the combined field. However, we have considered here only those that serve our immediate purpose.

4.3 Matrix Representation of the Combined Field

Considering the combined field at a point P, from (2.25) it is given by

$$\tilde{\vec{F}}_q = \tilde{\vec{E}} + qiZ_o\tilde{\vec{H}} \tag{4.15}$$

Similarly

$$\tilde{\vec{F}}_{-q} = \tilde{\vec{E}} - qiZ_o\tilde{\vec{H}} \tag{4.16}$$

Combining (4.15) and (4.16), we can write it in the matrix notation as

$$\begin{bmatrix} \tilde{\vec{F}}_q \\ \tilde{\vec{F}}_{-q} \end{bmatrix} = \begin{bmatrix} 1 & qi \\ 1 & -qi \end{bmatrix} \begin{bmatrix} \tilde{\vec{E}} \\ Z_o\tilde{\vec{H}} \end{bmatrix} \tag{4.17}$$

Taking the inverse of the 2 $\times$ 2 matrix on the right-hand side

$$\begin{bmatrix} \tilde{\vec{E}} \\ Z_o\tilde{\vec{H}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{qi}{2} & \frac{qi}{2} \end{bmatrix} \begin{bmatrix} \tilde{\vec{F}}_q \\ \tilde{\vec{F}}_{-q} \end{bmatrix} \tag{4.18}$$

This equation gives us a simple procedure to calculate $\tilde{\vec{E}}$ and $\tilde{\vec{H}}$ if $\tilde{\vec{F}}_q$ and $\tilde{\vec{F}}_{-q}$ are known. Similarly for the vector and scalar potentials $\tilde{\vec{C}}_q$ and $\tilde{\phi}_q$ we obtain

$$\begin{bmatrix} \tilde{\vec{C}}_q \\ \tilde{\vec{C}}_{-q} \end{bmatrix} = \begin{bmatrix} 1 & qi \\ 1 & -qi \end{bmatrix} \begin{bmatrix} \tilde{\vec{A}} \\ Z_o\tilde{\vec{A}}_m \end{bmatrix} \tag{4.19}$$

or

$$\begin{bmatrix} \tilde{\vec{A}} \\ Z_o \tilde{\vec{A}}_m \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{qi}{2} & \frac{qi}{2} \end{bmatrix} \begin{bmatrix} \tilde{\vec{C}}_q \\ \tilde{\vec{C}}_{-q} \end{bmatrix} \tag{4.20}$$

$$\begin{bmatrix} \tilde{\phi}_q \\ \tilde{\phi}_{-q} \end{bmatrix} = \begin{bmatrix} 1 & qi \\ 1 & -qi \end{bmatrix} \begin{bmatrix} \tilde{\phi} \\ Z_o \tilde{\phi}_m \end{bmatrix} \tag{4.21}$$

or

$$\begin{bmatrix} \tilde{\phi} \\ Z_o \tilde{\phi}_m \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{qi}{2} & \frac{qi}{2} \end{bmatrix} \begin{bmatrix} \tilde{\phi}_q \\ \tilde{\phi}_{-q} \end{bmatrix} \tag{4.22}$$

Similarly for the combined current and charge densities we obtain

$$\begin{bmatrix} \tilde{\vec{K}}_q \\ \tilde{\vec{K}}_{-q} \end{bmatrix} = \begin{bmatrix} 1 & qi \\ 1 & -qi \end{bmatrix} \begin{bmatrix} \tilde{\vec{J}} \\ \frac{1}{Z_o} \tilde{\vec{J}}_m \end{bmatrix} \tag{4.23}$$

or

$$\begin{bmatrix} \tilde{\vec{J}} \\ \frac{1}{Z_o} \tilde{\vec{J}}_m \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{qi}{2} & \frac{qi}{2} \end{bmatrix} \begin{bmatrix} \tilde{\vec{K}}_q \\ \tilde{\vec{K}}_{-q} \end{bmatrix} \tag{4.24}$$

$$\begin{bmatrix} \tilde{Q}_q \\ \\ \tilde{Q}_{-q} \end{bmatrix} = \begin{bmatrix} 1 & qi \\ \\ 1 & -qi \end{bmatrix} \begin{bmatrix} \tilde{\rho} \\ \\ \dfrac{\tilde{\rho}_m}{Z_o} \end{bmatrix} \tag{4.25}$$

or

$$\begin{bmatrix} \tilde{\rho} \\ \\ \dfrac{\tilde{\rho}_m}{Z_o} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \\ -\dfrac{qi}{2} & \dfrac{qi}{2} \end{bmatrix} \begin{bmatrix} \tilde{Q}_q \\ \\ \tilde{Q}_{-q} \end{bmatrix} \tag{4.26}$$

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