Mathematics Notes

Note 40

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Accurate Evaluation of the Impedance Matrix Elements in the Hallén Integral Equation for Wire Antennas

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Abstract

Formulas are developed for accurate evaluation of the normalized Hallén-System matrix that arises while solving for the current distribution on a wire antenna or scatterer. The technique developed here may be generalized to other similar electromagnetic problems. The thin wire example is chosen for illustrative purposes and also for comparison with previously known representations for the matrix elements. It is observed that the present method is applicable whenever the kernel functions are analytically Fourier (or Laplace) transformable.

I. Introduction

The Pocklington form[3] of the integral equation for the current distribution on a straight thin wire (receiving or driven) is well known and is given by

$$\left(\frac{\partial^2}{\partial z^2} + k_o^2\right) \int_0^L \int_0^{2\pi} J(\phi', z') \frac{e^{-jk_oR}}{4\pi R}\, a\, d\phi' dz' = -j\omega\epsilon_o E_z^{inc}(z, \phi) \tag{1.1}$$

With reference to figure 1, L and a are the length and radius of the wire. If we assume that the current density is azimuthally symmetric, the total current across the antenna cross section at location z is given by

$$I(z) = \int_0^{2\pi} J(z)\, a\, d\phi' = 2\pi a\, J(z) \tag{1.2}$$

Using equation 1.2 in equation 1.1, we get

$$\left(\frac{\partial^2}{\partial z^2} + k_o^2\right) \int_0^L I(z') K(z - z') dz' = -j\omega\epsilon_o E_z^{inc}(z) \tag{1.3}$$

where the kernel is given by

$$K(z - z') = \frac{1}{2\pi a} \int_0^{2\pi} \frac{e^{-jk_oR}}{4\pi R}\, a\, d\phi'$$

with

$$R = \left[(z - z')^2 + (2a \sin\phi'/2)^2\right]^{1/2} \tag{1.4}$$

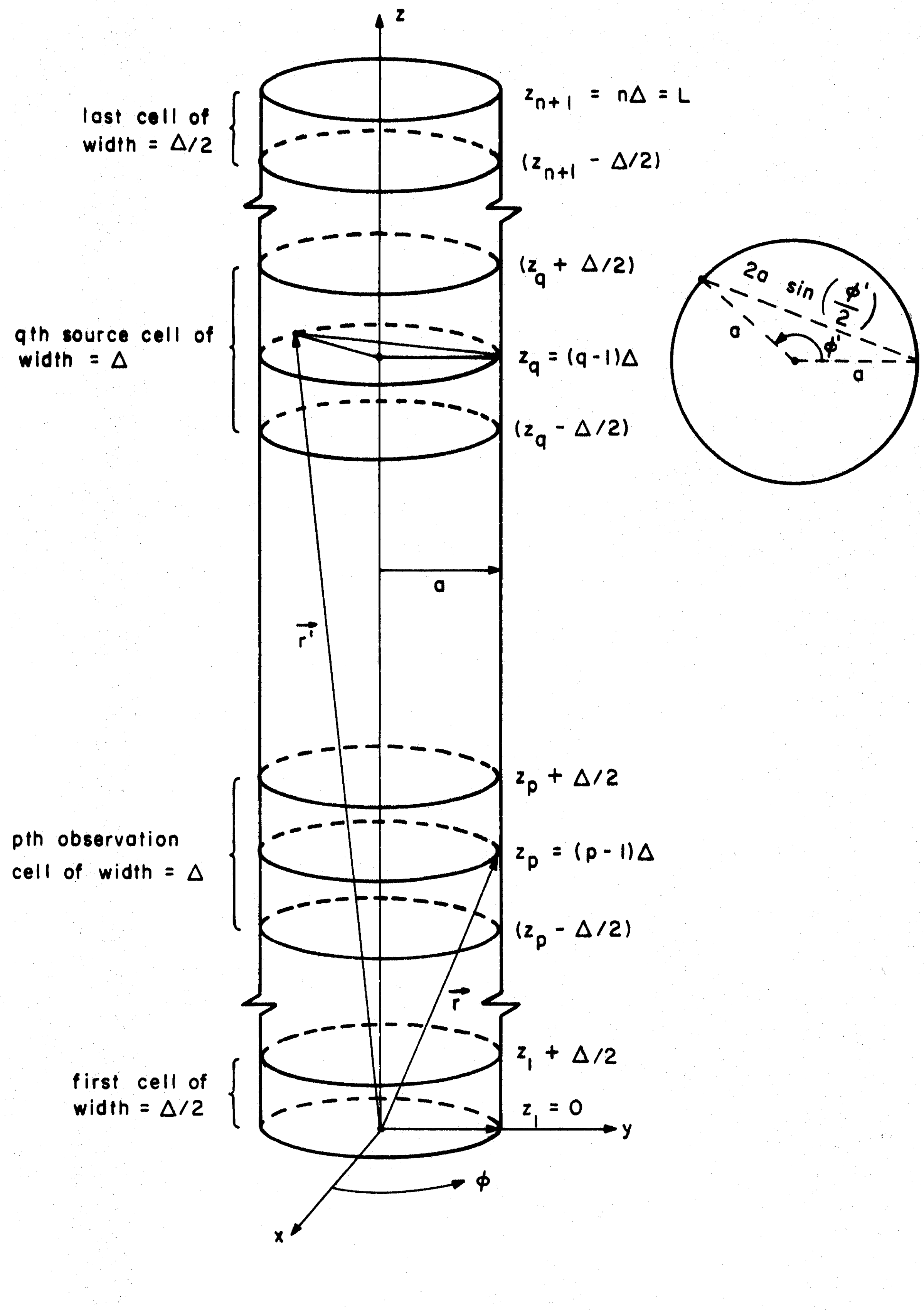

Figure 1. Geometry of the Problem Showing the Zoning of the Thin Wire

If we treat the integral in equation 1.3 as a function of z and solve the differential equation, Hallén's[1] form of integral equation is obtained as

$$\int_0^L I(z')K(z - z')dz' = A \sin(k_o z) + B \cos(k_o z)$$

$$\frac{-j\omega\epsilon_o}{k_o} \int_0^z E_z^{inc}(z') \sin[k_o(z - z')]dz' \qquad (1.5)$$

Implicit in equation 1.5 is a harmonic time dependence of the type exp(jωt) which has been Fourier transformed. Quite often, e.g., transient analysis, it is useful to introduce complex frequency s, in which case equations 1.2 and 1.5 become

$$\left(\frac{\partial^2}{\partial z^2} - \gamma^2\right) \int_0^L \tilde{I}(z')\tilde{K}(z - z')dz' = - s\epsilon_o E_z^{inc}(z) \qquad (1.6)$$

and

$$\int_0^L \tilde{I}(z')\tilde{K}(z - z')dz' = A \sinh(\gamma z) + B \cosh(\gamma z)$$

$$- \frac{1}{Z_o} \int_0^z E_z^{inc}(z') \sinh[\gamma(z - z')] dz' \qquad (1.7)$$

with

$$\tilde{K}(z - z') = \frac{1}{2\pi a} \int_0^{2\pi} \frac{e^{-\gamma R}}{4\pi R} a d\phi' \qquad (1.8)$$

where

$$\gamma = s/c = \text{propagation constant} \qquad (1.9)$$

$$z_o = \text{characteristic impedance of free space}$$

Implicit in equations 1.6 and 1.7 is a time dependence of the form exp(st) and the tilda signifies a two-sided Laplace transform. It can be easily verified that equations 1.3 and 1.5 are special cases of equations 1.6 and 1.7 by setting $s = j\omega$. Because of the differential operator in equation 1.6, equation 1.7 is more amenable for machine solution, which is usually achieved by the method of moments, e.g., Harrington.[2] This method converts the integral equation 1.7 into a system of linear equations

$$[Z_{p,q}][I_p] = [f_p(z)] \; ; \; p,q = 1, 2, 3, \cdots, (n+1)$$

or $$Z_o[Z_{N_{p,q}}][I_p] = [f_p(z)] \tag{1.10}$$

which is then solved for the unknown current distribution. We shall call this Z matrix as the Hallén-System (or simply H-S) matrix. This matrix is to be distinguished from the generalized impedance matrix[2] and associated eigen impedances[6], both of which have a more physical interpretation. If we use pulse functions for expanding the unknown current and delta functions as the testing functions, the elements of the normalized H-S matrix $[Z_N]$ are given by

$$Z_{N_{p,q}} = \frac{1}{2\pi a} \int_{z_q-(\Delta/2)}^{z_q+(\Delta/2)} dz' \int_0^{2\pi} \frac{e^{-\gamma R_p}}{4\pi R_p} \, a d\phi' \tag{1.11}$$

where $R_p = \left[(z_p - z')^2 + (2a \sin \phi'/2)^2\right]^{1/2}$ and $p, q = 1, 2 \cdots, (n+1)$. $[I_p]$ in equation 1.10 is a column matrix made up of zone currents $I_1, I_2 \cdots\cdots, I_{n+1}$. The right hand side of equation 1.10 is given by

$$f_p(z) = Z_o A \sinh(\gamma z_p) + Z_o B \sinh(\gamma z_p) - \int_0^{z_p} E_z^{inc}(z') \sinh[\gamma(z_p - z')]\, dz' \qquad (1.12)$$

A and B are constants to be determined by imposing the end conditions $I(z = 0) = I_1 = 0$ and $I(z = L) = I_{(n+1)} = 0$.

For purposes of this paper, we are concerned with the matrix $[Z_{N_{p,q}}]$ whose elements are given by equation 1.11, in which z_p and z' are respectively the observation and source points, and Δ is the cell width. It is observed that $[Z_{N_{p,q}}]$ is a Toeplitz symmetric matrix. However, if the current distribution is of interest, using $I_1 = I_{(n+1)} = 0$ and rearranging equation 1.10 becomes

$$Z_o [Z'_{N_{p,q}}][I'_p] = [V_p] \qquad (1.13)$$

where $[Z'_{N_{p,q}}]$ is $[Z_{N_{p,q}}]$ with its first and last columns replaced by $Z'_{N_{p,1}} = -\sinh(\gamma z_p)$ and $Z'_{N_{p,(n+1)}} = -\cosh(\gamma z_p)$. $[I'_p]$ is a column matrix made up of $A, I_2, I_3 \cdots I_n, B$, and the elements of the column matrix $[V_p]$ are given by

$$V_p = -\int_0^{z_p} E_z^{inc}(z') \sinh[\gamma(z_p - z')]\, dz' \qquad (1.14)$$

Now, however, $[Z'_{N_{p,q}}]$ is not a Toeplitz symmetric matrix.

Returning to equation 1.11, let us specialize it to the diagonal element. With a change of variable $x = (z' - z_q)$, it is given by

$$Z_{DN} = Z_{N_{p,p}} = \frac{1}{2\pi a} \int_{-\Delta/2}^{\Delta/2} \int_0^{2\pi} \frac{e^{-\gamma\sqrt{x^2 + (2a \sin\phi'/2)^2}}}{4\pi\sqrt{x^2 + (2a \sin\phi'/2)^2}}\, a dx\, d\phi' \qquad (1.15)$$

This integral is seemingly singular (as x and ϕ' approach 0) and the two approximations used by Harrington[2] and Tesche[3] will be reproduced below and later compared with a more accurate evaluation used in this paper.

$$Z_{DN}(\text{Harrington}^2) \simeq \frac{1}{2\pi} \ell n \left(\frac{\Delta}{a}\right) - \frac{\gamma\Delta}{2\pi} \tag{1.16}$$

$$Z_{DN}(\text{Tesche}^3) \simeq \left[\frac{\ell n\, 2}{2\pi} + \frac{1}{4\pi^2} \int_0^{2\pi} \ell n \left(\frac{\Delta}{4a} + \sqrt{\left(\frac{\Delta}{4a}\right)^2 + \sin^2\frac{\phi'}{2}}\right) d\phi' - \frac{\gamma\Delta}{4\pi}\right] \tag{1.17}$$

Equation 1.16 is derived from 1.15 by approximating

$$R(\phi') = \left[x^2 + (2a \sin\phi'/2)^2\right]^{1/2} \simeq \left[x^2 + a^2\right]^{1/2}$$

and keeping the first two terms in the expansion of $\exp(-\gamma R)$, whereas equation 1.17 retains the ϕ' dependence and also keeps only the first two terms.

II. Matrix Elements by Transform Method

In this section we shall develop formulas for more accurate evaluation of the general element in the normalized H-S matrix and later specialize it to the diagonal element and compare results with those of equations 1.16 and 1.17.

Consider the ϕ' integral of equation 1.11

$$\widetilde{K}(z_p - z') = \frac{1}{2\pi a} \int_0^{2\pi} \frac{e^{-\gamma R_p}}{4\pi R_p} a d\phi'$$

This has been shown by Hallén[1] to have a Fourier inverse transform representation given by

$$\widetilde{K}(z_p - z') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{j\xi |z_p - z'|} I_o\left(a\sqrt{\xi^2 + \gamma^2}\right) K_o\left(a\sqrt{\xi^2 + \gamma^2}\right) d\xi \tag{2.1}$$

Because of the symmetry property (z_p and z' are interchangeable), only the cosine part of the integral contributes so that

$$\widetilde{K}(z_p - z') = \frac{1}{2\pi^2} \int_0^{\infty} \cos\left[\xi(z_p - z')\right] I_o(u) K_o(u) d\xi$$

with $\quad u = a\sqrt{\xi^2 + \gamma^2}$

Using this result in equation 1.11 and performing the z' integral, a general matrix element is given by

$$Z_{N_{p,q}} = \frac{1}{\pi^2} \int_0^{\infty} I_o(u) K_o(u) \cos\left[\xi(z_p - z_q)\right] \frac{\sin(\xi\Delta/2)}{\xi} d\xi \tag{2.2}$$

If z_p and z_q are the mid points of pth and qth cells, respectively (see figure 1), they are given by

$$\left.\begin{aligned} z_p &= (p - 1)\Delta \qquad \text{with} \quad p = 1, 2, 3, \cdots (n + 1) \\ z_q &= (q - 1)\Delta \qquad \text{with} \quad q = 1, 2, 3, \cdots (n + 1) \end{aligned}\right\} \tag{2.3}$$

It is noted that the first and last cells are of width $\Delta/2$ and not Δ which makes the end points $z = 0$ and L to be the centers of imaginary cells of width Δ which extend beyond the physical ends of the antenna. This fact, if and when necessary, is easily accounted by dividing the first and last column elements by a factor of 2. Substituting equation 2.3 into equation 2.2 and with a change of variable $y = \xi\Delta/2$

$$Z_{N_{p,q}} = \frac{1}{\pi^2} \int_0^\infty \cos\,[2y(p - q)]\, \frac{\sin y}{y}\, I_o(v)\, K_o(v)\, dy = Z_{N_{q,p}} \tag{2.4}$$

where the argument v of the modified Bessel functions is given by

$$v = a\left[\gamma^2 + (4y^2/\Delta^2)\right]^{1/2} .$$

For adequate representation of the current, if we choose the cell width $\Delta \gg a$ and $\ll$ radian wavelength, then equation 2.4 is quite accurate. The radian wavelength $\lambda\!\!\!^{-}$ may be computed via $\lambda\!\!\!^{-} = c/|s|$. Equation 2.4 is now ready for machine integration and may be performed for real y or one can treat y as a complex variable and suitably deform the contour on the real axis to a contour that wraps around the branch cut. The branch points are at $y = \pm j\gamma\Delta/2$. We shall now specialize equation 2.4 for $p = q$ and obtain the normalized diagonal element as

$$Z_{DN} = \frac{1}{\pi^2} \int_0^\infty \frac{\sin y}{y}\, I_o(v)\, K_o(v)\, dy \tag{2.5}$$

with $\quad Z_o = \sqrt{\mu_o/\epsilon_o} \quad$ and $\quad v = a\left[\gamma^2 + (4y^2/\Delta^2)\right]^{1/2}$.

III. Numerical Results

To perform machine integration of equation 2.5, it is useful to investigate the singularities and the analytical structure of the integrand. As in equation 2.4, the integrand in 2.5 also has a pair of branch points at $y = \pm j\gamma\Delta/2$. If we exclude the DC frequency ($\gamma = 0$), the integration may be carried out on the positive real axis of the complex y plane. The infinite integral of equation 2.5 may be broken up as follows

$$Z_{DN} = \frac{1}{\pi^2}\left[\int_0^{UL} + \int_{UL}^{\infty} \frac{\sin y}{y} I_o(v) K_o(v)\, dy\right] \qquad (3.1)$$

If we choose an upper limit UL which permits a large argument approximation for the modified Bessel functions, e.g., $|v| = 10$ implies $UL = 5\Delta/a$ because of $|\gamma\Delta| \ll 1$. One can now write

$$Z_{DN} \simeq \frac{1}{\pi^2}\left[\int_0^{5\Delta/a} \frac{\sin y}{y} I_o(v) K_o(v)\, dy + \int_{5\Delta/a}^{\infty} \frac{\sin y}{y} \frac{1}{2v}\, dy\right] \qquad (3.2)$$

For a suitable change of variable, it is possible to write the second integral over a finite range of integration. Both of these integrals were carried out with a 40-point Gaussian quadrature routine. Care was taken to ensure outgoing wave nature by choosing the proper Bessel function product, depending on their arguments. The ranges of integration were continually subdivided until a convergence criterion of the type, the magnitude of the ratio of two successive answers was less than 10^{-3} was met. With this type of convergence, it was seen that the second integral was insignificant since it contributed values of the order of 10^{-6}. Requiring a 3-figure accuracy, the second integral was neglected. Numerical computations were made for cell sizes ranging from 1 to 100 radii for the case of 50 cells per wavelength. In order to be able to compare with previous results, the computation is done on the imaginary axis of the complex s plane. The results are found in table 1 and plotted in figure 2. The real part of Z_{DN} from Harrington's[2] approximate formula is a straight line in this figure and is valid only for $\Delta \geq 10a$. The real part from Tesche's[3] formula is not plotted because of its closeness with the values obtained by the transform method. This can be seen in table 1. Furthermore, the imaginary parts of both Harrington[2] and Tesche[3] are constant and equal to each other because of similar approximations involved. It is seen that the imaginary part from the transform method, as may be expected, is oscillatory and appears to have a mean value equal to the constant value $-\gamma\Delta/(4\pi)$ in the previous two methods. It must however be pointed out that both of the previous results are excellent approximations within the ranges of their validity and the transform method, although significantly more time consuming (10 times or more in several cases), does lead to improved accuracy.

Δ/a	Harrington		Tesche		Transform Method	
	Real	Imaginary	Real	Imaginary	Real	Imaginary
1.0	0.00000	-.01000	.09529	-.01000	.09447	-.00999
2.0	.11032	-.01000	.15460	-.01000	.15472	-.00982
3.0	.17485	-.01000	.20000	-.01000	.19974	-.01004
4.0	.22064	-.01000	.23661	-.01000	.23663	-.00956
5.0	.25615	-.01000	.26709	-.01000	.26660	-.00966
6.0	.28517	-.01000	.29309	-.01000	.29267	-.01028
7.0	.30970	-.01000	.31568	-.01000	.31568	-.01051
8.0	.33095	-.01000	.33561	-.01000	.33534	-.01040
9.0	.34970	-.01000	.35343	-.01000	.35281	-.00999
10.0	.36647	-.01000	.36952	-.01000	.36946	-.00931
20.0	.47679	-.01000	.47757	-.01000	.47691	-.00956
30.0	.54132	-.01000	.54167	-.01000	.54103	-.01073
40.0	.58710	-.01000	.58730	-.01000	.58630	-.01015
50.0	.62262	-.01000	.62274	-.01000	.62323	-.00834
60.0	.65164	-.01000	.65172	-.01000	.65053	-.01001
70.0	.67617	-.01000	.67623	-.01000	.67598	-.01168
80.0	.69742	-.01000	.69747	-.01000	.69848	-.00782
90.0	.71617	-.01000	.71621	-.01000	.71526	-.00879
100.0	.73294	-.01000	.73297	-.01000	.73150	-.00977

Table 1

Diagonal term of the normalized Hallén-System matrix for a thin wire as a function of normalized cell width; $\lambda/\Delta = 50.00$ or $k\Delta = .125664$.

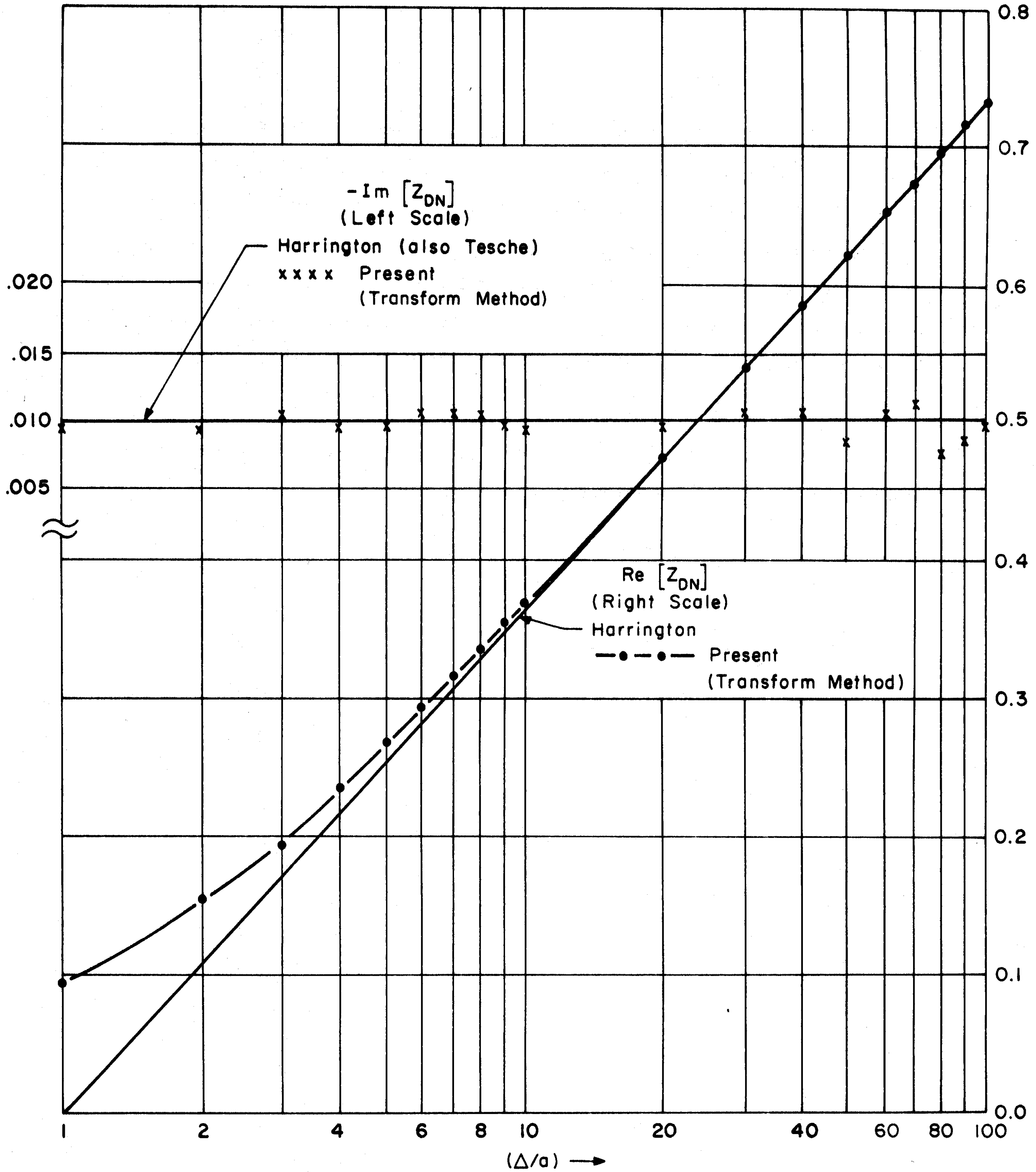

Figure 2. Plot of the Diagonal Term of the Normalized Hallén-System Matrix as a Function of Normalized Cell Width (Δ/a), $K\Delta = 0.125664$ (or 50 Cells Per Wavelength)

IV. Conclusions

An accurate way of computing the impedance matrix elements for a thin wire structure is outlined. This procedure is based on being able to analytically Fourier or Laplace transform the kernel function. The thin wire structure was only chosen for illustrative purposes, although the method is applicable if and whenever the kernel is analytically transformable. Another example where this method is usefully employed may be found in the treatment of three dimensional EM scattering from a finitely long circular cylinder by Kao.[4, 5]

Numerical results for a representative case of a thin wire structure are compared with previously available results.

V. References

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